Lax Embeddings of Generalized Quadrangles in Finite Projective Spaces

J. A. Thas H. Van Maldeghem*

1 Introduction

Definition 1.1 A (finite) *generalized quadrangle* (GQ) $\mathcal{S} = (P, B, \mathrm{I})$ is a point-line incidence geometry satisfying the following axioms.

(i) Every line is incident with $s + 1$ points for some integer $s \geq 1$ and two lines are incident with at most one common point.

(ii) Every point is incident with $t + 1$ lines for some integer $t \geq 1$ and two points are incident with at most one common line.

(iii) Given any point x and any line L not incident with x, i.e., $x \not\mathrm{I}\, L$, there exists a unique point y and a unique line M with $x \,\mathrm{I}\, M \,\mathrm{I}\, y \,\mathrm{I}\, L$.

Generalized quadrangles were introduced by Tits [14] as the geometric interpretation of certain algebraic and mixed groups of relative rank 2. The pair (s, t) is usually called the *order* or the *parameters* of $\mathcal{S}$. If $s, t \geq 2$, then $\mathcal{S}$ is called *thick*. If $t = 1$, then $\mathcal{S}$ is sometimes called a *grid*; if $s = 1$, we talk about a *dual grid*. In fact, there is a point-line duality for GQ. When $s \neq 1$, Axiom (ii) above can be weakened to

$(ii)'$ every point is on at least 2 lines.

For more information and properties of finite GQ we refer to the monograph Payne & Thas [10], for a recent survey see Thas [12], and for a treatment of some aspects of

*The second author is a Senior Research Associate of the Fund for Scientific Research – Flanders (Belgium).

infinite GQ see Van Maldeghem [15]. We restrict ourselves to introducing the finite classical quadrangles here.

The geometry of points and lines of a non-degenerate quadric of projective Witt index 1 in $\mathbf{PG}(d,q)$ is a GQ denoted by $Q(d,q)$. Here only the cases $d=3,4,5$ occur and $Q(d,q)$ has order (q,q^{d-3}). The geometry of all points of $\mathbf{PG}(3,q)$ together with all totally isotropic lines of a symplectic polarity in $\mathbf{PG}(3,q)$ is a GQ of order (q,q) denoted by $W(q)$. The geometry of points and lines of a hermitian variety of projective Witt index 1 in $\mathbf{PG}(d,q^2)$ is a GQ $H(d,q^2)$ of order (q^2,q^{2d-5}). Here $d=3$ or $d=4$. All these examples are called *classical*. However, $W(q)$ is the dual of $Q(4,q)$, and $H(3,q^2)$ is the dual of $Q(5,q)$. The classical and dual classical (finite) GQ are sometimes called the finite *Moufang* GQ.

Axiom (i) above implies that the set of points incident with a certain line L in a GQ $\mathcal{S}$ completely determines L. So we may identify L with the set of points of $\mathcal{S}$ incident with L. This way, we view the lines as subsets of P. This will be especially convenient for the purposes of this paper. Likewise, we view the lines of any projective space as subsets of the point set.

For notation and (standard) terminology not explained here, we refer to Payne & Thas [10]. Let us just mention that opposite points of a GQ are points which are not collinear and opposite lines are lines which are not concurrent. Also, if a GQ $\mathcal{S}$ has order (s,t), then a *full subquadrangle* $\mathcal{S}'$ of $\mathcal{S}$ is a subquadrangle of order (s,t'). We denote by $\mathcal{S}\setminus\mathcal{S}'$ the geometry of points and lines of $\mathcal{S}$ which do not belong to $\mathcal{S}'$.

Definition 1.2 A generalized quadrangle (GQ) $\mathcal{S}=(P,B,\mathrm{I})$ is *laxly embedded* in the projective space $\mathbf{PG}(d,q)$, $d\geq 2$, if the following conditions are satisfied:

(i) P is a point set of $\mathbf{PG}(d,q)$ which generates $\mathbf{PG}(d,q)$;

(ii) each line L of $\mathcal{S}$ is a subset of a line L' of $\mathbf{PG}(d,q)$, and distinct lines L_1,L_2 of $\mathcal{S}$ define distinct lines L_1',L_2' of $\mathbf{PG}(d,q)$.

There are two easy ways to obtain lax embeddings from another given lax embedding of a generalized quadrangle $\mathcal{S}$ in some projective space $\mathbf{PG}(d,q)$. Firstly, one can consider a proper field extension $\mathbf{GF}(q')$ of $\mathbf{GF}(q)$ and obtain a lax embedding in $\mathbf{PG}(d,q')$ in the obvious way; secondly, one might project the embedding from a point p of $\mathbf{PG}(d,q)$ onto some hyperplane H not containing p, where p is not on any line of $\mathbf{PG}(d,q)$ joining two arbitrary points of $\mathcal{S}$, and where p is not contained in the plane generated by the points of any two lines of $\mathcal{S}$ of which the corresponding lines in $\mathbf{PG}(d,q)$ intersect in $\mathbf{PG}(d,q)$ (and repeating this procedure, we may project from a subspace U of $\mathbf{PG}(d,q)$ onto some complementary space V (so $U\cap V=\emptyset$, while U and V together generate $\mathbf{PG}(q,q)$). In fact, a point p as in the second case can always be found if $d\geq 3$ and q is large enough with respect to the line size of $\mathcal{S}$ (the latter can be achieved by applying the first construction). A lax embedding which cannot be obtained from another embedding by a combination of the constructions just mentioned, will be called an *ultimate embedding* (as suggested by the referee). It is not clear to us whether an ultimate embedding should be — what one could call — *relatively universal*, i.e., it is conceivable that two different ultimate embeddings give rise to the same non-ultimate embedding by applying a number of times the two constructions mentioned above.

In order to classify all lax embeddings of a given geometry, it is enough to describe all ultimate embeddings. That is exactly what we will do below for all finite classical quadrangles (except for the symplectic quadrangle $W(s)$ with s odd) embedded in d-dimensional projective space, $d \geq 3$.

A lax embedding is called *full* if in *(ii)* above $L = L'$. Note that the description of the classical quadrangles above yields full embeddings of these. We call these full embeddings the *natural embeddings* of the classical GQ. All full embeddings of finite (Buekenhout & Lefèvre [1]) and infinite (Dienst [3, 4]) GQ are classified. In the finite case, only the natural embeddings of the classical GQ turn up. A lax embedding of a GQ $\mathcal{S}$ is called *weak* if the set of points of $\mathcal{S}$ collinear in $\mathcal{S}$ with any given point is contained in a hyperplane of $\mathbf{PG}(d, q)$. All weak embeddings in $\mathbf{PG}(3, q)$ of finite thick GQ are classified by Lefèvre-Percsy [7] and in $\mathbf{PG}(d, q)$, with $d > 3$, by Thas & Van Maldeghem [13] (although the former used a stronger definition for "weak embedding", proved by the latter to be equivalent with the notion in the present paper). Here, every weak embedding in $\mathbf{PG}(d, q)$ either turns out to be full in a subspace $\mathbf{PG}(d, q')$ of $\mathbf{PG}(d, q)$ over a subfield $\mathbf{GF}(q')$ of $\mathbf{GF}(q)$, or is the universal weak embedding of $W(2)$ in a projective 4-space over an odd characteristic finite field.

Hence for $s \neq 1$ "being fully or weakly embedded" characterizes the finite classical quadrangles amongst the others. This does not come as a surprise because from Lefèvre-Percsy [8] it immediately follows that for $s \neq 1$ weakly embedded quadrangles either admit non-trivial central collineations of the projective space, or have all their lines regular. This is not longer true for laxly embedded GQ; indeed, usually the projection of a weak embedding does not preserve the group action. Also, the quadrangles $T_2^*(O)$ of order $(q-1, q+1)$, $q > 2$, embed laxly in $\mathbf{PG}(3, q)$, do not admit central collineations and have non-regular lines. So different combinatorial and geometric methods are needed to handle laxly embedded GQ. Also, these combinatorial and geometric methods do not work in the case $d = 2$. Moreover, by projection, every quadrangle which admits an embedding in some projective space admits a lax embedding in a plane. This makes the classification problem very hard and probably impossible. Hence we will restrict our attention to the case $d \geq 3$. Notice also that by substituting $\mathbf{AG}(d, q)$ for $\mathbf{PG}(d, q)$ in Definition 1.2, we obtain the definition of laxly, weakly and fully embedded GQ in affine space. Every such embedding gives rise to a lax embedding in the corresponding projective space; the classification of all fully embedded GQ in finite affine space by THAS [11], see also PAYNE & THAS [10], Chapter 7, shows that we do not always have weak embeddings.

The elements of **PSL** will be called *special linear transformations*, those of **PGL** *linear transformations* and those of **PΓL** *semi-linear transformations.*

We split our main result in two parts. The first theorem characterizes some classical quadrangles by the fact that they admit certain embeddings.

Theorem 1.3 *If the generalized quadrangle $\mathcal{S}$ of order (s,t), $s>1$, is laxly embedded in* $\mathbf{PG}(d,q)$*, then $d \le 5$. Furthermore we have the following isomorphisms.*

- *(i) If $d=5$, then $\mathcal{S} \cong Q(5,s)$.*
- *(ii) If $d=4$, then $s \le t$.*
  - (a) *If $s=t$, then $\mathcal{S} \cong Q(4,s)$.*
  - (b) *If $t=s+2$, then $s=2$ and $\mathcal{S} \cong Q(5,2)$.*
  - (c) *If $t^2=s^3$, then $\mathcal{S} \cong H(4,s)$.*
- *(iii) If $d=3$ and $s=t^2$, then $\mathcal{S} \cong H(3,s)$.*

The second theorem characterizes the embeddings themselves of some classical quadrangles.

Theorem 1.4 *Suppose the laxly embedded generalized quadrangle $\mathcal{S}$ arises by extensions and projections from an ultimate embedding in* $\mathbf{PG}(d,q)$*, where $d \ge 3$ and $\mathcal{S} \cong Q(5,s)$, $Q(4,s)$, $H(4,s)$, $H(3,s)$ or the dual of $H(4,t)$. Then either the ultimate embedding is full, or one of the following holds.*

- *(i) $d=5$, $\mathcal{S} \cong Q(5,2)$, q is an odd prime number, the embedding is not weak and it is unique up to a special linear transformation; if $q=3$, then the embedding is full in an appropriate affine space. In all cases, the full automorphism group of $\mathcal{S}$ is induced by* $\mathbf{PGL}_6(q)$.
- *(ii) $d=4$, $\mathcal{S} \cong Q(4,2)$, q is an odd prime number, the embedding is weak and it is unique up to linear transformation; if $q=3$, then the embedding is full in an appropriate affine space. In all cases, the full automorphism group of $\mathcal{S}$ is induced by* $\mathbf{PGL}_5(q)$*, see* THAS & VAN MALDEGHEM [13].

(*iii*) $d = 4$, $\mathcal{S} \cong Q(4,3)$, $q \equiv 1 \bmod 3$, *q is either an odd prime number or the square of a prime number p with* $p \equiv -1 \bmod 3$, *the embedding is not weak, and it is unique up to a special linear transformation; if* $q = 4$, *then the embedding is full in an appropriate affine space. The group* $\mathbf{PSp}_4(3)$ *(which is not the full automorphism group of* $Q(4,3)$*) acting naturally as an automorphism group on* $W(3)$ *(which is dual to* $Q(4,3)$*) is induced on* $\mathcal{S}$ *by* $\mathbf{PSL}_5(q)$.

For more information about automorphism groups of, in particular, lax embeddings which are not ultimate, we refer to the statements in the next sections. For instance, it can happen that the automorphism group induced by $\mathbf{P\Gamma L}(d,q)$ is strictly contained in the one induced by $\mathbf{P\Gamma L}(d,q^2)$ (after field extension), see e.g. Theorem 5.1.

Note that $H(4,t)$ does not occur: it has no lax embedding at all in $\mathbf{PG}(d,q)$ for any q and any $d \geq 3$. Also, we did not consider $W(s)$ for s odd. The reason is that this quadrangle does not contain large enough grids or full subquadrangles (and these are essential for our techniques).

In two appendices we prove a characterization of the hermitian quadrangle $H(4,s^2)$ in terms of subquadrangles, and we show that any generalized quadrangle of order $(s,s+2)$ with $s > 2$ has at least one non-regular line.

We show Theorems 1.3 and 1.4 in a sequence of theorems.

2 Preliminary lemmas

If $\mathcal{S}$ is a laxly embedded GQ in $\mathbf{PG}(d,q)$, then for each line L of $\mathcal{S}$, we denote by L' the (set of points on the) corresponding line of $\mathbf{PG}(d,q)$. In particular, we have $L \subseteq L'$.

Lemma 2.1 *If the generalized quadrangle* $\mathcal{S}$ *is laxly embedded in* $\mathbf{PG}(d,q)$, $d \geq 2$, *and if L is any line of* $\mathcal{S}$, *then the points of L are the only points of* $\mathcal{S}$ *on the corresponding line* L' *of* $\mathbf{PG}(d,q)$.

Proof. Assume, by way of contradiction, that x is a point of $\mathcal{S}$ on $L' \setminus L$. If M is the line of $\mathcal{S}$ through x and concurrent with L, then also $M \subseteq L'$, contradicting (*ii*) in the definition of lax embedding. The lemma is proved.

Lemma 2.2 *If the generalized quadrangle* $\mathcal{S}$ *has* $s \neq 1$ *and is laxly embedded in* $\mathbf{PG}(d,q)$, *and if U is a subspace of* $\mathbf{PG}(d,q)$ *containing the points of two opposite lines* L, M *of* $\mathcal{S}$, *then the intersection of U with the point set of* $\mathcal{S}$ *yields a full subquadrangle laxly embedded in the subspace of U generated by the points of that intersection.*

Proof. Let $\mathcal{S}'$ be the subgeometry of $\mathcal{S}$ arising from the intersection of U with the point set of $\mathcal{S}$. Clearly $\mathcal{S}'$ satisfies (i) and (iii) of Definition 1.1. Let x be any point of $\mathcal{S}$. If $x \notin L \cup M$, then there are lines L_1, M_1 of $\mathcal{S}'$ through x meeting L, M respectively. Let N be a line of $\mathcal{S}$ concurrent with L and M, but not concurrent with either L_1 or M_1; as $s \neq 1$, such a line N exists. If N_1 is the line of $\mathcal{S}$ containing x and concurrent with N, then L_1 and N_1 are distinct lines of $\mathcal{S}'$ through x. If $x \in L \cup M$, say $x \in L$, then there is a line M_1 of $\mathcal{S}'$ through x meeting M, and $L \neq M_1$. Hence $(ii)'$ in the definition of a GQ is satisfied. So $\mathcal{S}'$ is a subquadrangle of $\mathcal{S}$, and it is clearly a full one. The lemma is proved.

For a subspace U as in the previous lemma, we say that U *induces* $\mathcal{S}'$.

Corollary 2.3 *If the generalized quadrangle $\mathcal{S}$ has $s \neq 1$ and is laxly embedded in* $\mathbf{PG}(d, q)$*, and if p is a point of $\mathcal{S}$ for which $p^\perp$ does not span* $\mathbf{PG}(d, q)$*, then no point of $\mathcal{S}$ opposite p is contained in the space $\langle p^\perp \rangle$.*

Proof. Let $\mathcal{S}$ have order (s, t). Suppose on the contrary that $\langle p^\perp \rangle$ contains a point x opposite p. Then $\langle p^\perp \rangle$ induces a full subquadrangle of order (s, t), a contradiction. The corollary follows.

This corollary shows that the definition of weakly embedded quadrangle in the present paper is equivalent to the definition of weakly embedded quadrangle in Thas & Van Maldeghem [13] (where it is required that no subspace $\langle p^\perp \rangle$ contains a point opposite p), which in turn is equivalent to the definition of weakly embedded quadrangle used in Lefèvre-Percsy [7].

3 Restrictions on the parameters

Theorem 3.1 *If the generalized quadrangle $\mathcal{S}$ of order (s, t), $s \neq 1$, is laxly embedded in* $\mathbf{PG}(d, q)$*, then $d \leq 5$. Also, if $d = 5$, then $t = s^2$ and $\mathcal{S}$ is isomorphic to the classical generalized quadrangle $Q(5, s)$. If $d = 4$, then $s \leq t$ and for $s = t$ the quadrangle $\mathcal{S}$ is isomorphic to the classical generalized quadrangle $Q(4, s)$.*

Proof. Let L, M be two opposite lines of $\mathcal{S}$. The subspace U of dimension ≤ 3 generated by L and M induces a full subquadrangle $\mathcal{S}'$ of order (s, t') in $\mathcal{S}$. Consider any point x of $\mathcal{S} \setminus \mathcal{S}'$. The subspace U' of dimension ≤ 4 generated by U and x induces a full subquadrangle $\mathcal{S}''$ of order (s, t''), with $t' < t'' \leq t$. If $t'' = t$, then $\mathcal{S}'' = \mathcal{S}$ and $d \leq 4$. By Payne & Thas [10](2.2.2), this must happen in particular if $s = t$. If $t'' < t$, then by Payne & Thas [10](2.2.2), $t = s^2 = t''^2$ and $t' = 1$. Considering a point $y \in \mathcal{S} \setminus \mathcal{S}''$, then, again by PAYNE & THAS [10](2.2.2), we see that the subspace (of dimension at most 5) generated by U' and y contains $\mathcal{S}$, hence $d \leq 5$.

If $d = 5$, then the proper full subquadrangles $\mathcal{S}'$ and $\mathcal{S}''$ of $\mathcal{S}$ in the previous paragraph do exist and hence $t = s^2$. Note that in this case every two opposite lines of $\mathcal{S}$ span a 3-dimensional subspace in $\mathbf{PG}(5, q)$. If $d = 4$, then $\mathcal{S}'$ is a proper subquadrangle and by PAYNE & THAS [10](2.2.2) we have $s \leq t$.

Now suppose $d = 4$ and $s = t$. By the foregoing, the space U generated by two non-intersecting lines of $\mathcal{S}$ meets $\mathcal{S}$ in a subquadrangle of order $(s, 1)$ (by PAYNE & THAS [10](2.2.2)). Hence every line is regular and $\mathcal{S} \cong Q(4, s)$ (see PAYNE & THAS [10](5.2.1)). If $d = 5$, then $t = s^2$ and by the previous paragraphs, every quintet of points $(x, y, z; u, v)$, with u opposite v and $x, y, z \in \{u, v\}^\perp$, is contained in a proper full subquadrangle. By PAYNE & THAS [10](5.3.5(ii)), we conclude $\mathcal{S} \cong Q(5, s)$. The theorem is proved.

Corollary 3.2 *Suppose the generalized quadrangle $\mathcal{S}$ of order (s, t), $s \neq 1$, is laxly embedded in $\mathbf{PG}(d, q)$, and let H be a hyperplane of $\mathbf{PG}(d, q)$ containing two opposite lines of $\mathcal{S}$. If $d = 5$, then H induces a subquadrangle of order (s, s) isomorphic to $Q(4, s)$. If $d = 4$ and $s = t$, then H induces a subquadrangle of order $(s, 1)$. Also, if $d = 4$ and $s = t$, then no plane of $\mathbf{PG}(4, q)$ contains two opposite lines of $\mathcal{S}$.*

Proof. This follows from the proof of the previous theorem.

4 The case $d = 3$, $s = t^2$

Theorem 4.1 *If the generalized quadrangle $\mathcal{S}$ of order (s^2, s) is laxly embedded in $\mathbf{PG}(3, q)$, then $\mathcal{S}$ is a full embedding of the classical generalized quadrangle $H(3, s^2)$ in a subspace $\mathbf{PG}(3, s^2)$ of $\mathbf{PG}(3, q)$, for the subfield $\mathbf{GF}(s^2)$ of $\mathbf{GF}(q)$.*

Proof. Assume, by way of contradiction, that the lines L, M are not concurrent in $\mathcal{S}$, but are coplanar in $\mathbf{PG}(3, q)$. Then the plane $\langle L, M\rangle$ induces a subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order (s^2, t'). By PAYNE & THAS [10](2.2.1), we then have $s^2 t' \leq t = s$, a contradiction.

Let $L_1, \ldots, L_4$, respectively $M_1, \ldots, M_4$, be four mutually non-concurrent lines of $\mathcal{S}$, with $L_i \sim M_j$ for all $i, j \in \{1, \ldots, 4\}$ with $(i, j) \neq (4, 4)$ (here " $\sim$" means concurrent in the GQ $\mathcal{S}$). As no two of the lines $L_1, \ldots, L_4$, respectively $M_1, \ldots, M_4$, are coplanar in $\mathbf{PG}(3, q)$, the corresponding lines $L_1', \ldots, L_4', M_1', \ldots, M_4'$ of $\mathbf{PG}(3, q)$ are lines of a hyperbolic quadric. So L_4' intersects M_4' in $\mathbf{PG}(3, q)$. Hence by the preceding paragraph $L_4 \sim M_4$. Now by PAYNE & THAS [10](5.3.2) we have $\mathcal{S} \cong H(3, s^2)$.

First, assume $s > 2$.
Let x and y be non-collinear points of $\mathcal{S}$. As $\mathcal{S} \cong H(3, s^2)$, we have $|\{x, y\}^{\perp\perp}| = s + 1$. Let $z \in \{x, y\}^{\perp\perp} \setminus \{x, y\}$, and let $x \in L$, $y \in M$, $z \in N$ with L, M, N pairwise non-concurrent lines of $\mathcal{S}$. As $\mathcal{S} \cong H(3, s^2)$, we have $|\{L, M, N\}^{\perp\perp}| = s + 1$. Also, each line of $\{L, M, N\}^{\perp\perp}$ contains a point of $\{x, y\}^{\perp\perp}$. Clearly the lines $L', M', N', \ldots$ of $\mathbf{PG}(3, q)$ which correspond to the lines of $\{L, M, N\}^{\perp\perp}$, belong to a regulus R of $\mathbf{PG}(3, q)$. Next, let N_1 be a line of $\mathcal{S}$ through z, with $N \neq N_1$, $N_1 \not\sim L$, $N_1 \not\sim M$ (as $s > 2$, the line N_1 exists). Again, each line of $\{L, M, N_1\}^{\perp\perp}$ contains a point of $\{x, y\}^{\perp\perp}$, and the lines $L', M', N_1', \ldots$ of $\mathbf{PG}(3, q)$ which correspond to the lines of $\{L, M, N_1\}^{\perp\perp}$ belong to a regulus R_1 of $\mathbf{PG}(3, q)$. Let U' be the line of $\mathbf{PG}(3, q)$ containing z and intersecting the lines L' and M' non-trivially. Then the lines L', M' and U' of $\mathbf{PG}(3, q)$, are common lines of the hyperbolic quadrics Q and Q_1 defined respectively by R and R_1. Hence $Q \cap Q_1 = L' \cup M' \cup U' \cup V'$, with V' a line which possibly coincides with U'.

As $\{x, y\}^{\perp\perp}$ belongs to $Q \cap Q_1$, the $s - 1$ points $z, u, \ldots$ of $\{x, y\}^{\perp\perp} \setminus \{x, y\}$ belong to $U' \cup V'$. Since $\{x, y\}^{\perp}$ contains a point of each of L, M, N, N_1, also $\{x, y\}^{\perp}$ belongs to $Q \cap Q_1$. Let $n_1 \in N_1$, $n_1 \sim x$ and $n \in N$, $n \sim x$. Then $n, n_1 \in Q \cap Q_1$. Clearly, $n, n_1 \notin U' \cup L' \cup M'$, and so, V' is the line nn_1 of $\mathbf{PG}(3, q)$. If $u \in \{x, y\}^{\perp\perp} \setminus \{x, y, z\}$ belongs to V', then the lines un and un_1 of $\mathcal{S}$ coincide, a contradiction. It follows that the $s - 1$ points of $\{x, y\}^{\perp\perp} \setminus \{x, y\}$ belong to U'. Consequently any $s - 1$ points of any hyperbolic line of $\mathcal{S}$ (that is, a point set of the form $\{v, w\}^{\perp\perp}$, with $v \not\sim w$) are collinear in $\mathbf{PG}(3, q)$. It easily follows that for $s \geq 4$ any hyperbolic line of $\mathcal{S}$ is a subset of a line of $\mathbf{PG}(3, q)$. Now let $s = 3$, and consider a line $L_1 \neq L$ of $\mathcal{S}$ through x, with $L_1 \not\sim M$. As $zu = U'$, the line U' is independent of the choice of L, M, N through respectively x, y, z. Hence U' intersects the line $L_1' \supseteq L_1$ of $\mathbf{PG}(3, q)$.

Consequently, $x \in U'$. Analogously, $y \in U'$. Hence $\{x, y\}^{\perp\perp} \subseteq U'$, and so also for $s = 3$ any hyperbolic line of $\mathcal{S}$ is a subset of a line of $\mathbf{PG}(3, q)$.

Let v be any point of $\mathcal{S}$, and let $w_1 \sim v \sim w_2, w_1 \not\sim w_2$. Then the $s + 1$ points of $\{w_1, w_2\}^{\perp\perp}$ are collinear in $\mathbf{PG}(3, q)$. Consequently the $s + 1$ lines of $\mathcal{S}$ containing v belong to a common plane π of $\mathbf{PG}(3, q)$. So $\mathcal{S}$ is weakly embedded in $\mathbf{PG}(3, q)$. Now by Lefèvre-Percsy [7] and Thas & Van Maldeghem [13], $\mathcal{S}$ is a full embedding of the GQ $H(3, s^2)$ in a subspace $\mathbf{PG}(3, s^2)$ of $\mathbf{PG}(3, q)$, for the subfield $\mathbf{GF}(s^2)$ of $\mathbf{GF}(q)$.

Next, let $s = 2$. We give an explicit description of any lax non-weak embedding of $H(3, 4)$ in $\mathbf{PG}(3, q)$. To that end, we first need an explicit description of the points and lines of $H(3, 4)$. We use coordinatization. By Hanssens & Van Maldeghem [5], we can define $H(3, 4)$ as follows. The point set of $H(3, 4)$ is the set

$$\begin{aligned}&\{(\infty)\} \cup \{(a) : a \in \mathbf{GF}(4)\}\\ &\quad \cup \{(k, b) : k \in \mathbf{GF}(2), b \in \mathbf{GF}(4)\} \cup \{(a, l, a') : a, a' \in \mathbf{GF}(4), l \in \mathbf{GF}(2)\},\end{aligned}$$

where ∞ is a symbol not contained in $\mathbf{GF}(4)$, the line set of $H(3,4)$ is the set

$$\begin{aligned}&\{[\infty]\} \cup \{[k] : k \in \mathbf{GF}(2)\}\\ &\quad \cup \{[a,l] : a \in \mathbf{GF}(4), l \in \mathbf{GF}(2)\} \cup \{[k,b,k'] : k,k' \in \mathbf{GF}(2), b \in \mathbf{GF}(4)\},\end{aligned}$$

and incidence is given by

$$[k,b,k'] \mathrm{~I~} (k,b) \mathrm{~I~} [k] \mathrm{~I~} (\infty) \mathrm{~I~} [\infty] \mathrm{~I~} (a) \mathrm{~I~} [a,l] \mathrm{~I~} (a,l,a')$$

and no other cases occur, except that $(a,l,a') \mathrm{~I~} [k,b,k']$ if and only if (viewing $\mathbf{GF}(2)$ as a subfield of $\mathbf{GF}(4)$)

$$\left\{\begin{array}{lcl} b & = & ak + a', \\ l & = & a^3k + k' + a^2b + ab^2. \end{array}\right.$$

We can now coordinatize $\mathbf{PG}(3,q)$ in such a way that, without loss of generality, the following points of $H(3,4)$ are given the following corresponding coordinates in $\mathbf{PG}(3,q)$:

in $H(3,4)$	in $\mathbf{PG}(3,q)$	in $H(3,4)$	in $\mathbf{PG}(3,q)$
(∞)	$(1,0,0,0)$	$(1,0,0)$	$(0,1,0,1)$
(0)	$(0,0,1,0)$	$(1,0,1)$	$(1,1,1,1)$
(1)	$(1,0,1,0)$	(ϵ)	$(a,0,1,0)$
$(0,0)$	$(0,0,0,1)$	$(\epsilon,0,0)$	$(0,1,0,b)$
$(0,1)$	$(1,0,0,1)$	$(0,\epsilon)$	$(1,0,0,c)$
$(0,0,0)$	$(0,1,0,0)$	$(0,0,\epsilon)$	$(0,d,1,0)$
$(0,0,1)$	$(0,1,1,0)$	$(\epsilon,0,\epsilon)$	$(a,d,1,ac)$

where $\mathbf{GF}(4) = \{0,1,\epsilon,\epsilon^2\}$, with $a,b,c,d \in \mathbf{GF}(q) \setminus \{0,1\}$ and with $ac = bd$. By PAYNE & THAS [10](5.3.5), there is a (unique) subquadrangle $\mathcal{S}'$ of order $(2,2)$ containing the points (∞), (ϵ), (0), $(0,\epsilon)$, $(\epsilon,0,\epsilon)$ and $(0,0,\epsilon)$. Since in $\mathcal{S}'$ there are three lines concurrent with $[0]$, $[\epsilon,0]$ and $[0,0]$, and since in $\mathcal{S}$ only $[\infty]$, $[0,\epsilon,0]$ and $[0,0,0]$ are concurrent with all three of $[0]$, $[\epsilon,0]$ and $[0,0]$, also the line $[0,0,0]$, and hence the points $(0,0)$, $(\epsilon,0,0)$ and $(0,0,0)$ belong to $\mathcal{S}'$. We now calculate the coordinates in $\mathbf{PG}(3,4)$ of all points of $\mathcal{S}'$. First, we must determine all points of $\mathcal{S}'$. There remains to determine six points. Clearly the point $(1,0)$ belongs to $\mathcal{S}'$. As $H(3,4)$ is non-weakly embedded in $\mathbf{PG}(3,q)$, we may assume that the point $(1,0)$ does not belong to the plane $\langle(\infty),(0),(0,0)\rangle$. Hence we may assume that $(1,0) = (x,1,y,z)$. Since the line $[1,0,0]$ belongs to $\mathcal{S}'$, also the point $(\epsilon,1,\epsilon)$ collinear with (ϵ) belongs to $\mathcal{S}'$. We can assign it the coordinates (x,u,y,z), with $u \in \mathbf{GF}(q) \setminus \{0,1\}$ (if $u=0$, then the plane $\langle(\infty),(0),(0,0)\rangle$ induces a proper full subquadrangle, yielding a contradiction). The line $[1,\epsilon,0]$ is the unique line in $\mathcal{S}'$ containing $(0,0,\epsilon)$ and concurrent with both $[\epsilon,1]$ (which contains (ϵ) and $(\epsilon,1,\epsilon)$) and $[1]$ (which contains (∞) and $(1,0)$). Since this also determines the line $[1,\epsilon,0]$ in $\mathbf{PG}(3,q)$, we can calculate that this line intersects $[1]$ in the point $(dx+ua-a, d, dy, dz)$ (which is the point $(1, \epsilon)$ of $\mathcal{S}'$) and the line $[\epsilon, 1]$ in the point $(dx+ua-a, du, dy+u-1, dz)$ (which is the point $(\epsilon, 1, 0)$ of $\mathcal{S}'$). Similarly, we see that the point $(0, 1, \epsilon)$ belongs to $\mathcal{S}'$ and that it has coordinates $(cx+ub-b, cu, cy, cz+ubc-bc)$ in $\mathbf{PG}(3, q)$. Only one point of $\mathcal{S}'$ remains, and that is $(0, 1, 0)$, which is the intersection of $[0, 1]$ (containing the points (0) and $(0, 1, \epsilon)$) and $[0, 0, 1]$ (containing the points $(0, 0)$ and $(\epsilon, 1, 0)$). We easily calculate that $(0, 1, 0)$ has coordinates $(dx + ua - a, du, dy + u - 1, dz + ubd - bd)$. But $(0, 1, 0)$ also belongs to the subquadrangle $\mathcal{S}$ obtained from $H(3, 4)$ by restricting coordinates (in the sense of Hanssens & Van Maldeghem [5]) to $\mathbf{GF}(2) \cup \{\infty\}$. We obtain all points of $\mathcal{S}$ from the points of $\mathcal{S}'$ by putting $a = b = c = d = 1$, and writing u' for u, with $u' \in \mathbf{GF}(q) \setminus \{0, 1, u\}$. Hence the point $(0, 1, 0)$ has also coordinates $(x + u' - 1, u', y + u' - 1, z + u' - 1)$. It easily follows that

$$\begin{cases} uu'(a-d) + u(d-xd) + u'(xd-a) = 0, \\ uu'(d-1) + u(yd-d) + u'(1-yd) = 0, \\ uu'(b-1) + u(1-z) + u'(z-b) = 0. \end{cases}$$

Clearly

$$\begin{vmatrix} a-d & d-xd & xd-a \\ d-1 & yd-d & 1-yd \\ b-1 & 1-z & z-b \end{vmatrix} = 0.$$

If the rank of the matrix

$$\begin{bmatrix} d-1 & yd-d & 1-yd \\ b-1 & 1-z & z-b \end{bmatrix}$$

is equal to 2, then $u = u' = uu' = \ell(byd + dz + 1 - bd - yd - z)$ for some $\ell \neq 0$, clearly a contradiction as $u \neq u'$. Consequently the rank of the above 2×3-matrix is at most 1 and so $byd + dz + 1 - bd - yd - z = 0$. As $ac = bd$, we have $acy + dz + 1 - ac - yd - z = 0$. It follows that the points $(1, 0) = (x, 1, y, z)$, $(\epsilon, 0, \epsilon) = (a, d, 1, ac)$, $(1, 0, 1) = (1, 1, 1, 1)$ and $(\infty) = (1, 0, 0, 0)$ are coplanar. As $(\epsilon, 0, \epsilon)$, $(1, 0, 1)$ and (∞) are points of $(1, 0)^\perp$, we have that $(1, 0)^\perp$ belongs to a plane. It follows that if p is any point of $H(3, 4)$ for which the lines of $H(3, 4)$ through it are non-coplanar in $\mathbf{PG}(3, q)$, then for any point $p' \in H(3, 4)$ with $p' \sim p$ and $p' \neq p$, the lines of $H(3, 4)$ through p' are coplanar.

Let p be a point of $H(3, 4)$ and assume that the lines L_1, L_2, L_3 of $H(3, 4)$ through p are not coplanar. Further, let p' be a point of $H(3, 4)$ with $p' \notin p^\perp$. If $m_i \in L_i$ with $m_i \sim p'$, then the lines of $H(3, 4)$ through m_i are coplanar, $i = 1, 2, 3$. So the points p, p', p'' of $\{p, p'\}^{\perp\perp}$ are on a common line of $\mathbf{PG}(3, q)$. As m_1, m_2, m_3 are not on a common line of $\mathbf{PG}(3, q)$, for at least one of the points p', p'' the lines of $H(3, 4)$ through it are not coplanar. It follows that the number of points r of $H(3, 4)$ for which $r^\perp$ is not contained in a plane, is at least $1 + 16 = 17$. As no two distinct such points are collinear in $H(3, 4)$, we have $17 \leq st + 1 = 9$, a contradiction.

Hence every lax embedding of $H(3, 4)$ in $\mathbf{PG}(3, q)$ is weak and the result follows.

5 The case $d = 4$ and $(s,t) \in \{(s,s),(s,s+2),(s^2,s^3)\}$

If the GQ $\mathcal{S}$ of order (s,t) is laxly embedded in $\mathbf{PG}(4,q)$, then by Theorem 3.1 we have $s \le t$. Also, every known GQ of order (s,t), with $1 \ne s \le t$, has an order of the form $(s,t) \in \{(s,s),(s,s+2),(s^2,s^3),(s,s^2)\}$. In this section we will determine, in each of the cases $(s,s),(s,s+2),(s^2,s^3)$, all lax embeddings in $\mathbf{PG}(4,q)$.

We start with an exceptional non-weak lax embedding of a small GQ.

Theorem 5.1 *Let q be a power of the prime p. If a generalized quadrangle $\mathcal{S}$ of order $(3,3)$ is laxly embedded in $\mathbf{PG}(4,q)$, then $\mathcal{S} \cong Q(4,3)$ and either $\mathcal{S}$ is weakly embedded in $\mathbf{PG}(4,q)$, or $q \equiv 1 \bmod 3$ and, up to a special linear transformation, there exists a unique (non-weak) lax embedding, which is contained in a subspace $\mathbf{PG}(4,p)$ of $\mathbf{PG}(4,q)$, if $p \equiv 1 \bmod 3$, and in a subspace $\mathbf{PG}(4,p^2)$ of $\mathbf{PG}(4,q)$, if $p \equiv 2 \bmod 3$.*

Let $\mathcal{S}$ be non-weakly lax embedded in $\mathbf{PG}(4,q)$. Then the case $q = 4$ corresponds to a full affine embedding; the case q even corresponds to a full affine embedding in an affine subspace over the subfield $\mathbf{GF}(4)$ of $\mathbf{GF}(q)$. In each case, the automorphism group $\mathbf{PSp}_4(3)$ of $\mathcal{S}$ (the group generated by all root elations of $\mathcal{S}$) is the group induced on $\mathcal{S}$ by $\mathbf{PSL}_5(q)$ and by $\mathbf{PGL}_5(q)$. If q is a perfect square and if $\sqrt{q} \equiv -1 \bmod 3$, then the full automorphism group $\mathbf{PGSp}_4(3)$ of $\mathcal{S}$ is the group induced by $\mathbf{P\Gamma L}_5(q)$; otherwise, $\mathbf{P\Gamma L}_5(q)$ just induces $\mathbf{PSp}_4(3)$.

Proof. By Theorem 3.1, $\mathcal{S} \cong Q(4,3)$. We describe $\mathcal{S}$ with coordinates as follows (see Hanssens & Van Maldeghem [5]). The points are the elements of the set

$$\{(\infty)\} \cup \{(a) : a \in \mathbf{GF}(3)\} \cup \{(k,b) : k,b \in \mathbf{GF}(3)\} \cup \{(a,l,a') : a,a',l \in \mathbf{GF}(3)\},$$

the lines are the elements of the set

$$\{[\infty]\} \cup \{[k] : k \in \mathbf{GF}(3)\} \cup \{[a,l] : a,l \in \mathbf{GF}(3)\} \cup \{[k,b,k'] : k,k',b \in \mathbf{GF}(3)\},$$

incidence is given by the general sequence

$$(a,l,a') \,\mathrm{I}\, [a,l] \,\mathrm{I}\, (a) \,\mathrm{I}\, [\infty] \,\mathrm{I}\, (\infty) \,\mathrm{I}\, [k] \,\mathrm{I}\, (k,b) \,\mathrm{I}\, [k,b,k'] \,\mathrm{I}\, (a, ak+k', b+ak^2-kk').$$

As $\mathcal{S}$ is not contained in a hyperplane, the lines $[\infty]$, $[0]$ and $[1]$ through (∞) are not contained in a plane (otherwise the proper subspace containing these lines and a line of $\mathcal{S}$ concurrent with $[\infty]$ but not containing (∞) would induce a subquadrangle of order $(3,t')$ with $3t' \le 3$ and $t' \ne 1$, a contradiction). Also, the 3-dimensional space $\langle [\infty],[0],[1] \rangle$ does not contain the point $(0,0,0)$. Hence, without loss of generality, we can choose coordinates in $\mathbf{PG}(4,q)$ as follows:

in $\mathcal{S}$	in $\mathbf{PG}(4,q)$	in $\mathcal{S}$	in $\mathbf{PG}(4,q)$
(∞)	$(1,0,0,0,0)$	$(0,0,0)$	$(0,0,1,0,0)$
(0)	$(0,1,0,0,0)$	$(0,0,1)$	$(0,1,1,0,0)$
(1)	$(1,1,0,0,0)$	$(0,0,-1)$	$(0,b,1,0,0)$
(-1)	$(a,1,0,0,0)$	$(1,0,0)$	$(0,0,1,1,0)$
$(0,0)$	$(0,0,0,1,0)$	$(1,0,1)$	$(1,1,1,1,0)$
$(0,1)$	$(1,0,0,1,0)$	$(1,0,-1)$	$(b,b,1,1,0)$
$(0,-1)$	$(b,0,0,1,0)$	$(-1,0,0)$	$(0,0,1,a,0)$
$(1,0)$	$(0,0,0,0,1)$	$(-1,0,1)$	$(a,1,1,a,0)$
$(1,1)$	$(1,0,0,0,1)$	$(-1,0,-1)$	$(ab,b,1,a,0)$

with $a,b \in \mathbf{GF}(q) \setminus \{0,1\}$. These coordinates can easily be computed, because we have grids in $\mathcal{S}$. Now also all points of the subquadrangle of order $(3,1)$ induced by the subspace generated by $[1]$ and $[0,0]$ can be computed. We obtain:

in $\mathcal{S}$	in $\mathbf{PG}(4,q)$
$(1,-1)$	$(b,0,0,0,1)$
$(1,1,0)$	$(b,b,1,0,1)$
$(1,1,1)$	$(0,0,1,0,1)$
$(1,1,-1)$	$(1,1,1,0,1)$
$(-1,-1,0)$	$(a,1,1,0,a)$
$(-1,-1,1)$	$(ab,b,1,0,a)$
$(-1,-1,-1)$	$(0,0,1,0,a)$

The point $(-1,0)$ is incident with the line $[-1,0,1]$, which contains furthermore the points $(1,0,-1)$ and $(-1,-1,0)$. But the same point $(-1,0)$ is also on $[-1,0,-1]$, which contains $(1,1,0)$ and $(-1,0,1)$. Looking at the above coordinates in $\mathbf{PG}(4,q)$ of these points, we conclude that $(-1,0)$ has coordinates $(ab+a, ab+1, a+1, a, a)$ in $\mathbf{PG}(4,q)$. Similarly, one calculates that $(-1,1)$ (on the lines $[-1,1,1] = \langle(1,0,0),(-1,-1,1)\rangle$ and $[-1,1,-1] = \langle(1,1,1),(-1,0,-1)\rangle$) has coordinates $(ab,b,a+1,a,a)$. Since (∞), $(-1,0)$ and $(-1,1)$ are collinear in $\mathbf{PG}(4,q)$, we obtain $ab+1=b$. Similarly, one calculates that $(-1,-1)$ (on the lines $[-1,-1,1] = \langle(1,0,1),(-1,-1,-1)\rangle$ and $[-1,-1,-1] = \langle(1,1,-1),(-1,0,0)\rangle$) has coordinates $(a,a,a+1,a,a)$. Since (∞), $(-1,0)$ and $(-1,-1)$ are collinear in $\mathbf{PG}(4,q)$, we obtain $ab+1=a$. Hence $a=b$ and $a^2-a+1=0$. This equation has no solution if $q \equiv -1 \bmod 3$. If $q \equiv 0 \bmod 3$, then the only solution is $a=b=-1$ and $(\infty)^\perp$ is contained in the hyperplane with equation $X_2=0$ in $\mathbf{PG}(4,q)$ (working with $X_0,\ldots,X_4$-coordinates). Hence, since (∞) is basically an arbitrary point of $\mathcal{S}$, we obtain a weak embedding.

Now suppose that $q \equiv 1 \bmod 3$. Then we can calculate all other points of $\mathcal{S}$ in $\mathbf{PG}(4,q)$ and we obtain:

in $\mathcal{S}$	in $\mathbf{PG}(4,q)$
$(-1,0)$	$(1+a,1,2-a,1,1)$
$(-1,1)$	$(a,1,2-a,1,1)$
$(-1,-1)$	$(1,1,2-a,1,1)$
$(0,1,0)$	$(a,a,1,a,1)$
$(0,1,1)$	$(a,0,1,a,1)$
$(0,1,-1)$	$(a,1,1,a,1)$
$(0,-1,0)$	$(1,1-a,1-a,1-a,1)$
$(0,-1,1)$	$(1,1,1-a,1-a,1)$
$(0,-1,-1)$	$(1,0,1-a,1-a,1)$
$(1,-1,0)$	$(1,1-a,1-a,1,1)$
$(1,-1,1)$	$(1+a,1,1-a,1,1)$
$(1,-1,-1)$	$(a,0,1-a,1,1)$
$(-1,1,0)$	$(a,a,1,1,1)$
$(-1,1,1)$	$(1,0,1,1,1)$
$(-1,1,-1)$	$(1+a,1,1,1,1)$

Now it is a tedious, but very easy calculation, to check that this representation of $Q(4,3)$ in $\mathbf{PG}(4,q)$ is indeed a lax embedding of $Q(4,3)$, and we denote that lax embedding by $\mathcal{S}[a]$. It is clear that, if $p \equiv 1 \bmod 3$, then $a \in \mathbf{GF}(p)$ and all points of the quadrangle belong to the subspace $\mathbf{PG}(4,p)$. Likewise, if $p \equiv 2 \bmod 3$, then $a \in \mathbf{GF}(p^2)$ and the embedding happens in the subspace $\mathbf{PG}(4,p^2)$.

The following linear transformations in $\mathbf{PG}(4,q)$, given by the matrices

$$\begin{bmatrix} a-1 & 0 & 0 & 1 & 1 \\ 0 & a-1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} a-1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a-1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -a & 0 & 1 \\ 1 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -a & 0 & 0 & 0 \\ 0 & 0 & -a & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 & 1+a \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 2-a \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

preserve $\mathcal{S}[a]$ (one only needs to check that these matrices map the 18 points given in the first table of this proof onto points of $\mathcal{S}[a]$; this is again a tedious but very easy job) and induce root elations which generate a flag transitive automorphism group of

$\mathcal{S}[a]$ isomorphic to $\mathbf{PSp}_4(3)$. Hence the latter is a subgroup G of $\mathbf{PSL}_5(q)$ (indeed, all determinants are fifth powers in $\mathbf{GF}(q)$ observing that $a = a^{-5}$) acting flag transitively on $\mathcal{S}[a]$.

Now it is easily checked that the special linear transformation with matrix

$$\begin{bmatrix} a^4 & 0 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & 0 & -a^2 \end{bmatrix}$$

interchanges $\mathcal{S}[a]$ with $\mathcal{S}[-a^2]$, where a and $-a^2$ are the two solutions of the equation $x^2-x+1=0$. Hence, up to a special linear transformation, we have a unique (non-weak) lax embedding. Moreover, a direct computation shows that there is an automorphism of $\mathbf{PG}(4,q)$ stabilizing $\mathcal{S}[a]$, fixing the points (∞), (0), $(0,0)$, $(0,0,0)$ and $(1,0)$ and mapping (1) to (-1) (which would induce an element of $\mathbf{PGSp}_4(3) \setminus \mathbf{PSp}_4(3)$) if and only if q is a perfect square and $a^{1+\sqrt{q}} = 1$. If q is odd, then a has order 6 and the condition is equivalent to $\sqrt{q} \equiv -1 \bmod 6$, hence to $\sqrt{q} \equiv -1 \bmod 3$. If q is even, then $a^3 = 1$ and again this is equivalent to $\sqrt{q} \equiv -1 \bmod 3$.

If q is even, then the hyperplane β with equation

$$aX_0 + X_1 + a^2X_2 + X_3 + X_4 = 0$$

is preserved by G (this can easily be checked with the above matrices). By the transitivity, no point of $\mathcal{S}[a]$ belongs to β. Hence $\mathcal{S}[a]$ lies in an affine space. If $q=4$, then the lines of $\mathcal{S}[a]$ must be full lines of that affine space and we obtain the well-known embedding of $Q(4,3)$ in $\mathbf{AG}(4,4)$, see PAYNE & THAS [10](7.4.1(*iii*)). Since $q \equiv 1 \bmod 3$ if and only if q is an even power of 2, this lax embedding of $Q(4,3)$ in $\mathbf{PG}(4,q)$ for q even is just the full embedding of $Q(4,3)$ in $\mathbf{AG}(4,4)$, with $\mathbf{AG}(4,4)$ an affine subspace of some affine space $\mathbf{AG}(4,q)$ associated to $\mathbf{PG}(4,q)$.

The theorem is proved.

We make the following observation. With the notation of the previous proof, the ten points (∞), $(a,l,-a)$, $a,l,\in \mathbf{GF}(3)$, form an ovoid of $\mathcal{S}[a]$ which is obtained by intersecting $Q(4,3)$ in its natural embedding in $\mathbf{PG}(4,3)$ with a certain hyperplane. These ten points of the (non-weak) lax embedding in $\mathbf{PG}(4,q)$ clearly generate $\mathbf{PG}(4,q)$. Any ovoid O of $\mathcal{S}[a]$ obtained by intersecting $Q(4,3)$ in its natural embedding in $\mathbf{PG}(4,3)$ with some hyperplane (that is, O is an elliptic quadric of $Q(4,3)$ in its natural embedding), will be called a *classical ovoid*. Any two classical ovoids of $\mathcal{S}[a]$ are equivalent under the subgroup G of $\mathbf{PSL}_5(q)$.

We now handle the general case $Q(4,s)$.

Theorem 5.2 *If the generalized quadrangle $\mathcal{S}$ of order (s, s), with $s \neq 1$ and where $s \neq 3$ for $q \equiv 1 \bmod 3$, is laxly embedded in $\mathbf{PG}(4, q)$, then $\mathcal{S} \cong Q(4, s)$ and the lax embedding is a weak embedding.*

Proof. If $s = 3$, then the result follows from Theorem 5.1. If $s = 2$, then there are only three lines through each point of $Q(4, 2)$, hence they cannot generate $\mathbf{PG}(4, q)$ and so the lax embedding is necessarily a weak embedding. Henceforth, we may assume $s \geq 4$.

By Corollary 3.2, we know that every two non-intersecting lines of $\mathcal{S}$ generate a 3-dimensional space in $\mathbf{PG}(4, q)$ which induces a subquadrangle of order $(s, 1)$ in $\mathcal{S}$. We also know that every line is regular. Now we fix a line L in $\mathcal{S}$ and a plane U skew to the line L' of $\mathbf{PG}(4, q)$ containing L. Let M be any line of $\mathcal{S}$ intersecting L. The plane $\langle L, M\rangle$ meets U in a point p_M. If N is a line of $\mathcal{S}$ opposite L, then the 3-dimensional space $\langle L, N\rangle$ meets U in a line which contains all points p_M such that M is a line of $\mathcal{S}$ in $\{L, N\}^{\perp}$. So we obtain a (lax) embedding of the dual affine plane defined by the regular line L, in the projective plane U. By Limbos [9], $s \geq 4$ implies that the points p_M of U such that M meets L in a fixed point x are contained in a line L_x of U, for all x of $\mathcal{S}$ on L. Hence $x^{\perp}$ is contained in the hyperplane generated by L and L_x. So we have a weak embedding and the theorem is proved.

Theorem 5.3 *No generalized quadrangle $\mathcal{S}$ of order $(s, s{+}2)$ with $s > 2$, is laxly embedded in $\mathbf{PG}(4, q)$.*

Proof. Assume, by way of contradiction, that $\mathcal{S}$ is a GQ of order $(s, s + 2)$, with $s > 2$, laxly embedded in $\mathbf{PG}(4, q)$. Let L, M be two non-concurrent lines of $\mathcal{S}$ and let $\mathcal{S}'$ be the subquadrangle of order (s, t') induced by $\mathbf{PG}(m, q) = \langle L, M\rangle$; clearly $m \leq 3$. By Payne & Thas [10](2.2), we have $st' \leq s + 2$, and so $t' = 1$. It follows that all lines of $\mathcal{S}$ are regular, contradicting Theorem 8.10 (see Appendix A). The theorem is proved.

Theorem 5.4 *If a generalized quadrangle $\mathcal{S}$ of order (s^2, s^3), $s \neq 1$, is laxly embedded in $\mathbf{PG}(4, q)$, then $\mathcal{S}$ is a full embedding of the classical generalized quadrangle $H(4, s^2)$ in a subspace $\mathbf{PG}(4, s^2)$ of $\mathbf{PG}(4, q)$, for the subfield $\mathbf{GF}(s^2)$ of $\mathbf{GF}(q)$.*

Proof. Suppose that the GQ $\mathcal{S}$ of order (s^2, s^3), $s \neq 1$, is laxly embedded in $\mathbf{PG}(4, q)$. Let L, M be two non-concurrent lines of $\mathcal{S}$. First, assume that $\langle L, M\rangle$ is a plane $\mathbf{PG}(2, q)$. Then in $\mathbf{PG}(2, q)$ a subquadrangle of order (s^2, t'), with $t' < s^3$, is induced. Now let N be a line of $\mathcal{S}$ which is concurrent with L but not contained in $\mathbf{PG}(2, q)$. Then in $\mathbf{PG}(3, q) = \langle L, M, N\rangle$ a subquadrangle of order (s^2, t''), with $t' < t'' < s^3$, is induced. This contradicts Payne & Thas [10](2.2.2). Hence $\langle L, M\rangle$ is always a $\mathbf{PG}(3, q)$. By such a $\mathbf{PG}(3, q)$ a subquadrangle $\mathcal{S}'$ of order (s^2, t') is induced, and, by Payne & Thas [10](2.2.2), $t' \in \{1, s\}$.

Assume, by way of contradiction, that for any two non-concurrent lines of $\mathcal{S}$ the space $\langle L, M\rangle$ induces a subquadrangle of order $(s^2, 1)$. Then all lines of $\mathcal{S}$ are regular. So by Payne & Thas [10](1.5.1), s^2+1 divides $(s^6-1)s^6$, clearly a contradiction. It follows that by at least one space $\mathbf{PG}(3,q) = \langle L, M\rangle$ a subquadrangle $\mathcal{S}'$ of order (s^2, s) is induced. As $\mathcal{S}'$ is laxly embedded in $\mathbf{PG}(3,q)$, we have by Theorem 4.1 that $\mathcal{S}' \cong H(3,s^2)$ and that it is fully embedded in a subspace $\mathbf{PG}(3,s^2)$ of $\mathbf{PG}(3,q)$, for the subfield $\mathbf{GF}(s^2)$ of $\mathbf{GF}(q)$.

Let y be a point of $\mathcal{S}$ which is not contained in $\mathcal{S}'$. By Payne & Thas [10](2.2.1), the point y is collinear with the s^3+1 points of an ovoid O of $\mathcal{S}'$. Let $O = \{z_1, z_2, \ldots, z_{s^3+1}\}$. The maximum number of points z_i, $i \neq 1,2$, for which there is a plane through z_1, z_2, z_i containing $s+1$ lines of $\mathcal{S}'$ equals $(s+1)(s-1)$. So there is a point z_i, $i \neq 1,2$, say z_3, such that no plane through z_1, z_2, z_3 contains $s+1$ lines of $\mathcal{S}'$. For such a point z_3 the points z_1, z_2, z_3 are not collinear in $\mathbf{PG}(4,q)$, and the plane $z_1z_2z_3$ contains exactly s^3+1 points of $\mathcal{S}'$. Let $\mathbf{PG}'(3,q) = \langle z_1, z_2, z_3, y\rangle$. First, assume that y is collinear (in $\mathcal{S}$) with the s^3+1 points of $\mathcal{S}'$ in $z_1z_2z_3$. Then the s^3+1 lines of $\mathcal{S}$ through y are contained in a 3-dimensional space. Next, assume that y is not collinear with the s^3+1 points of $\mathcal{S}'$ in the plane $z_1z_2z_3$. Then $\mathbf{PG}'(3,q)$ induces a subquadrangle $\mathcal{S}''$ of $\mathcal{S}$ of order (s^2, s). As $\mathcal{S}''$ is fully embedded in a subspace $\mathbf{PG}'(3,s^2)$ of $\mathbf{PG}'(3,q)$, the $s+1$ lines of $\mathcal{S}''$ through y are coplanar, so yz_1, yz_2, yz_3 are coplanar so z_1, z_2, z_3 are collinear in $\mathbf{PG}(3,q)$, a contradiction. So we conclude that the lines of $\mathcal{S}$ through y are contained in a 3-dimensional space.

Next, assume that z is a point of $\mathcal{S}'$. Consider a plane π of $\mathbf{PG}(3,q)$, with $z \notin \pi$, which contains s^3+1 mutually non-collinear points of $\mathcal{S}'$. Let u_1, u_2 be points of $\mathcal{S}'$ in π. If each point of $\mathcal{S} \setminus \mathcal{S}'$ collinear with u_1 and u_2, is collinear with all points of $\mathcal{S}'$ in π, then by Payne & Thas [10](1.4.2) we have $(s^3-s)s^3 \leq s^4$, a contradiction. So let v be a point of $\mathcal{S} \setminus \mathcal{S}'$ collinear with u_1 and u_2, but not with all points of $\mathcal{S}'$ in π. Then $\langle \pi, v\rangle = \mathbf{PG}''(3,q)$ induces a subquadrangle $\mathcal{S}'''$ of order (s^2, t'). The intersection of $\mathcal{S}'$ and $\mathcal{S}'''$ is an ovoid of $\mathcal{S}'''$, or is the union of $t'+1$ concurrent lines of $\mathcal{S}'''$. As the s^3+1 points of $\mathcal{S}'$ in π belong also to $\mathcal{S}'''$, the latter case would imply $t' \geq s^3$, a contradiction. Hence the intersection of $\mathcal{S}'$ and $\mathcal{S}'''$ is an ovoid of $\mathcal{S}'''$, which contains at least s^3+1 points. So $s^2t'+1 \geq s^3+1$, that is, $t' \geq s$, and consequently $t' = s$. Interchanging roles of $\mathcal{S}'$ and $\mathcal{S}'''$, and as z is not contained in $\mathcal{S}'''$, the previous paragraph shows that all lines of $\mathcal{S}$ through z are contained in a 3-dimensional space.

It follows that $\mathcal{S}$ is weakly embedded in $\mathbf{PG}(4,q)$. Now by Thas & Van Maldeghem [13] $\mathcal{S}$ is a full embedding of the GQ $H(4,s^2)$ in a subspace $\mathbf{PG}(4,s^2)$ of $\mathbf{PG}(4,q)$, for the subfield $\mathbf{GF}(s^2)$ of $\mathbf{GF}(q)$.

The theorem is proved.

6 The case $d = 5$

For $d = 5$, a complete classification is possible. We start with an exceptional lax non-weak embedding of a small GQ.

Theorem 6.1 *Up to a linear transformation, there exists a unique lax embedding of $Q(5,2)$ in $\mathbf{PG}(5,q)$ with q odd. This lax embedding is not weak and the full collineation group of $Q(5,2)$ is induced by $\mathbf{PGL}_6(q)$. Also, this collineation group fixes a hyperplane in $\mathbf{PG}(5,q)$ if and only if q is a power of 3, in which case the lax embedding is a full embedding in some affine subspace $\mathbf{AG}(5,3)$ over the subfield $\mathbf{GF}(3)$ of $\mathbf{GF}(q)$. In each case, the lax embedding is contained in a subspace $\mathbf{PG}(5,p)$ of $\mathbf{PG}(5,q)$ over the subfield $\mathbf{GF}(p)$ of $\mathbf{GF}(q)$, with p the characteristic of $\mathbf{GF}(q)$.*

Proof. We use the description of (the dual of) $Q(5,2)$ in terms of coordinates, introduced in the last part of the proof of Theorem 4.1. By Corollary 3.2, we know that every subspace $\mathbf{PG}(4,q)$ containing two opposite lines of $Q(5,2)$ induces a subquadrangle $Q(4,2)$. Since $Q(4,2)$ is laxly embedded in $\mathbf{PG}(4,q)$ and q is odd, we know by Theorem 5.2 combined with Theorem 1 of Thas & Van Maldeghem [13] that the embedding of $Q(4,2)$ is universal in $\mathbf{PG}(4,q)$. Without loss of generality, we may assume that $Q(4,2)$ is obtained from $Q(5,2)$ by restricting coordinates to $\mathbf{GF}(2)$ in the coordinatization of Hanssens & Van Maldeghem [5]. It is an elementary exercise to write down explicitly the universal embedding of $Q(4,2)$ (see also Thas & Van Maldeghem [13]), and one obtains, up to a linear transformation,

in $\mathcal{S}$	in $\mathbf{PG}(5,q)$
(∞)	$(1,0,0,0,0,0)$
(0)	$(0,0,1,0,0,0)$
(1)	$(1,0,1,0,0,0)$
$(0,0)$	$(1,0,0,1,0,0)$
$(0,1)$	$(0,0,0,1,0,0)$
$(1,0)$	$(0,1,0,0,1,0)$
$(1,1)$	$(-1,1,0,0,1,0)$

$(0,0,0)$	$(0,0,0,0,1,0)$
$(0,0,1)$	$(0,0,1,0,1,0)$
$(1,0,0)$	$(1,0,0,1,-1,0)$
$(1,0,1)$	$(0,0,1,-1,1,0)$
$(0,1,0)$	$(0,1,-1,1,0,0)$
$(0,1,1)$	$(0,1,0,1,0,0)$
$(1,1,0)$	$(1,-1,1,0,0,0)$
$(1,1,1)$	$(0,1,0,0,0,0)$

Let $\mathbf{GF}(4) = \{0, 1, \epsilon, \epsilon^2\}$. We may now choose, without loss of generality, the coordinates of the point $(\epsilon, 0)$ as $(0,0,0,0,0,1)$ and those of $(\epsilon, 1)$ as $(1,0,0,0,0,1)$, since (∞), $(\epsilon, 0)$ and $(\epsilon, 1)$ are collinear in $Q(5,2)$, and hence in $\mathbf{PG}(5,q)$ as well. With the same elementary technique as in the proof of Theorem 5.1, we deduce the coordinates of all other points of $Q(5,2)$ and we obtain:

in $\mathcal{S}$	in $\mathbf{PG}(5,q)$	in $\mathcal{S}$	in $\mathbf{PG}(5,q)$
$(\epsilon,0)$	$(0,0,0,0,0,1)$	$(1,\epsilon,0)$	$(1,0,1,0,1,1)$
$(\epsilon,1)$	$(1,0,0,0,0,1)$	$(1,\epsilon,1)$	$(0,0,0,0,1,1)$
$(\epsilon^2,0)$	$(1,-1,1,-1,1,1)$	$(0,\epsilon^2,0)$	$(1,-1,0,0,0,1)$
$(\epsilon^2,1)$	$(0,-1,1,-1,1,1)$	$(0,\epsilon^2,1)$	$(1,-1,1,0,0,1)$
$(0,\epsilon,0)$	$(0,0,1,-1,1,1)$	$(1,\epsilon^2,0)$	$(0,1,0,1,0,-1)$
$(0,\epsilon,1)$	$(0,0,0,1,-1,-1)$	$(1,\epsilon^2,1)$	$(1,-1,1,-1,0,1)$

The following linear transformations given by their matrices, preserve the point set of $Q(5,2)$, induce root elations and generate a subgroup (isomorphic to $\mathbf{PGU}_4(2)$) of index 2 of the full automorphism group $\mathbf{PGU}_4(2):2$ of $Q(5,2)$:

$$\begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Also, the linear transformation given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

fixes $Q(4,2)$ pointwise and preserves $Q(5,2)$. Hence we now have the full automorphism group $\mathbf{PGU}_4(q):2$ of $Q(5,2)$ which is induced by $\mathbf{PGL}_6(q)$. So we do have a lax embedding of $Q(5,2)$ in $\mathbf{PG}(5,q)$, and, obviously, it is unique and contained in $\mathbf{PG}(5,p)$ (with p the characteristic of q).

One can check easily that all the above transformations fix a hyperplane if and only if they fix a unique hyperplane if and only if q is a power of 3. In such a case the hyperplane has equation

$$X_0 + X_1 + X_2 + X_3 + X_4 + X_5 = 0.$$

Then $\mathcal{S}$ lies in an affine space and the smallest case, $q = 3$, again corresponds to a full affine embedding, see Payne & Thas [10](7.5.1(ii)). Now, up to some tedious but easy calculations which, having all the necessary information above, can be done by the reader, the theorem is completely proved.

Theorem 6.2 *If the generalized quadrangle $\mathcal{S}$ of order (s,t), $s \neq 1$, is laxly embedded in $\mathbf{PG}(5,q)$, then $t = s^2$ and $\mathcal{S} \cong Q(5,s)$. If q is even for $s = 2$, then the lax embedding is a weak embedding and hence a full one in a subspace $\mathbf{PG}(5,s)$ of $\mathbf{PG}(5,q)$ over the subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$.*

Proof. By Theorem 3.1 we have $t = s^2$ and $\mathcal{S} \cong Q(5,s)$.

Suppose first that $s = 3$. If $q \not\equiv 1 \bmod 3$, then, by Theorem 5.1, every subquadrangle of order $(3,3)$ induced by a hyperplane is weakly embedded in that hyperplane. Now let $q \equiv 1 \bmod 3$. Let $\mathcal{S}'$ be a subquadrangle of order $(3,3)$ induced by a hyperplane. Every classical ovoid in $\mathcal{S}'$ is contained in exactly two subquadrangles $\mathcal{S}'$, $\mathcal{S}''$ of order $(3,3)$. By Theorem 3.1, $\mathcal{S}''$ is laxly embedded in some hyperplane of $\mathbf{PG}(5,q)$. But from the observation following the proof of Theorem 5.1, we have that a classical ovoid of $\mathcal{S}'$ generates a hyperplane in $\mathbf{PG}(5,q)$. Hence the two subquadrangles lie in the same hyperplane, a contradiction.

So by Theorem 5.2, and for all s, every subquadrangle of order (s,s) induced by a hyperplane is weakly embedded in that hyperplane. As $\mathcal{S} \cong Q(5,s)$, every point x of $\mathcal{S}$ is collinear with every point of some classical ovoid in any given subquadrangle of order (s,s) not containing x; also, every line of $\mathcal{S}$ through x contains a unique point of the ovoid. But since every subquadrangle of order (s,s) is weakly embedded in a hyperplane, and since q is even if $s = 2$, every classical ovoid lies in a 3-dimensional space. Hence the set of lines through a point of $\mathcal{S}$ is contained in a hyperplane and so $\mathcal{S}$ is weakly embedded in $\mathbf{PG}(5,q)$.

The theorem is proved.

7 The case $d = 4$ and $\mathcal{S}$ is isomorphic to $Q(5,s)$

We start with an exceptional lax embedding of a small GQ.

Theorem 7.1 *If the generalized quadrangle $\mathcal{S} \cong Q(5,2)$ is laxly embedded in $\mathbf{PG}(4,q)$, q odd, then there exists a $\mathbf{PG}(5,q)$ containing $\mathbf{PG}(4,q)$ and a point $x \in \mathbf{PG}(5,q)\backslash\mathbf{PG}(4,q)$ such that $\mathcal{S}$ is the projection from x onto $\mathbf{PG}(4,q)$ of a generalized quadrangle $\widetilde{\mathcal{S}} \cong Q(5,2)$ which is laxly embedded in $\mathbf{PG}(5,q)$, and hence determined by* Theorem 6.1.

Proof. We start by making the following crucial remark: if L is a line of $\mathcal{S}$ opposite the two lines M and M', where M and M' meet in a point of $\mathcal{S}$, then at least one of the spaces $\langle L, M\rangle$ or $\langle L, M'\rangle$ is 3-dimensional. Indeed, if they were both 2-dimensional, then they would coincide and the plane $\langle L, M, M'\rangle$ would induce a subquadrangle $\mathcal{S}'$ in $\mathcal{S}$, necessarily of order $(2,2)$; considering then the 3-space generated by L, M and some further point $z \in \mathcal{S} \setminus \mathcal{S}'$, we see that $\mathcal{S}$ would be contained in a hyperplane of $\mathbf{PG}(4,q)$, a contradiction.

Hence there exist opposite lines L, M of $\mathcal{S}$ such that $\langle L, M\rangle$ is 3-dimensional. Let u be a point of L. If all lines of $\mathcal{S}$ through u were contained in $\langle L, M\rangle$, then $\mathcal{S}$ would be contained in $\langle L, M\rangle$, a contradiction. Let N be a line of $\mathcal{S}$ incident with u, but not contained in $\langle L, M\rangle$. As $\mathcal{S} \cong Q(5,2)$, there is a unique subquadrangle $\mathcal{S}''$ of order $(2,2)$ of $\mathcal{S}$ containing L, M, N. If we use the description of $Q(5,2)$ in the proof of Theorem 6.1, then we may take for $\mathcal{S}''$ the subquadrangle obtained by restricting coordinates (in the sense of Hanssens & Van Maldeghem [5]) to $\mathbf{GF}(2)$. Hence, without loss of generality, we may assume that the following points have coordinates in $\mathbf{PG}(4,q)$ as shown in the table:

in $\mathcal{S}$	in $\mathbf{PG}(4,q)$		
(∞)	$(1,0,0,0,0)$	$(0,0,0)$	$(0,0,0,0,1)$
(0)	$(0,0,1,0,0)$	$(0,0,1)$	$(0,0,1,0,1)$
(1)	$(1,0,1,0,0)$	$(1,0,0)$	$(1,0,0,1,-1)$
$(0,0)$	$(1,0,0,1,0)$	$(1,0,1)$	$(0,0,1,-1,1)$
$(0,1)$	$(0,0,0,1,0)$	$(0,1,0)$	$(0,1,-1,1,0)$
$(1,0)$	$(0,1,0,0,1)$	$(0,1,1)$	$(0,1,0,1,0)$
$(1,1)$	$(-1,1,0,0,1)$	$(1,1,0)$	$(1,-1,1,0,0)$
		$(1,1,1)$	$(0,1,0,0,0)$

Now we put $(\epsilon,0)$ and $(\epsilon,1)$ equal to (a,b,c,d,e) and (a',b,c,d,e), respectively, with $a,a',b,c,d,e \in \mathbf{GF}(q)$, $a \neq a'$ and $(b,c,d,e) \neq (0,0,0,0)$. By our first remark in this proof, we have that the plane generated by $[\epsilon]$ and $[\infty]$ does not contain both $[0,0]$ and $[0,1]$. By recoordinatizing, we may assume that it does not contain $[0,0]$. Hence the space $U := \langle[\epsilon],[0,0]\rangle$ is 3-dimensional and so the lines

$$[\infty],[\epsilon,1,0],[\epsilon,0,0] \text{ and } [\epsilon],[1,\epsilon],[0,0]$$

form a grid (hence lie on a hyperbolic quadric in U). Since we know the coordinates in $\mathbf{PG}(4,q)$ of 7 of the 9 points of that grid, we can calculate the coordinates of the other two points (the line $[1, \epsilon]$ containing the points $(1, \epsilon, 0)$ and $(1, \epsilon, 1)$ is the unique line through $(1) = (1, 0, 1, 0, 0)$ intersecting the lines $[\epsilon, 1, 0]$ and $[\epsilon, 0, 0]$ non-trivially). We obtain:

$$(1, \epsilon, 0) = (a', b, c + a' - a, d, e + a' - a), \qquad (1, \epsilon, 1) = (a, b, c, d, e + a' - a).$$

Now we consider the line $[0, \epsilon]$. The plane $\langle [0, \epsilon], [\infty] \rangle$ does not contain both $[0]$ and $[\epsilon]$, and we can calculate in both cases the coordinates of the points $(0, \epsilon, 0)$ and $(0, \epsilon, 1)$ similarly as above. In both cases, we obtain the same coordinates. Continuing like this, we eventually get the coordinates in $\mathbf{PG}(4, q)$ of all points of $\mathcal{S}$, and we summarize this in the following table (putting $a - a' = f$):

in $\mathcal{S}$	in $\mathbf{PG}(4, q)$
$(\epsilon, 0)$	(a, b, c, d, e)
$(\epsilon, 1)$	$(a - f, b, c, d, e)$
$(\epsilon^2, 0)$	$(a - f, b + f, c - f, d + f, e - f)$
$(\epsilon^2, 1)$	$(a, b + f, c - f, d + f, e - f)$
$(0, \epsilon, 0)$	$(a, b, c - f, d + f, e - f)$
$(0, \epsilon, 1)$	$(a, b, c, d + f, e - f)$
$(1, \epsilon, 0)$	$(a - f, b, c - f, d, e - f)$
$(1, \epsilon, 1)$	$(a, b, c, d, e - f)$
$(0, \epsilon^2, 0)$	$(a - f, b + f, c, d, e)$
$(0, \epsilon^2, 1)$	$(a - f, b + f, c - f, d, e)$
$(1, \epsilon^2, 0)$	$(a, b + f, c, d + f, e)$
$(1, \epsilon^2, 1)$	$(a - f, b + f, c - f, d + f, e)$

Now let $\mathbf{PG}(4, q)$ be embedded as the hyperplane with equation $X_6 = 0$ in $\mathbf{PG}(5, q)$, and let $\widetilde{\mathcal{S}}$ be the lax embedding of $Q(5, 2)$ in $\mathbf{PG}(5, q)$ described in the proof of Theorem 6.1. Then it is clear that $\mathcal{S}$ is the projection of $\widetilde{\mathcal{S}}$ onto $\mathbf{PG}(4, q)$ from the point x with coordinates (a, b, c, d, e, f). The theorem is proved.

Theorem 7.2 *If the generalized quadrangle $\mathcal{S} \cong Q(5, s)$ is laxly embedded in $\mathbf{PG}(4, q)$, where $s \neq 2$ for q odd, then there exists a $\mathbf{PG}(5, q)$ containing $\mathbf{PG}(4, q)$ and a point $x \in \mathbf{PG}(5, q) \backslash \mathbf{PG}(4, q)$ such that $\mathcal{S}$ is the projection from x onto $\mathbf{PG}(4, q)$ of a generalized quadrangle $\widetilde{\mathcal{S}} \cong Q(5, s)$ which is fully embedded in a subspace $\mathbf{PG}(5, s)$ of $\mathbf{PG}(5, q)$, for the subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$.*

Proof. Assume, by way of contradiction, that a subquadrangle $\mathcal{S}' \cong Q(4, s)$ of $\mathcal{S}$ is contained in a plane $\mathbf{PG}(2, q)$ of $\mathbf{PG}(4, q)$. Let L be a line of $\mathcal{S}$ not contained in $\mathbf{PG}(2, q)$ but containing a point of $\mathcal{S}'$. Then the 3-dimensional space $\langle \mathbf{PG}(2, q), L \rangle$ induces a subquadrangle of order (s,t) of $\mathcal{S}$, with $s < t < s^2$. This contradicts Payne & Thas [10](2.2.2).

Assume that the subquadrangle $\mathcal{G}$ of order $(s,1)$ of $\mathcal{S}$ is contained in a plane $\mathbf{PG}(2,q)$. Let L be a line of $\mathcal{S}$ not contained in $\mathbf{PG}(2,q)$ but containing a point of $\mathcal{G}$. Further, let M be a line of $\mathcal{G}$ not concurrent with L. So $\mathcal{S}$ always contains lines L, M which are not coplanar. As $\mathcal{S} \cong Q(5,s)$ the lines L, M are contained in $s+1$ subquadrangles of order (s,s). By Payne & Thas [10](2.2.2) at most one of these subquadrangles is contained in $\langle L, M\rangle$. Hence at least s of these subquadrangles generate $\mathbf{PG}(4,q)$.

Let $\mathcal{S}'$ be a subquadrangle of order (s,s) of $\mathcal{S}$ which generates $\mathbf{PG}(4,q)$. If for $q \equiv 1 \bmod 3$ we have $s \neq 3$ and if for q odd we have $s \neq 2$, then by Thas & Van Maldeghem [13] and by Theorem 5.2 the subquadrangle $\mathcal{S}'$ is fully embedded in some subspace $\mathbf{PG}(4,s)$ of $\mathbf{PG}(4,q)$. Now assume, by way of contradiction, that $s = 3$ with $q \equiv 1 \bmod 3$. Let L, M be two non-coplanar lines of $\mathcal{S}$. Further, let $\{L, M\}^{\perp} = \{M_1, M_2, M_3, M_4\}$ and $\{L, M\}^{\perp\perp} = \{L_1, L_2, L_3, L_4\}$ with $L_i \cap M_j = \{x_{ij}\}$. Consider a point y collinear with x_{12} and x_{21}, $y \neq x_{11}, x_{22}$, and consider a point z collinear with y, x_{23}, x_{32}. For the point z we have 4 different choices. For given y there is at most one point z for which the lines yz and $x_{12}x_{32}$ respectively $x_{21}x_{23}$ are coplanar. So we may assume that $\langle yz, x_{12}x_{32}\rangle$ and $\langle yz, x_{21}x_{23}\rangle$ are hyperplanes of $\mathbf{PG}(4,q)$. As $\{yz, x_{12}x_{32}\}^{\perp\perp}$ is a set of 4 lines on a hyperbolic quadric $Q^+(3,q)$ the cross-ratios $\{x_{12}, x_{22}; x_{32}, x_{42}\}$ and $\{y, u; z, v\}$ are equal, where u is the point of yz collinear with x_{22} and v is the point of yz collinear with x_{42}. Analogously, we have $\{x_{21}, x_{22}; x_{23}, x_{24}\} = \{y, u; z, v\}$. Hence $\{x_{12}, x_{22}; x_{32}, x_{42}\} = \{x_{21}, x_{22}; x_{23}, x_{24}\}$. It follows that the points $x_{11}, x_{22}, x_{33}, x_{44}$ of $Q^+(3,q)$ are coplanar. Consequently, any plane containing 3 pairwise non-collinear points of the grid defined by L, M, contains exactly 4 points of this grid. Now let π be a plane of $\mathbf{PG}(4,q)$ not containing x_{11} and let $x_{11}x_{ij} \cap \pi = \{x'_{ij}\}, i, j = 1,2,3,4$ and $(i,j) \neq (1,1)$. Any line containing two distinct points of the set $P = \{x'_{22}, x'_{32}, x'_{42}, x'_{23}, x'_{33}, x'_{43}, x'_{24}, x'_{34}, x'_{44}\}$ contains exactly three points of P. Hence there arises an affine plane $\mathrm{AG}(2,3)$. Also the lines $x'_{22}x'_{23}, x'_{32}x'_{33}, x'_{42}x'_{43}$ all contain the point $x'_{21} = x'_{31} = x'_{41}$. Now by Limbos [9] we have $q \equiv 0 \bmod 3$, a contradiction. Consequently, if for q odd we have $s \neq 2$, then $\mathcal{S}'$ is always fully embedded in some subspace $\mathbf{PG}(4,s)$ of $\mathbf{PG}(4,q)$.

Choose a line N of $\mathcal{S}$ not contained in $\mathcal{S}'$. Then N contains just one point n of $\mathcal{S}'$. By the foregoing it is clear that N is contained in a subquadrangle $\mathcal{S}'' \cong Q(4,s)$ which generates $\mathbf{PG}(4,q)$. Again $\mathcal{S}''$ is fully embedded in some subspace $\mathbf{PG}'(4,s)$ of $\mathbf{PG}(4,q)$. Hence N is a subline $\mathbf{PG}(1,s)$ of the corresponding line N' of $\mathbf{PG}(4,q)$.

Now we show that not all points of N belong to $\mathbf{PG}(4,s)$ (that is, N contains at most two points of $\mathbf{PG}(4,s)$). Suppose the contrary. Let u be a point of $\mathcal{S}$ not in $\mathcal{S}'$, with $u \notin N$. Assume first that $u \not\sim n$. Let w be the point of N for which $w \sim u$. Then the line wu of $\mathcal{S}$ has a point n' in common with $\mathcal{S}'$. As $\mathcal{S}' \cong Q(4,s)$ is fully embedded in $\mathbf{PG}(4,s)$, there is a line W of $\mathcal{S}'$ through n which is not contained in the plane $\langle N, u\rangle$. Now we consider a subquadrangle $\mathcal{S}''' \cong Q(4,s)$ of $\mathcal{S}$ containing W, N, u, n' and generating $\mathbf{PG}(4,q)$ (by the second paragraph of this proof such a subquadrangle exists). Then $\mathcal{S}'''$ is fully embedded in a subspace $\mathbf{PG}''(4,s)$ of $\mathbf{PG}(4,q)$. As $\mathcal{S}'$ and $\mathcal{S}'''$ share the line W, they intersect either in a grid or in $s+1$ concurrent lines. In either case $\mathbf{PG}(4,s)$ and $\mathbf{PG}''(4,s)$ share a $\mathbf{PG}(3,s)$. So the line nn' of $\mathbf{PG}(4,s)$ coincides with the line nn' of $\mathbf{PG}''(4,s)$. In $\mathbf{PG}''(4,s)$ there is a plane $\mathbf{PG}(2,s)$ containing N, u, n'. This plane contains the lines N and nn' of $\mathbf{PG}(4,s)$, hence $\mathbf{PG}(2,s)$ is a plane of $\mathbf{PG}(4,s)$. It follows that u is a point of $\mathbf{PG}(4,s)$. Next, assume that $u \sim n$. Choose distinct lines T, T' of $\mathcal{S}$ through u, with $T \neq un \neq T'$. By the preceding case the line T respectively T' contains at least s points of $\mathbf{PG}(4,s)$. So the common point u of the lines T, T' belongs to $\mathbf{PG}(4,s)$. Consequently, $\mathcal{S}$ is fully embedded in $\mathbf{PG}(4,s)$, contradicting the Theorem of Buekenhout & Lefèvre [1]. We conclude that at most one point of $N \setminus \{n\}$ belongs to $\mathbf{PG}(4,s)$.

Now we choose a $\mathbf{PG}(5,q)$ containing $\mathbf{PG}(4,q)$, and in $\mathbf{PG}(5,q)$ we choose a line $\bar{N}$ having just n in common with $\mathbf{PG}(4,q)$. Next, we choose on $\bar{N}$ a subline $\widetilde{N}$ over $\mathbf{GF}(s)$ which contains n, and such that N is the projection of $\widetilde{N}$ onto $\mathbf{PG}(4,q)$ from some point x of $\mathbf{PG}(5,q)$. We call $\mathbf{PG}(5,s)$ the subspace of $\mathbf{PG}(5,q)$ defined by $\mathbf{PG}(4,s)$ and $\widetilde{N}$. If x would belong to $\mathbf{PG}(5,s)$, then N would belong to $\mathbf{PG}(4,s)$, a contradiction. It follows that $x \notin \mathbf{PG}(5,s)$.

Let u be a point of $\mathcal{S}$ not in $\mathcal{S}'$ and not on N. Assume first that $u \not\sim n$. The line M of $\mathcal{S}$ through u and concurrent with N contains a point m of $\mathcal{S}'$. By a foregoing argument the line nm of $\mathbf{PG}(4,s)$ coincides with the line nm of the plane $\mathbf{PG}(2,s)$ containing N, u, m. Let $\widetilde{\mathbf{PG}}(2,s)$ be the plane over $\mathbf{GF}(s)$ defined by $\widetilde{N}$ and the line nm of $\mathbf{PG}(4,s)$. Projecting this $\widetilde{\mathbf{PG}}(2,s)$ from x onto $\mathbf{PG}(4,q)$, we clearly obtain the plane $\mathbf{PG}(2,s)$. Hence the line ux intersects $\widetilde{\mathbf{PG}}(2,s)$ in a point $\widetilde{u}$ (if ux would contain two distinct points of $\widetilde{\mathbf{PG}}(2,s)$, then x would be a point of $\widetilde{\mathbf{PG}}(2,q)$, so N would contain n and m, a contradiction). If ux intersects $\mathbf{PG}(5,s)$ in more than one point, then u belongs to $\mathbf{PG}(4,s)$. As x is not in $\mathbf{PG}(5,s)$, for at most one point r of $\mathcal{S}$ the line rx has more than one point in common with $\mathbf{PG}(5,s)$. Assume that the line ux has more than one point in common with $\mathbf{PG}(5,s)$. The line M of $\mathcal{S}$ is the projection from x onto $\mathbf{PG}(4,q)$ of some line $\widetilde{M}$ of the plane $\widetilde{\mathbf{PG}}(2,s)$ of $\mathbf{PG}(5,s)$. Interchanging roles of M and N, and of $\widetilde{M}$ and $\widetilde{N}$, we then see that, without loss of generality, we may always assume that for any point $u \not\sim n$, u not in $\mathcal{S}'$, the line ux has just one point in common with $\mathbf{PG}(5,s)$. Next, assume that $u \sim n$. Choose a line $W \neq un$ of $\mathcal{S}$ through u. Then for any point v of $W \setminus \{u\}$ not in $\mathcal{S}'$, the line xv has exactly one point $\widetilde{v}$ in common with $\mathbf{PG}(5,s)$. Let T be a line of $\mathcal{S}$ concurrent with W and N, but neither containing n nor the common point w of W and $\mathcal{S}'$. Then T is the projection from x onto $\mathbf{PG}(4,q)$ of some line $\widetilde{T}$ of $\mathbf{PG}(5,s)$; the line $\widetilde{T}$ has a unique point in common with $\mathbf{PG}(4,s)$ and this point coincides with the intersection of T and $\mathcal{S}'$. Interchanging roles of N and T, and also of $\widetilde{N}$ and $\widetilde{T}$, we then see that W is the projection from x onto $\mathbf{PG}(4,q)$ of a line $\widetilde{W}$ of $\mathbf{PG}(5,s)$. Hence the line xu intersects $\mathbf{PG}(5, s)$ in at least one point. Assume that the line ux has more than one point in common with $\mathbf{PG}(5, s)$. Interchanging roles of W and N, and also of $\widetilde{W}$ and $\widetilde{N}$, we then see that without loss of generality we may always assume that for any point $u \sim n$, u not in $\mathcal{S}'$ and u not on N, the line ux has just one point in common with $\mathbf{PG}(5, s)$. Now it is clear that there arises an injection θ from the point set of $\mathcal{S}$ into $\mathbf{PG}(5, s)$; also, for any point d of $\mathcal{S}$ the points x, d, d^θ are collinear; the points of $\mathcal{S}'$ are fixed by θ, the points of N are mapped onto the points of $\widetilde{N}$, and any point d of $\mathcal{S}$ not in $\mathcal{S}'$ and not on N is mapped onto the unique common point of dx and $\mathbf{PG}(5, s)$.

Let M be a line of $\mathcal{S}$ which is concurrent with N, but does not contain n. Then we have already shown that M^θ is a line $\widetilde{M}$ of $\mathbf{PG}(5, s)$. If A is a line of $\mathcal{S}'$, then trivially $A^\theta = A$ is a line of $\mathbf{PG}(5, s)$. Next, let W be a line of $\mathcal{S}$ which is not concurrent with N and which does not belong to $\mathcal{S}'$. Further, let T be a line of $\mathcal{S}$ which is concurrent with N and W, but which neither contains n nor the common point w of W and $\mathcal{S}'$. Then $T^\theta = \widetilde{T}$ and $\mathbf{PG}(4, s)$ determine uniquely $\mathbf{PG}(5, s)$, and so by a previous argument the point w together with the s points u^θ, with $u \in W \setminus \{w\}$, form a line $W^\theta = \widetilde{W}$ of $\mathbf{PG}(5, s)$. Finally, let $S \neq N$ be a line of $\mathcal{S}$ which does not belong to $\mathcal{S}'$ but contains n. Let W be a line of $\mathcal{S}$ which is concurrent with S, but does not contain n. Then $W^\theta = \widetilde{W}$ and $\mathbf{PG}(4, s)$ determine uniquely $\mathbf{PG}(5, s)$, and so by a previous argument the common point n of S and $\mathcal{S}'$ together with the s points e^θ, with $e \in S \setminus \{n\}$, form a line $S^\theta = \widetilde{S}$ of $\mathbf{PG}(5, s)$.

Hence θ is an injection from the point set of $\mathcal{S}$ into $\mathbf{PG}(5, s)$, which maps the lines of $\mathcal{S}$ onto lines of $\mathbf{PG}(5, s)$. So there arises a GQ $\widetilde{\mathcal{S}} \cong Q(5, s)$ which is fully embedded in $\mathbf{PG}(5, s)$. Also, $\mathcal{S}$ is the projection of $\widetilde{\mathcal{S}}$ from x onto $\mathbf{PG}(4, q)$. The theorem is proved.

8 The case $d = 3$ and $\mathcal{S}$ is isomorphic to one of $Q(4, s)$, $Q(5, s)$, $H(4, s^2)$, or the dual of $H(4, s^2)$

Theorem 8.1 *If the generalized quadrangle $\mathcal{S} \cong H(4, s^2)$ is laxly embedded in $\mathbf{PG}(3, q)$, then there exists a $\mathbf{PG}(4, q)$ containing $\mathbf{PG}(3, q)$ and a point $x \in \mathbf{PG}(4, q) \backslash \mathbf{PG}(3, q)$ such that $\mathcal{S}$ is the projection from x onto $\mathbf{PG}(3, q)$ of a generalized quadrangle $\widetilde{\mathcal{S}} \cong H(4, s^2)$ which is fully embedded in a subspace $\mathbf{PG}(4, s^2)$ of $\mathbf{PG}(4, q)$, for the subfield $\mathbf{GF}(s^2)$ of $\mathbf{GF}(q)$.*

Proof. Assume, by way of contradiction, that all the lines of $\mathcal{S}$ are mutually coplanar in $\mathbf{PG}(3, q)$. Then the corresponding lines of $\mathbf{PG}(3, q)$ all contain a common point y of $\mathbf{PG}(3, q)$; by Lemma 2.1 the point y does not belong to $\mathcal{S}$. Let L, M be lines of $\mathcal{S}$

and let $z \in L$. If N is the line of $\mathcal{S}$ incident with z and concurrent with M, then the corresponding line N' of $\mathbf{PG}(3,q)$ does not contain y, a contradiction. So there exist lines L, M of $\mathcal{S}$ for which $\langle L, M \rangle = \mathbf{PG}(3,q)$. As $\mathcal{S} \cong H(4,s^2)$ there exists a subquadrangle $\mathcal{S}' \cong H(3,s^2)$ of $\mathcal{S}$ for which L, M are lines. By Theorem 4.1 $\mathcal{S}'$ is fully embedded in a subspace $\mathbf{PG}(3,s^2)$ of $\mathbf{PG}(3,q)$.

Choose a line N of $\mathcal{S}$ not contained in $\mathcal{S}'$. Then N contains just one point n of $\mathcal{S}'$. Assume, by way of contradiction, that all the lines of $\mathcal{S}'$ are coplanar with a common line Z of $\mathbf{PG}(3,q)$. As the $s+1$ lines of $\mathcal{S}'$ containing a given point of $\mathcal{S}'$ are coplanar, it then easily follows that $\mathcal{S}'$ is contained in a plane through Z, a contradiction. Hence there exists a line U in $\mathcal{S}'$ such that $\langle U, N \rangle$ is 3-dimensional. Again there exists a subquadrangle $\mathcal{S}'' \cong H(3,s^2)$ of $\mathcal{S}$ which contains the lines U, N. By Theorem 4.1 $\mathcal{S}''$ is fully embedded in a subspace $\mathbf{PG}'(3,s^2)$ of $\mathbf{PG}(3,q)$. Hence N is a subline $\mathbf{PG}(1,s^2)$ of the corresponding line N' of $\mathbf{PG}(3,q)$.

Now we show that not all points of N belong to $\mathbf{PG}(3,s^2)$ (that is, N contains at most two points of $\mathbf{PG}(3,s^2)$). Suppose the contrary. Let u be a point of $\mathcal{S}$ not in $\mathcal{S}'$, with $u \notin N$. Assume first that $u \not\sim n$. Let w be the point of N for which $w \sim u$. Then the line wu of $\mathcal{S}$ has a point n' in common with $\mathcal{S}'$. As $\mathcal{S}' \cong H(3,s^2)$ is fully embedded in $\mathbf{PG}(3,s^2)$, the lines of $\mathcal{S}'$ through n respectively n' are contained in a plane π respectively π' of $\mathbf{PG}(3,q)$. At least one of π, π' is distinct from the plane $\langle N, u \rangle$, and so there is a line W of $\mathcal{S}'$ through n or n', which is not contained in $\langle N, u \rangle$. Now we consider a subquadrangle $\mathcal{S}''' \cong H(3,s^2)$ of $\mathcal{S}$ containing W, N, u and n'. As $\mathcal{S}'''$ is fully embedded in a $\mathbf{PG}''(3,s^2)$, there exists a plane $\mathbf{PG}(2,s^2)$ in $\mathbf{PG}''(3,s^2)$ containing N, u, n'. The subquadrangles $\mathcal{S}'$ and $\mathcal{S}'''$ share the line W, so they intersect in $s+1$ concurrent lines. Consequently the hyperbolic line containing n, n' of $\mathcal{S}'$ coincides with the hyperbolic line through n, n' of $\mathcal{S}'''$, and further this hyperbolic line is a $\mathbf{PG}(1,s)$ which belongs to the plane $\mathbf{PG}(2,s^2)$. It follows that $\mathbf{PG}(2,s^2)$ belongs to $\mathbf{PG}(3,s^2)$, hence u belongs to $\mathbf{PG}(3,s^2)$. Next, assume that $u \sim n$. Choose a line $T \neq un$ of $\mathcal{S}$ through u. By the preceding case the line T contains at least s^2 points of $\mathbf{PG}(3,s^2)$. As T is a line over $\mathbf{GF}(s^2)$ in $\mathbf{PG}(3,q)$, also u belongs to $\mathbf{PG}(3,s^2)$. Consequently, $\mathcal{S}$ is fully embedded in $\mathbf{PG}(3,s^2)$, contradicting the Theorem of Buekenhout & Lefèvre [1]. We conclude that at most one point of $N \setminus \{n\}$ belongs to $\mathbf{PG}(3,s^2)$.

Now we choose a $\mathbf{PG}(4,q)$ containing $\mathbf{PG}(3,q)$, and in $\mathbf{PG}(4,q)$ we choose a line $\bar{N}$ having just n in common with $\mathbf{PG}(3,q)$. Next, we choose on $\bar{N}$ a subline $\widetilde{N}$ over $\mathbf{GF}(s^2)$ which contains n, and such that N is the projection of $\widetilde{N}$ onto $\mathbf{PG}(3,q)$ from some point x of $\mathbf{PG}(4,q)$. We call $\mathbf{PG}(4,s^2)$ the subspace of $\mathbf{PG}(4,q)$ defined by $\mathbf{PG}(3,s^2)$ and $\widetilde{N}$. If x would belong to $\mathbf{PG}(4,s^2)$, then N would belong to $\mathbf{PG}(3,s^2)$, a contradiction. It follows that $x \notin \mathbf{PG}(4,s^2)$.

Let u be a point of $\mathcal{S}$ not in $\mathcal{S}'$ and not on N. Assume first that $u \not\sim n$. The line M of $\mathcal{S}$ through u and concurrent with N contains a point m of $\mathcal{S}'$. By a foregoing argument, the hyperbolic line Y of $\mathcal{S}'$ through m and n is a $\mathbf{PG}(1,s)$ of $\mathbf{PG}(3,q)$, and in the plane $\langle mu, N\rangle$ $\mathcal{S}$ has $s+1$ lines intersecting $\mathcal{S}'$ in the points of Y. Let $\widetilde{\mathbf{PG}}(2,s^2)$ be the plane over $\mathbf{GF}(s^2)$ defined by $\widetilde{N}$ and Y (then $\widetilde{\mathbf{PG}}(2,s^2)$ is a plane of the space $\mathbf{PG}(4,s^2)$). Projecting this $\widetilde{\mathbf{PG}}(2,s^2)$ from x onto $\mathbf{PG}(3,q)$, we obtain the unique plane $\mathbf{PG}(2,s^2)$ over $\mathbf{GF}(s^2)$ containing N and Y. By the foregoing paragraphs the point u also belongs to this plane $\mathbf{PG}(2,s^2)$. Hence the line ux intersects $\widetilde{\mathbf{PG}}(2,s^2)$ in a unique point $\widetilde{u}$. If ux intersects $\mathbf{PG}(4,s^2)$ in more than one point, then u belongs to $\mathbf{PG}(3,s^2)$. As x is not in $\mathbf{PG}(4,s^2)$, for at most one point r of $\mathcal{S}$ the line rx has more than one point in common with $\mathbf{PG}(4,s^2)$. Assume that the line ux has more than one point in common with $\mathbf{PG}(4,s^2)$. The line M of $\mathcal{S}$ is the projection from x onto $\mathbf{PG}(3,q)$ of some line $\widetilde{M}$ of the plane $\widetilde{\mathbf{PG}}(2,s^2)$ of $\mathbf{PG}(4,s^2)$. Interchanging roles of M and N, and of $\widetilde{M}$ and $\widetilde{N}$, we then see that, without loss of generality, we may always assume that for any point $u \not\sim n, u$ not in $\mathcal{S}'$, the line ux has just one point in common with $\mathbf{PG}(4,s^2)$. Next, assume that $u \sim n$. Choose a line $W \neq un$ of $\mathcal{S}$ through u. Then for any point v of $W \setminus \{u\}$ not in $\mathcal{S}'$, the line xv has exactly one point $\widetilde{v}$ in common with $\mathbf{PG}(4,s^2)$. Let T be a line of $\mathcal{S}$ concurrent with W and N, but neither containing n nor the common point w of W and $\mathcal{S}'$. Then T is the projection from x onto $\mathbf{PG}(3,q)$ of some line $\widetilde{T}$ of $\mathbf{PG}(4,s^2)$; the line $\widetilde{T}$ has a unique point in common with $\mathbf{PG}(3,s^2)$ and this point coincides with the intersection of T and $\mathcal{S}'$. Interchanging roles of N and T, and also of $\widetilde{N}$ and $\widetilde{T}$, we then see that W is the projection from x onto $\mathbf{PG}(3,s)$ of a line $\widetilde{W}$ of $\mathbf{PG}(4,s^2)$. Hence the line xu intersects $\mathbf{PG}(4,q^2)$ in at least one point. Assume that the line ux has more than one point in common with $\mathbf{PG}(4,s^2)$. Interchanging roles of W and N, and also of $\widetilde{W}$ and $\widetilde{N}$, we then see that without loss of generality we may always assume that for any point $u \sim n, u$ not in $\mathcal{S}'$ and u not on N, the line ux has just one point in common with $\mathbf{PG}(4,s^2)$. Now it is clear that there arises an injection θ from the point set of $\mathcal{S}$ into $\mathbf{PG}(4,s^2)$; also, for any point d of $\mathcal{S}$ the points x, d, d^θ are collinear; the points of $\mathcal{S}'$ are fixed by θ, the points of N are mapped onto the points of $\widetilde{N}$, and any point d of $\mathcal{S}$ not in $\mathcal{S}'$ and not on N is mapped onto the unique common point of dx and $\mathbf{PG}(4,s^2)$.

Let M be a line of $\mathcal{S}$ which is concurrent with N, but does not contain n. Then we have already shown that M^θ is a line $\widetilde{M}$ of $\mathbf{PG}(4,s^2)$. If A is a line of $\mathcal{S}'$, then trivially $A^\theta = A$ is a line of $\mathbf{PG}(4,s^2)$. Next, let W be a line of $\mathcal{S}$ which is not concurrent with N and which does not belong to $\mathcal{S}'$. Further, let T be a line of $\mathcal{S}$ which is concurrent with N and W, but which neither contains n nor the common point w of W and $\mathcal{S}'$. Then $T^\theta = \widetilde{T}$ and $\mathbf{PG}(3,s^2)$ determine uniquely $\mathbf{PG}(4,s^2)$, and so by a previous argument the point w together with the s^2 points u^θ, with $u \in W \setminus \{w\}$, form a line $W^\theta = \widetilde{W}$ of $\mathbf{PG}(4,s^2)$. Finally, let $S \neq N$ be a line of $\mathcal{S}$ which does not belong to $\mathcal{S}'$ but contains n. Let W be a line of $\mathcal{S}$ which is concurrent with S, but does not contain n. Then $W^\theta = \widetilde{W}$ and $\mathbf{PG}(3,s^2)$ determine uniquely $\mathbf{PG}(4,s^2)$, and so by a previous argument the common point n of S and $\mathcal{S}'$ together with the s^2 points e^θ, with $e \in S \setminus \{n\}$, form a line $S^\theta = \widetilde{S}$

of $\mathbf{PG}(4, s^2)$.

Hence θ is an injection from the point set of $\mathcal{S}$ into $\mathbf{PG}(4, s^2)$, which maps the lines of $\mathcal{S}$ onto lines of $\mathbf{PG}(4, s^2)$. So there arises a GQ $\widetilde{\mathcal{S}} \cong H(4, s^2)$ which is fully embedded in $\mathbf{PG}(4, s^2)$. Also, $\mathcal{S}$ is the projection of $\widetilde{\mathcal{S}}$ from x onto $\mathbf{PG}(3, q)$.

The theorem is proved.

A *perspectivity* $[L; M]$ in a GQ is a map from a line L to an opposite line M which maps any point x on L onto the unique point y of M collinear with x. A *projectivity* is the composition of a finite number of perspectivities. We denote $[L_1; L_2; L_3; \ldots; L_i] = [L_1; L_2][L_2; L_3] \cdots [L_{i-1}; L_i]$, for L_j opposite L_{j+1}, $j = 1, \ldots, i-1$.

Theorem 8.2 *A generalized quadrangle $\mathcal{S}$ isomorphic to the dual of $H(4, s^2)$ cannot be laxly embedded in $\mathbf{PG}(d, q)$, for $d \geq 3$.*

Proof. By Theorem 3.1, we only have to deal with the case $d = 3$.

Let L and M be two arbitrary opposite lines of $\mathcal{S}$ and suppose that $\mathcal{S}$ is laxly embedded in $\mathbf{PG}(3, q)$. Then $\langle L, M \rangle$ is 3-dimensional, otherwise $\mathcal{S}$ has a subquadrangle of order (s^3, t), with $s^3 t \leq s^2$, a contradiction. Also, $|\{L, M\}^{\perp\perp}| = s + 1$. Clearly, the perspectivity in $\mathcal{S}$ mapping a point x on L to the point y on M collinear with x is the restriction of a linear transformation of $\mathbf{PG}(3, q)$ to L (because the lines xy of $\mathbf{PG}(3, q)$ are generators of the hyperbolic quadric containing the extensions of the $s + 1$ lines of $\{L, M\}^{\perp\perp}$ to $\mathbf{GF}(q)$). Hence the full group G of projectivities of a line L of $\mathcal{S}$ is contained in the group $\mathbf{PGL}_2(q)$. By Knarr [6], the group G is permutation equivalent to $\mathbf{PGU}_3(s)$ acting naturally on a hermitian unital. It follows that G contains a non-trivial element fixing three distinct points of L. On the other hand $\mathbf{PGL}_2(q)$ acts sharply 3-transitive on the extension L' of L to $\mathbf{GF}(q)$. This contradiction proves the theorem.

The following lemma follows immediately from the classification of all (maximal) subgroups of $\mathbf{PGL}_2(q)$, see Dickson [2].

Lemma 8.3 *Consider the natural action of $\mathbf{PGL}_2(q)$ on the projective line $\mathbf{PG}(1, q)$. If $\mathbf{PSL}_2(s)$ is a subgroup of $\mathbf{PGL}_2(q)$ and if it has an orbit of length $s + 1$, then either this orbit is a projective subline $\mathbf{PG}(1, s)$ over the subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$, or $s = 3$ and $q \equiv 1 \bmod 3$, or $s = 2$ and q is odd. Also, if $s = 3$ and $\mathbf{PGL}_2(3)$ is a subgroup of $\mathbf{PGL}_2(q)$ with an orbit of length 4, then q is a power of 3.*

Theorem 8.4 *If the generalized quadrangle $\mathcal{S} \cong Q(4, s)$ of order (s, s), with $s \neq 2$ for q odd and $s \neq 3$ for $q \equiv 1 \bmod 3$, is laxly embedded in $\mathbf{PG}(3, q)$, then either s is even and $\mathcal{S} \cong Q(4, s) \cong W(s)$ is fully embedded in a subspace $\mathbf{PG}(3, s)$ of $\mathbf{PG}(3, q)$, or there exists a $\mathbf{PG}(4, q)$ containing $\mathbf{PG}(3, q)$ and a point $x \in \mathbf{PG}(4, q) \setminus \mathbf{PG}(3, q)$ such that $\mathcal{S}$ is the projection from x onto $\mathbf{PG}(3, q)$ of a generalized quadrangle $\widetilde{\mathcal{S}} \cong Q(4, s)$ which is fully embedded in a subspace $\mathbf{PG}(4, s)$ of $\mathbf{PG}(4, q)$, for the subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$.*

Proof. First we claim that, if $s > 2$, and if L is a line of $\mathcal{S}$, then the group $\mathbf{PGL}_2(q)$ of the linear transformations of the corresponding line L' of $\mathbf{PG}(3,q)$ admits a subgroup isomorphic to $\mathbf{PSL}_2(s)$ acting on the $s+1$ points of L.

Suppose first that no two opposite lines of $\mathcal{S}$ are contained in a plane of $\mathbf{PG}(3,q)$. Let L and M be two such lines. The lines L and M are contained in a subquadrangle $\mathcal{S}'$ of order $(s,1)$, and $\mathcal{S}'$ is contained in a unique hyperbolic quadric of $\mathbf{PG}(3,q)$. It follows easily that the perspectivity $[L;M]$ from L to M in $\mathcal{S}$ is the restriction of a linear transformation $L \to M$ in $\mathbf{PG}(3,q)$. Hence the group $\mathbf{PSL}_2(s)$ of projectivities of a line of $\mathcal{S}$ (see KNARR [6]) is contained in the group $\mathbf{PGL}_2(q)$ of all linear transformations of a line in $\mathbf{PG}(3,q)$. The claim follows.

Now suppose that the two opposite lines L and M of $\mathcal{S}$ are contained in a plane π of $\mathbf{PG}(3,q)$. Let N_∞ be a line of $\mathcal{S}$ concurrent with both L,M and let x,y be points on N_∞ not incident with either L or M. Let $\{N_i | i \in \{\infty\} \cup \mathbf{GF}(s)\}$ be the set of lines of $\mathcal{S}$ through x, with $N_0 \in \{L,M\}^{\perp\perp}$, and let $\{R_i | i \in \{\infty\} \cup \mathbf{GF}(s)\}$ be the set of lines of $\mathcal{S}$ through y, with $R_0 \in \{L,M\}^{\perp\perp}$. Consider the projectivity $\theta = [L;N_j;M;R_k;L]$, $j,k \in \mathbf{GF}(s)^\times$. Note that the lines L' and N'_j, $j \neq 0$, are skew in $\mathbf{PG}(3,q)$ because otherwise $\mathcal{S}$ would be contained in the plane π. Similarly, the lines M' and N'_j, $j \neq 0$, the lines M' and R'_k, $k \neq 0$, and the lines L' and R'_k, $k \neq 0$, are skew. Hence, as in the previous paragraph, the projectivity θ is the restriction to L of a linear transformation in $\mathbf{PG}(3,q)$ of L'. Let $Q(4,s)$ be the generalized quadrangle arising from the quadric in $\mathbf{PG}(4,s)$ with equation $X_4^2 = X_0X_1 + X_2X_3$ and let γ be an isomorphism of $\mathcal{S}$ onto $Q(4,s)$. If we take for L^γ, M^γ, N_∞^γ the lines with respective equations $X_1 = X_2 = X_4 = 0$, $X_0 = X_3 = X_4 = 0$ and $X_1 = X_3 = X_4 = 0$, then we can choose for x^γ and y^γ the points with coordinates $(a,0,1,0,0)$ and $(b,0,1,0,0)$, respectively, $a,b \in \mathbf{GF}(s) \setminus \{0\}$, and for N_j^γ, respectively R_k^γ, the line through x^γ and $(j^2,1,0,-a,j)$, respectively through y^γ and $(k^2,1,0,-b,k)$. One calculates that θ maps the point $(x_0,0,0,1,0)^{\gamma^{-1}}$ onto the point $(x_0 - a^{-1}j^2 + b^{-1}k^2, 0,0,1,0)^{\gamma^{-1}}$. Since the element $a^{-1}j^2 - b^{-1}k^2$ is arbitrary in $\mathbf{GF}(s)$ (by appropriate choices of a,b,j,k), we easily see that, by varying N_∞, these elements θ generate $\mathbf{PSL}_2(s)$, which acts on the $s+1$ points of L. Our claim now follows.

By the previous lemma, and under the assumptions of the theorem, we now know that the points of a line L of $\mathcal{S}$ form a projective subline of the corresponding line L' of $\mathbf{PG}(3,q)$. Suppose now that L,M are two opposite lines of $\mathcal{S}$ which span $\mathbf{PG}(3,q)$. The unique subquadrangle $\mathcal{S}'$ of order $(s,1)$ of $\mathcal{S}$ containing L and M is clearly contained in a hyperbolic quadric $Q^+(3,q)$ of $\mathbf{PG}(3,q)$. Let N be a line of $\mathcal{S}'$ concurrent with both L and M. By the foregoing, it is clear that L,M,N are contained in a unique subquadric $Q^+(3,s)$ of $Q^+(3,q)$ in a projective subspace $\mathbf{PG}(3,s)$ of $\mathbf{PG}(3,q)$ defined over the field $\mathbf{GF}(s)$. Since any line K of $\mathcal{S}'$ not concurrent with L is contained in the unique line of $\mathbf{PG}(3,q)$ through $K \cap N$ meeting every line X' of $\mathbf{PG}(3,q)$, where X is a line of $\mathcal{S}$ meeting both L and M, we see that all points of K belong to $Q^+(3,s)$. Hence we have shown that, if L,M are lines of $\mathcal{S}$ with $\langle L,M\rangle$ 3-dimensional, then the subquadrangle of order $(s,1)$ of $\mathcal{S}$ containing L, M, is fully embedded in a subspace $\mathbf{PG}(3,s)$ of $\mathbf{PG}(3,q)$.

Let L, M, respectively L, M^*, be opposite lines of $\mathcal{S}$, with M and M^* concurrent in $\mathcal{S}$ and $M \neq M^*$. If L, M^* are contained in a plane π, then π induces a subquadrangle $\mathcal{S}'$ of order $(s,1)$, and so M is not contained in π. Hence $\langle L, M \rangle$ is 3-dimensional. Let us now fix lines L, M, with $\langle L, M \rangle$ 3-dimensional, and denote by $\mathcal{G}$ the unique grid of $\mathcal{S}$ for which L, M are lines. By the previous paragraph $\mathcal{G}$ is fully embedded in a subspace $\mathbf{PG}(3,s)$ of $\mathbf{PG}(3,q)$.

Choose a line N of $\mathcal{S}$ not contained in $\mathcal{G}$. Then N contains just one point n of $\mathcal{G}$. By the foregoing N is a subline $\mathrm{PG}(1,s)$ of the corresponding line N' of $\mathbf{PG}(3,q)$.

First, assume that all points of N belong to $\mathbf{PG}(3,s)$. Let u be a point of $\mathcal{S}$ not in $\mathcal{G}$, with $u \notin N$. Assume first that $u \not\sim n$. Let w be the point of N for which $w \sim u$. Then the line wu of $\mathcal{S}$ has a point n' in common with $\mathcal{G}$. Since the point n' of the plane $\langle N, u \rangle$ clearly is not collinear with n, it does not belong to the plane of $\mathbf{PG}(3,q)$ generated by the two lines of $\mathcal{G}$ through n. Hence the two lines of $\mathcal{G}$ through n are not both contained in the plane $\langle N, u \rangle$. Consequently there is a line W of $\mathcal{G}$ through n which is not contained in the plane $\langle N, u \rangle$. Now we consider the grid $\mathcal{G}'$ of $\mathcal{S}$ containing W, N, u, n'; clearly $\mathcal{G}'$ generates $\mathbf{PG}(3,q)$. Then $\mathcal{G}'$ is fully embedded in a subspace $\mathbf{PG}'(3,s)$ of $\mathbf{PG}(3,q)$. As $\mathcal{G}$ and $\mathcal{G}'$ share the line W, they intersect in 2 concurrent lines. It follows that $\mathbf{PG}(3,s)$ and $\mathbf{PG}'(3,s)$ share a $\mathbf{PG}(2,s)$. So the line nn' of $\mathbf{PG}(3,s)$ coincides with the line nn' of $\mathbf{PG}'(3,s)$. In $\mathbf{PG}'(3,s)$ there is a plane $\mathbf{PG}'(2,s)$ containing N, u, n'. This plane contains the lines N and nn' of $\mathbf{PG}(3,s)$, hence $\mathbf{PG}'(2,s)$ is a plane of $\mathbf{PG}(3,s)$. It follows that u is a point of $\mathbf{PG}(3,s)$. Next, assume that $u \sim n$. Choose distinct lines T, T' of $\mathcal{S}$ through u, with $T \neq un \neq T'$. By the preceding case the line T respectively T' contains at least s points of $\mathbf{PG}(3,s)$. So the common point u of the lines T, T' belongs to $\mathbf{PG}(3,s)$. Consequently, $\mathcal{S}$ is fully embedded in $\mathbf{PG}(3,s)$. Now by the Theorem of Buekenhout & Lefèvre [1], s is necessarily even and so $\mathcal{S} \cong W(s)$.

Next, assume that not all points of N belong to $\mathbf{PG}(3,s)$, that is, assume that at most one point of $N \setminus \{n\}$ belongs to $\mathbf{PG}(3,s)$. Now from here on the proof is completely analogous to the second half of the proof of Theorem 7.2.

The theorem is proved.

Remark 8.5 If the generalized quadrangle $\mathcal{S} \cong W(s)$, s even, is fully embedded in a subspace $\mathbf{PG}(3,s)$ of $\mathbf{PG}(3,q)$, then there also exists a $\mathbf{PG}(4,q)$ containing $\mathbf{PG}(3,q)$ and a point $x \in \mathbf{PG}(4,q) \setminus \mathbf{PG}(3,q)$ such that $\mathcal{S}$ is the projection from x onto $\mathbf{PG}(3,q)$ of a generalized quadrangle $\widetilde{\mathcal{S}} \cong Q(4,s) \cong W(s)$ which is fully embedded in a subspace $\mathbf{PG}(4,s)$ of $\mathbf{PG}(4,q)$. Here x is the nucleus of the quadric Q defining $Q(4,s)$.

Theorem 8.6 *If the generalized quadrangle $\mathcal{S} \cong Q(4,s)$ of order (s,s), with $s = 2$ and q odd, or $s = 3$ and $q \equiv 1 \bmod 3$, is laxly embedded in $\mathbf{PG}(3,q)$, then there exists a*

$\mathbf{PG}(4,q)$ *containing* $\mathbf{PG}(3,q)$ *and a point* $x \in \mathbf{PG}(4,q) \setminus \mathbf{PG}(3,q)$ *such that* $\mathcal{S}$ *is the projection from* x *onto* $\mathbf{PG}(3,q)$ *of a generalized quadrangle* $\widetilde{\mathcal{S}} \cong Q(4,s)$ *which is laxly embedded in* $\mathbf{PG}(4,q)$ *(and hence weakly embedded for* $s=2$*).*

Proof. First let $s=3$.

We use the same description, in terms of Hanssens and Van Maldeghem coordinates, of $Q(4,3)$ as in the proof of Theorem 5.1. Since there must be at least one ordinary quadrangle in $\mathcal{S}$ spanning $\mathbf{PG}(3,q)$ (otherwise $\mathcal{S}$ is contained in a plane), we may assume that the points (∞), (0), $(0,0)$ and $(0,0,0)$ span $\mathbf{PG}(3,q)$. Also, not both lines $[1]$ and $[-1]$ are contained in the plane $\langle(\infty),(0),(0,0,0)\rangle$, since otherwise again $\mathcal{S}$ would be contained in this plane. So we may assume that $[1]$ is skew to $[0,0]$ in $\mathbf{PG}(3,q)$. Hence, without loss of generality we can choose coordinates in $\mathbf{PG}(3,q)$ as follows:

in $\mathcal{S}$	in $\mathbf{PG}(3,q)$	in $\mathcal{S}$	in $\mathbf{PG}(3,q)$
(∞)	$(1,0,0,0)$	$(0,0,0)$	$(0,0,1,0)$
(0)	$(0,1,0,0)$	$(0,0,1)$	$(0,1,1,0)$
(1)	$(1,1,0,0)$	$(0,0,-1)$	$(0,b,1,0)$
(-1)	$(a,1,0,0)$	$(1,0,0)$	$(0,0,1,1)$
$(0,0)$	$(0,0,0,1)$	$(1,0,1)$	$(1,1,1,1)$
$(0,1)$	$(1,0,0,1)$	$(1,0,-1)$	$(b,b,1,1)$
$(0,-1)$	$(b,0,0,1)$	$(-1,0,0)$	$(0,0,1,a)$
$(1,0)$	(A,B,C,D)	$(-1,0,1)$	$(a,1,1,a)$
$(1,1)$	(A',B,C,D)	$(-1,0,-1)$	$(ab,b,1,a)$

with $a,b \in \mathbf{GF}(q)\setminus\{0,1\}$, and $A,A',B,C,D \in \mathbf{GF}(q)$, $A \neq A'$, $D \neq 0$. These coordinates can be easily computed because the grids of $\mathcal{S}$ we must use to that purpose, are not contained in a plane of $\mathbf{PG}(3,q)$. In the same way, we can calculate the following points of $\mathcal{S}$ (where we put $E = A - A'$):

in $\mathcal{S}$	in $\mathbf{PG}(3,q)$
$(1,-1)$	$(A-bE,B,C,D)$
$(1,1,0)$	$(A-bE,B-bE,C-E,D)$
$(1,1,1)$	$(A,B,C-E,D)$
$(1,1,-1)$	$(A',B-E,C-E,D)$
$(-1,-1,0)$	$(A'a,aB-E,aC-E,aD)$
$(-1,-1,1)$	$(aA-aEb,aB-bE,aC-E,aD)$
$(-1,-1,-1)$	$(aA,aB,aC-E,aD)$

In exactly the same way as we calculated the coordinates of the points $(-1,0)$ and $(-1,1)$ in the proof of Theorem 5.1, we can do this here and we obtain respectively $(Aa-E(ab+a), Ba-E(ab+1), Ca-E(a+1), Da-Ea)$ and $(Aa-Eab, Ba-Eb, Ca-E(a+1), Da-Ea)$. This again implies $ab+1=b$. Also, exactly as in the proof of Theorem 5.1, it follows that $ab+1=b$ and so we conclude that $a=b$. Now using the same technique as in the proof of Theorem 7.1 (noting that, if two opposite lines of $\mathcal{S}$ are contained in a plane of $\mathbf{PG}(3,q)$, then no line not belonging to the grid defined by these two lines is contained in that plane), we see that we can calculate the coordinates of all points uniquely. Embedding $\mathbf{PG}(3,q)$ into $\mathbf{PG}(4,q)$ as the hyperplane with equation $X_4=0$, we see that $\mathcal{S}$ is the projection from the point (A,B,C,D,E) onto $\mathbf{PG}(3,q)$ of the quadrangle $Q(4,3)$ laxly embedded in $\mathbf{PG}(4,q)$ as in the proof of Theorem 5.1.

Now let $s=2$. We obtain a description of $Q(4,2)$ by restricting coordinates (in the sense of Hanssens & Van Maldeghem [5]) to $\mathbf{GF}(2)$ in the representation of $H(3,4)$ in the proof of Theorem 4.1. After an elementary exercise, we see that we can choose coordinates in $\mathbf{PG}(3,q)$ in such a way that, for some $A,A',B,C,D\in\mathbf{GF}(q)$, with $A\neq A'$, $D\neq 0$, and putting $E=A-A'$, we have:

in $\mathcal{S}$	in $\mathbf{PG}(3,q)$		
(∞)	$(1,0,0,0)$	$(0,0,0)$	$(0,0,1,0)$
(0)	$(0,1,0,0)$	$(0,0,1)$	$(0,1,1,0)$
(1)	$(1,1,0,0)$	$(1,0,0)$	$(0,0,1,1)$
$(0,0)$	$(0,0,0,1)$	$(1,0,1)$	$(1,1,1,1)$
$(0,1)$	$(1,0,0,1)$	$(1,1,1)$	$(A,B,C-E,D)$
$(1,0)$	(A,B,C,D)	$(1,1,0)$	$(A',B-E,C-E,D)$
$(1,1)$	(A',B,C,D)	$(0,1,0)$	$(A',B-E,C-E,D-E)$
		$(0,1,1)$	$(A',B,C-E,D-E)$

Let $\mathbf{PG}(4,q)$ be a projective space containing $\mathbf{PG}(3,q)$ as the hyperplane $X_4=0$. Then it is readily checked that $\mathcal{S}$ is the projection from the point (A,B,C,D,E) of $\mathbf{PG}(4,q)$ onto $\mathbf{PG}(3,q)$ of the quadrangle $\mathcal{S}'\cong Q(4,2)$, weakly embedded in $\mathbf{PG}(4,q)$ as follows:

in $\mathcal{S}'$	in $\mathbf{PG}(4,q)$		
(∞)	$(1,0,0,0,0)$	$(0,0,0)$	$(0,0,1,0,0)$
(0)	$(0,1,0,0,0)$	$(0,0,1)$	$(0,1,1,0,0)$
(1)	$(1,1,0,0,0)$	$(1,0,0)$	$(0,0,1,1,0)$
$(0,0)$	$(0,0,0,1,0)$	$(1,0,1)$	$(1,1,1,1,0)$
$(0,1)$	$(1,0,0,1,0)$	$(1,1,1)$	$(0,0,1,0,1)$
$(1,0)$	$(0,0,0,0,1)$	$(1,1,0)$	$(1,1,1,0,1)$
$(1,1)$	$(1,0,0,0,1)$	$(0,1,0)$	$(1,1,1,1,1)$
		$(0,1,1)$	$(1,0,1,1,1)$

This completes the proof of the theorem.

Theorem 8.7 *If the generalized quadrangle $\mathcal{S} \cong Q(5,s)$ of order (s,s^2), with $s \neq 2$ for q odd, is laxly embedded in $\mathbf{PG}(3,q)$, then there exists a $\mathbf{PG}(5,q)$ containing $\mathbf{PG}(3,q)$ and a line L of $\mathbf{PG}(5,q)$ skew to $\mathbf{PG}(3,q)$ such that $\mathcal{S}$ is the projection from L onto $\mathbf{PG}(3,q)$ of a generalized quadrangle $\widetilde{S} \cong Q(5,s)$ which is fully embedded in a subspace $\mathbf{PG}(5,s)$ of $\mathbf{PG}(5,q)$, for the subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$.*

Proof. Let $\mathcal{S}'$ be a subquadrangle of order (s,s) of $\mathcal{S}$ containing a given line M of $\mathcal{S}$ ($\mathcal{S}'$ exists as $\mathcal{S} \cong Q(5,s)$). Assume that $\mathcal{S}'$ is contained in a plane $\mathbf{PG}(2,q)$. Let y be a point of $\mathcal{S}'$ not on the line M. Further, let N be a line of $\mathcal{S}$ containing y but not contained in the plane $\mathbf{PG}(2,q)$. As $\mathcal{S} \cong Q(5,s)$, the lines M and N are contained in a subquadrangle $\mathcal{S}''$ of order (s,s) of $\mathcal{S}$. This subquadrangle $\mathcal{S}''$ generates $\mathbf{PG}(3,q)$. Hence any line M of $\mathcal{S}$ is contained in a subquadrangle of order (s,s) of $\mathcal{S}$ which generates $\mathbf{PG}(3,q)$.

Let z_1 and z_2 be non-collinear points of $\mathcal{S}$. If $y_1, y_2 \in \{z_1,z_2\}^\perp$, $y_1 \neq y_2$, and if the s^2+1 lines of $\mathcal{S}$ through y_i, $i=1,2$, are contained in a plane π_i of $\mathbf{PG}(3,q), i=1,2$, then $\pi_1 \neq \pi_2$ as otherwise $\mathcal{S}$ is contained in a plane; also, $\pi_1 \cap \pi_2$ contains $\{y_1,y_2\}^\perp$. As $\mathcal{S} \cong Q(5,s)$ and so $|\{y,z\}^{\perp\perp}| = 2$ for any two non-collinear points y,z, it follows that there are at least s^2-1 points u in $\{z_1,z_2\}^\perp$ for which $u^\perp$ is not contained in a plane of $\mathbf{PG}(3,q)$. Let $U_1 = uz_1, U_2 = uz_2, U_3$ be lines of $\mathcal{S}$ through u which are not coplanar. Then U_1, U_2, U_3 are contained in a subquadrangle of order (s,s) of $\mathcal{S}$ which generates $\mathbf{PG}(3,q)$ and is not fully embedded in a subspace $\mathbf{PG}(3,s)$ of $\mathbf{PG}(3,q)$.

So without loss of generality we may assume that $\mathcal{S}'$ generates $\mathbf{PG}(3,q)$ and is not fully embedded in a subspace $\mathbf{PG}(3,s)$ of $\mathbf{PG}(3,q)$. Also, any point of $\mathcal{S}$ is contained in such a subquadrangle $\mathcal{S}'$.

First, assume that $s \neq 2$ for q odd and $s \neq 3$ for $q \equiv 1 \bmod 3$. By Theorem 8.4 there exists a $\mathbf{PG}(4,q)$ containing $\mathbf{PG}(3,q)$ and a point $x \in \mathbf{PG}(4,q) \setminus \mathbf{PG}(3,q)$ such that $\mathcal{S}'$ is the projection from x onto $\mathbf{PG}(3,q)$ of a generalized quadrangle $\widetilde{S}' \cong Q(4,s)$ which is fully embedded in a subspace $\mathbf{PG}(4,s)$ of $\mathbf{PG}(4,q)$, for the subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$; as $\mathcal{S}'$ is not fully embedded in a 3-dimensional subspace over $\mathbf{GF}(s)$ we have $x \notin \mathbf{PG}(4,s)$. Let U be a line of $\mathcal{S}$ not contained in $\mathcal{S}'$. Then U contains exactly one point u of $\mathcal{S}'$. By the first paragraph of the proof there is a subquadrangle of order (s,s) of $\mathcal{S}$ which contains U and generates $\mathbf{PG}(3,q)$. By Theorem 8.4 U is a subline $\mathbf{PG}(1,s)$ of the corresponding line U' of $\mathbf{PG}(3,q)$. Further, let $\widetilde{u}$ be the unique common point of the line ux of $\mathbf{PG}(4,q)$ and $\widetilde{S}'$.

Assume, by way of contradiction, that U is the projection from x onto $\mathbf{PG}(3,q)$ of a line $\widetilde{U}$ through $\widetilde{u}$ of $\mathbf{PG}(4,s)$. Let v be a point of $\mathcal{S}$ not in $\mathcal{S}'$, with $v \notin U$. Assume first that $v \not\sim u$. Let w be the point of U with $w \sim v$. Then the line vw of $\mathcal{S}$ has a point u' in common with $\mathcal{S}'$. If all lines of $\mathcal{S}'$ through u are contained in the plane $\langle v,U\rangle$, then the plane $\langle v,U\rangle$ induces a proper subquadrangle of order (s,t') of $\mathcal{S}$, with $t' > s$, a contradiction as $\mathcal{S} \cong Q(5,s)$. Hence there is a line W of $\mathcal{S}'$ through u which is not contained in the plane $\langle v, U\rangle$. Now we consider a subquadrangle $\mathcal{S}'' \cong Q(4,s)$ of order (s,s) of $\mathcal{S}$ containing W, U, v, u'. By Theorem 8.4 and Remark 8.5 $\mathcal{S}''$ is the projection from some point x' onto $\mathbf{PG}(3,q)$ of a generalized quadrangle $\widetilde{\widetilde{\mathcal{S}}}''$ fully embedded in some subspace $\mathbf{PG}'(4,s)$ of $\mathbf{PG}(4,q)$. As $\mathcal{S}'$ and $\mathcal{S}''$ share the line W, they intersect either in a grid or in $s+1$ concurrent lines. Let V be the line of $\mathcal{S}$ containing u' and concurrent with W; then V is a common line of $\mathcal{S}'$ and $\mathcal{S}''$. Let $\widetilde{V}$ respectively $\widetilde{W}$ be the line of $\widetilde{\mathcal{S}}'$ which corresponds to V respectively W, and let $\widetilde{\widetilde{V}}$ respectively $\widetilde{\widetilde{W}}$ be the line of $\widetilde{\widetilde{\mathcal{S}}}''$ which corresponds to V respectively W. The plane $\widetilde{\mathbf{PG}}(2,s)$, respectively $\widetilde{\widetilde{\mathbf{PG}}}(2,s)$, defined by $\widetilde{V}, \widetilde{W}$, respectively $\widetilde{\widetilde{V}}, \widetilde{\widetilde{W}}$, is projected from x, respectively x', onto the plane $\mathbf{PG}(2,s)$ of $\mathbf{PG}(3,q)$ defined by V and W. If ux' intersects $\widetilde{\widetilde{\mathcal{S}}}''$ in $\widetilde{\widetilde{u}}$, if $u'x$ intersects $\widetilde{\mathcal{S}}'$ in $\widetilde{u}'$, and if $u'x'$ intersects $\widetilde{\widetilde{\mathcal{S}}}''$ in $\widetilde{\widetilde{u}}'$, then the line uu' of $\mathbf{PG}(2,s)$ is the projection from x, respectively x', onto $\mathbf{PG}(3,q)$ of the line $\widetilde{u}\widetilde{u}'$, respectively $\widetilde{\widetilde{u}}\widetilde{\widetilde{u}}'$, of $\widetilde{\mathbf{PG}}(2,s)$, respectively $\widetilde{\widetilde{\mathbf{PG}}}(2,s)$. Hence the plane π over $\mathbf{GF}(s)$ defined by U, v, u' contains the projection of the line $\widetilde{u}\widetilde{u}'$ of $\mathbf{PG}(4,s)$ from x onto the plane $\langle U, v, u'\rangle$ of $\mathbf{PG}(3,q)$. It follows that π is the projection from x onto $\mathbf{PG}(3,q)$ of a plane of $\mathbf{PG}(4,s)$. So v is the projection from x onto $\mathbf{PG}(3,q)$ of a point $\widetilde{v}$ of $\mathbf{PG}(4,s)$. Also the projection of the line $\widetilde{u}'\widetilde{v}$ of $\mathbf{PG}(4,s)$ from x onto $\mathbf{PG}(3,q)$ is the line $u'v$ of $\mathcal{S}$. As $x \notin \mathbf{PG}(4,s)$ there is at most one point r of $\mathcal{S}$ for which the line rx of $\mathbf{PG}(4,q)$ contains more than one point of $\mathbf{PG}(4,s)$. If such a point r exists, we may assume it belongs to $\mathcal{S}'$. Next, assume that $v \sim u$. Now we consider a line U_1 of $\mathcal{S}$ not in $\mathcal{S}'$ where $U \sim U_1$ and where the common point u_1 of $\mathcal{S}'$ and the line U_1 is not one of the s^2+1 points of $\mathcal{S}'$ collinear with v. Interchanging roles of U and U_1, we see that also in this case v is the projection from x onto $\mathbf{PG}(3,q)$ of a point $\widetilde{v}$ of $\mathbf{PG}(4,s)$. Now let T be a line of $\mathcal{S}$ not in $\mathcal{S}'$ which is not concurrent with U. Let U_1 be a line of $\mathcal{S}$ concurrent with U and T, which contains neither the point u nor the common point t of T and $\mathcal{S}'$. Interchanging roles of U and U_1 we then see that the common point $\widetilde{t}$ of $\widetilde{\mathcal{S}}'$ and xt, together with the common points of $\mathbf{PG}(4,s)$ and xm, with $m \in T - \{t\}$, form a line $\widetilde{T}$ of $\mathbf{PG}(4,s)$. Finally, let $T_1 \neq U$ be a line of $\mathcal{S}$ not in $\mathcal{S}'$ through u. Let U_1 be a line of $\mathcal{S}$ not in $\mathcal{S}'$, which does not contain u. Interchanging roles of U and U_1, we see again that with T_1 there corresponds a line $\widetilde{T}_1$ of $\mathbf{PG}(4,s)$.

Now it is clear that $\mathcal{S}$ is isomorphic to a GQ $\widetilde{\mathcal{S}} \cong Q(5,s)$ which is fully embedded in $\mathbf{PG}(4,s)$, contradicting the Theorem of Buekenhout & Lefèvre [1]. We conclude that U is not the projection from x onto $\mathbf{PG}(3,q)$ of a line of $\mathbf{PG}(4,s)$ through $\widetilde{u}$.

Consider a line $\widetilde{U}$ of $\mathbf{PG}(4,q)$ through $\widetilde{u}$ such that the projection from x onto $\mathbf{PG}(3,q)$ is the line U of $\mathcal{S}$. Then $\widetilde{U}$ has at most two points in common with $\mathbf{PG}(4,s)$. Now we consider a space $\mathbf{PG}(5,q)$ containing $\mathbf{PG}(4,q)$, a point $\bar{x} \in \mathbf{PG}(5,q) \setminus \mathbf{PG}(4,q)$ and a line $\overline{U}$ through $\widetilde{u}$ in $\mathbf{PG}(5,q)$ but not in $\mathbf{PG}(4,q)$, in such a way that $\widetilde{U}$ is the projection from $\bar{x}$ onto $\mathbf{PG}(4,q)$ of the line $\overline{U}$. Further, let $\mathbf{PG}(5,s)$ be the 5-dimensional subspace of

$\mathbf{PG}(5,q)$ defined by $\mathbf{PG}(4,s)$ and $\overline{U}$. As $\widetilde{U}$ is not a line of $\mathbf{PG}(4,s)$ we have $\bar{x} \notin \mathbf{PG}(5,s)$. Let $\mathcal{R}$ be the projection of $\mathbf{PG}(5,s)$ from $\bar{x}$ onto $\mathbf{PG}(4,q)$.

First, assume that $\widetilde{U}$ contains two points $\widetilde{u}, \widetilde{h}$ of $\mathbf{PG}(4,s)$; then the line $\bar{x}\widetilde{h}$ of $\mathbf{PG}(5,q)$ contains a line of $\mathbf{PG}(5,s)$. Consider any line $\widetilde{N}$, $\widetilde{h} \notin \widetilde{N}$, of $\mathbf{PG}(4,s)$ through $\widetilde{u}$, and let $\pi_{\widetilde{N}}$ be the plane over $\mathbf{GF}(s)$ defined by $\widetilde{N}$ and $\widetilde{U}$. We have $|\mathcal{R}| = s^5+s^4+s^3+s^2+1$, and the subset $\mathcal{N} = \bigcup_{\widetilde{N}} \pi_{\widetilde{N}}$ of $\mathcal{R}$ has size $s^5+s^4+s^3+s+1$. The s^2-s points of $\mathcal{R} \setminus \mathcal{N}$ are on the line $\widetilde{u}\widetilde{h}$ of $\mathbf{PG}(4,q)$ and are the intersections of the line $\widetilde{u}\widetilde{h}$ of $\mathbf{PG}(4,q)$ with the lines of $\mathbf{PG}(4,q)$ containing exactly s points of $\mathcal{N}$.

Next, assume that $\widetilde{U}$ contains exactly one point of $\mathbf{PG}(4,s)$ and that the extension $\widetilde{U}'$ of $\widetilde{U}$ to $\mathbf{GF}(q)$ does not contain a line of $\mathbf{PG}(4,s)$. If $\widetilde{N}$ is any line of $\mathbf{PG}(4,s)$ through $\widetilde{u}$ and if $\pi_{\widetilde{N}}$ is the plane over $\mathbf{GF}(s)$ defined by $\widetilde{N}$ and $\widetilde{U}$, then $\mathcal{R} = \bigcup_{\widetilde{N}} \pi_{\widetilde{N}}$. If there is no line of $\mathbf{PG}(5,s)$ whose extension to $\mathbf{GF}(q)$ contains $\bar{x}$, then $|\mathcal{R}| = s^5+s^4+s^3+s^2+s+1$; if $\mathbf{PG}(5,s)$ contains a line whose extension to $\mathbf{GF}(q)$ contains $\bar{x}$, then $|\mathcal{R}| = s^5+s^4+s^3+s^2+1$.

Finally, assume that $\widetilde{U}$ contains exactly one point of $\mathbf{PG}(4,s)$ and that the extension $\widetilde{U}'$ of $\widetilde{U}$ to $\mathbf{GF}(q)$ contains a line $\widetilde{W}$ of $\mathbf{PG}(4,s)$. Consider any line $\widetilde{N}, \widetilde{N} \neq \widetilde{W}$, of $\mathbf{PG}(4,s)$ through $\widetilde{u}$, and let $\pi_{\widetilde{N}}$ be the plane over $\mathbf{GF}(s)$ defined by $\widetilde{U}$ and $\widetilde{N}$. Further, let $\mathcal{N} = \bigcup_{\widetilde{N}} \pi_{\widetilde{N}}$. The points of $\mathcal{R} \setminus \mathcal{N}$ are on the line $\widetilde{U}'$ and are the intersections of the line $\widetilde{U}'$ with the lines of $\mathbf{PG}(4,q)$ containing exactly s points of $\mathcal{N}$. If there is no line of $\mathbf{PG}(5,s)$ whose extension to $\mathbf{GF}(q)$ contains $\bar{x}$, then $|\mathcal{R}| = s^5+s^4+s^3+s^2+s+1$, and $|\mathcal{N}| = s^5+s^4+s^3+s+1$; if there is a line H of $\mathbf{PG}(5,s)$ whose extension H' to $\mathbf{GF}(q)$ contains $\bar{x}$, then H contains $\widetilde{u}$, $|\mathcal{R}| = s^5+s^4+s^3+s^2+1$, and $|\mathcal{N}| = s^5+s^4+s^3+s+1$.

So in each case $\mathcal{R}$ can be easily constructed from $\mathbf{PG}(4,s)$ and $\widetilde{U}$. Also, if $\widetilde{U}_1$ is a $\mathbf{PG}(1,s)$ contained in $\mathcal{R}$, but not in $\mathbf{PG}(4,s)$, which is the projection of a line of $\mathbf{PG}(5,s)$ from $\bar{x}$ onto $\mathbf{PG}(4,q)$, then $\mathcal{R}$ can be analogously constructed from $\mathbf{PG}(4,s)$ and $\widetilde{U}_1$; in particular this is the case when $\widetilde{U}_1$ is a $\mathbf{PG}(1,s)$ contained in $\mathcal{R}$, but not in $\mathbf{PG}(4,s)$, for which the extension $\widetilde{U}_1'$ to $\mathbf{GF}(q)$ intersects $\mathcal{R}$ exactly in $\widetilde{U}_1$.

Let v be a point of $\mathcal{S}$ not in $\mathcal{S}'$ and not on U. Assume first that $v \not\sim u$, with u the unique common point of $\mathcal{S}'$ and U. The line M of $\mathcal{S}$ through v and concurrent with U contains a point m of $\mathcal{S}'$. If $\widetilde{u}$ respectively $\widetilde{m}$ is the common point of $\widetilde{\mathcal{S}}'$ and the line xu respectively xm of $\mathbf{PG}(4,q)$, then let T be the projection from x onto $\mathbf{PG}(3,q)$ of the line $\widetilde{u}\widetilde{m}$ of $\mathbf{PG}(4,s)$. By a foregoing argument the line T over $\mathbf{GF}(s)$ belongs to the plane $\mathbf{PG}(2,s)$ defined by U, v, m. Let $\widetilde{\mathbf{PG}}(2,s)$ be the plane over $\mathbf{GF}(s)$ defined by $\widetilde{U}$ and the line $\widetilde{u}\widetilde{m} = \widetilde{T}$ of $\mathbf{PG}(4,s)$. Projecting this $\widetilde{\mathbf{PG}}(2,s)$ from x onto $\mathbf{PG}(3,q)$ we clearly obtain the plane $\mathbf{PG}(2,s)$. The plane $\widetilde{\mathbf{PG}}(2,s)$ belongs to the set $\mathcal{R}$. Hence the line vx of $\mathbf{PG}(4,q)$ intersects $\widetilde{\mathbf{PG}}(2,s)$ in a point $\widetilde{v}$ (if vx would contain distinct points of $\widetilde{\mathbf{PG}}(2,s)$, then x would be a point of $\widetilde{\mathbf{PG}}(2,q)$, so M and U would be on a common line of $\mathbf{PG}(3,q)$, clearly a contradiction). Let v' be the common point of M and U. If $\widetilde{v}'$ is the point of $\widetilde{U}$ which corresponds to v', then the points $\widetilde{m}, \widetilde{v}'$ and the $s-1$ points $\widetilde{v}$ which correspond to the $s-1$ points v on $M \setminus \{m, v'\}$ form a line $\widetilde{M}$ over $\mathbf{GF}(s)$ in $\mathcal{R}$. As $\widetilde{M}$ belongs to the plane $\widetilde{\mathbf{PG}}(2,s)$, it is the projection from $\bar{x}$ onto $\mathbf{PG}(4,q)$ of some line $\overline{M}$ through $\widetilde{m}$ of $\mathbf{PG}(5,s)$. Next, assume that $v \sim u$. Now we consider a line U_1 of $\mathcal{S}$ not in $\mathcal{S}'$ where $U \sim U_1$ and where the common point u_1 of $\mathcal{S}'$ and the line U_1 is not one of the s^2+1 points of $\mathcal{S}'$ collinear with v. Then with U_1 there corresponds a line $\widetilde{U}_1$ in $\mathcal{R}$. Interchanging roles of U and U_1, and of $\widetilde{U}$ and $\widetilde{U}_1$, we see again that v is the projection from x onto $\mathbf{PG}(3,q)$ of a point $\widetilde{v}$ of $\mathcal{R}$. Hence every point of $\mathcal{S}$ is the projection from x onto $\mathbf{PG}(3,q)$ of some point of $\mathcal{R}$. Now let W be a line of $\mathcal{S}$ not in $\mathcal{S}'$ which is not concurrent with U. Let U_1 be a line of $\mathcal{S}$ concurrent with U and W, which contains neither the point u nor the common point w of W and $\mathcal{S}'$. Interchanging roles of U and U_1 we then see that W is the projection from x onto $\mathbf{PG}(3,q)$ of some line $\widetilde{W}$ in $\mathcal{R}$. Finally, let $W_1 \neq U$ be a line of $\mathcal{S}$ not in $\mathcal{S}'$ through u. Let U_1 be a line of $\mathcal{S}$ not in $\mathcal{S}'$, which does not contain u. Interchanging roles of U and U_1, we see again that W_1 is the projection from x onto $\mathbf{PG}(3,q)$ of some line $\widetilde{W}_1$ in $\mathcal{R}$. We conclude that for any line R of $\mathcal{S}$ not in $\mathcal{S}'$, there is a line $\widetilde{R}$ in $\mathcal{R}$ whose projection from x onto $\mathbf{PG}(3,q)$ is R. Also, if r is the common point of R and $\mathcal{S}'$, then $\widetilde{R}$ contains the unique common point $\widetilde{r}$ of the line xr of $\mathbf{PG}(4,q)$ and $\widetilde{\mathcal{S}}'$. Further, $\widetilde{R}$ is the projection from $\bar{x}$ onto $\mathbf{PG}(4,q)$ of some line $\overline{R}$ through $\widetilde{r}$ of $\mathbf{PG}(5,s)$.

Let v be any point of $\mathcal{S}$ not in $\mathcal{S}'$ and let T be any line of $\mathcal{S}$ containing v. If the line $\widetilde{T}$ in $\mathcal{R}$ corresponds to T, then let v^θ be the unique common point of $\widetilde{T}$ and the line xv of $\mathbf{PG}(4,q)$. We will show that v^θ is uniquely defined by v.

First assume, by way of contradiction, that with T there correspond distinct lines $\widetilde{T}$ and $\widetilde{T}^*$ in $\mathcal{R}$; these lines contain a common point of $\widetilde{\mathcal{S}}'$. Let w be any point of $T \setminus \{t\}$, where t is the common point of $\mathcal{S}'$ and T. Then the line wx of $\mathbf{PG}(4,q)$ contains a point $\widetilde{w}$ of $\widetilde{T}$ and a point $\widetilde{w}^*$ of $\widetilde{T}^*$. It follows that the plane $\widetilde{\mathbf{PG}}(2,s)$ over $\mathbf{GF}(s)$ defined by $\widetilde{T}$ and $\widetilde{T}^*$ contains x. Suppose that the line $\overline{T}$, respectively $\overline{T}^*$, of $\mathbf{PG}(5,s)$ corresponds to $\widetilde{T}$, respectively $\widetilde{T}^*$; then $\overline{T}$ and $\overline{T}^*$ both contain $\tilde{t}$. If $\overline{\mathbf{PG}}(2,s)$ is the plane over $\mathbf{GF}(s)$ defined by $\overline{T}$ and $\overline{T}^*$, then $\widetilde{\mathbf{PG}}(2,s)$ is the projection of $\overline{\mathbf{PG}}(2,s)$ from $\bar{x}$ onto $\mathbf{PG}(4,q)$. As $\overline{\mathbf{PG}}(2,s)$ contains a line $\overline{A}$ of $\mathbf{PG}(4,s)$, also $\widetilde{\mathbf{PG}}(2,s)$ contains $\overline{A}$; clearly $\overline{A}$ contains $\tilde{t}$. As $\bar{x} \notin \mathbf{PG}(4,q)$, it follows that T is the projection of $\overline{A}$ from x onto $\mathbf{PG}(3,q)$. At the beginning of the proof we have shown that this implies that some GQ $\widetilde{\mathcal{S}} \cong Q(5,s)$ is fully embedded in $\mathbf{PG}(4,s)$, giving a contradiction.

Next, let T and T_1 be distinct lines of $\mathcal{S}$ containing v. Assume that with T, respectively T_1, there corresponds the line $\widetilde{T}$, respectively $\widetilde{T}_1$, in $\mathcal{R}$. By the beginning of the proof the uniquely defined line $\widetilde{T}_1$ necessarily contains $\widetilde{v}$.

Consequently the point v^θ is uniquely defined by v. Further, for any point u in $\mathcal{S}'$ we put $u^\theta = u$. So there arises an injection from the point set of $\mathcal{S}$ into $\mathcal{R}$. Also, it is now clear that for any line W of $\mathcal{S}$ the set W^θ is a line over $\mathbf{GF}(s)$. Further, for distinct lines W, W_1 of $\mathcal{S}$ the extensions $(W^\theta)', (W_1^\theta)'$ of W^θ, W_1^θ to $\mathbf{GF}(q)$ are distinct. So θ is an isomorphism of $\mathcal{S}$ onto a GQ $\widetilde{\mathcal{S}} \cong Q(5,s)$ which is laxly embedded in $\mathbf{PG}(4,q)$; also $\mathcal{S}$ is the projection of $\widetilde{\mathcal{S}}$ from x onto $\mathbf{PG}(3,q)$. From Theorem 11 it now follows that there exists a $\widetilde{\widetilde{\mathbf{PG}}}(5,q)$ containing $\mathbf{PG}(4,q)$ and a point $\widetilde{\widetilde{x}} \in \widetilde{\widetilde{\mathbf{PG}}}(5,q) \setminus \mathbf{PG}(4,q)$ such that $\widetilde{\mathcal{S}}$ is the projection from $\widetilde{\widetilde{x}}$ onto $\mathbf{PG}(4,q)$ of a GQ $\widetilde{\widetilde{\mathcal{S}}} \cong Q(5,s)$ which is fully embedded in a subspace $\widetilde{\widetilde{\mathbf{PG}}}(5,s)$ of $\widetilde{\widetilde{\mathbf{PG}}}(5,q)$; in fact, as $\mathcal{R}$ is the projection of $\mathbf{PG}(5,s)$ from $\bar{x}$ onto $\mathbf{PG}(4,q)$, it is clear that we can put $\widetilde{\widetilde{\mathbf{PG}}}(5,q) = \mathbf{PG}(5,q), \widetilde{\widetilde{x}} = \bar{x}$ and $\widetilde{\widetilde{\mathbf{PG}}}(5,s) = \mathbf{PG}(5,s)$. We conclude that $\mathcal{S}$ is the projection of $\widetilde{\widetilde{\mathcal{S}}}$ from $L = x\widetilde{\widetilde{x}}$ onto $\mathbf{PG}(3,q)$.

Finally, let $s = 3$ and $q \equiv 1 \bmod 3$. If every two opposite lines of $\mathcal{S}$ span $\mathbf{PG}(3,q)$, then, since the full group of projectivities of a line of $\mathcal{S}$ is isomorphic to $\mathbf{PGL}_2(s)$ (by Knarr [6]), as in the first part of the proof of Theorem 8.4, we obtain that for any line L of $\mathcal{S}$ the group $\mathbf{PGL}_2(q)$ of the linear transformations of the corresponding line L' of $\mathbf{PG}(3,q)$ admits a subgroup isomorphic to $\mathbf{PSL}_2(s)$ acting on the $s+1$ points of L. Hence in such a case we have that q is a power of 3 by Lemma 8.3, contradicting $q \equiv 1 \bmod 3$.

Let L, M be two opposite lines of $\mathcal{S}$ contained in a plane π of $\mathbf{PG}(3,q)$, let x, y, $x \neq y$, be two points on L, let N be the unique line of $\mathcal{S}$ through y concurrent with M, and let $\{L_r | r \in \mathbf{GF}(3)^2\}$ be the set of lines of $\mathcal{S}$ through x, different from L. Since π induces a proper subquadrangle, there are at most four lines of $\mathcal{S}$ through x lying in π. We now consider the naturally embedded generalized quadrangle $Q(5,3)$ in $\mathbf{PG}(5,3)$; the point set of $Q(5,3)$ is a non-singular elliptic quadric Q of $\mathbf{PG}(5,3)$. We can take as equation of Q

$$X_2X_3 + X_4X_5 = X_0^2 + X_1^2.$$

Let γ be an isomorphism of $\mathcal{S}$ onto $Q(5,3)$. Then we can take for L^γ the line through the two points $x^\gamma(0,0,0,1,0,0)$ and $y^\gamma(0,0,0,0,1,0)$, for N^γ the line through the points y^γ and $z^\gamma(0,0,1,0,0,0)$, and for M^γ the line through the points z^γ and $(0,0,0,0,0,1)$. Furthermore, we can take for L_r^γ, $r = (r_0, r_1) \in \mathbf{GF}(3)^2$, the line through x^γ and the point $(r_0, r_1, 0, 0, r_0^2 + r_1^2, 1)$, and we also may assume that the lines L_r with $r_0 \neq 0$ are not contained in π. Then the projectivity $\theta = [N; L_a; M; L_b; N]$, $a, b \in \mathbf{GF}(3)^\times \times \mathbf{GF}(3)$ of the line N of $\mathcal{S}$ extends to a unique linear transformation of the line N' of $\mathbf{PG}(3,q)$ containing N. One easily calculates that in $\mathbf{PG}(5,3)$, the projectivity $\gamma^{-1}\theta\gamma$ of N^γ, with $a = (1,1)$ and $b = (1,0)$, maps the point with coordinates $(0,0,c,0,1,0)$ onto the point with coordinates $(0,0,-c,0,1,0)$. Hence by varying y and z on N, we now see that the symmetric group $\mathbf{S}_4$, which is isomorphic to $\mathbf{PGL}_2(3)$, is induced on N' as a subgroup of $\mathbf{PGL_2}(q)$ and acting on the four points of N. By Lemma 8.3, q is divisible by 3, a contradiction.

Now the theorem is completely proved.

Theorem 8.8 *If the generalized quadrangle $\mathcal{S}$ of order $(2,4)$ is laxly embedded in $\mathbf{PG}(3,q)$, q odd, then there exists a $\mathbf{PG}(5,q)$ containing $\mathbf{PG}(3,q)$ and a line L of $\mathbf{PG}(5,q)$ skew to $\mathbf{PG}(3,q)$ such that $\mathcal{S}$ is the projection from L onto $\mathbf{PG}(3,q)$ of a generalized quadrangle $\widetilde{S} \cong Q(5,2)$ which is laxly embedded in $\mathbf{PG}(5,q)$, and hence determined by* Theorem 6.1.

Proof. We consider two opposite lines L_0 and L_1 in $\mathcal{S}$ which are not coplanar in $\mathbf{PG}(3,q)$ (these lines exist by the first paragraph of the proof of Theorem 8.7). Using the description of $\mathcal{S}$ in terms of Hanssens and Van Maldeghem coordinates, as in the proof of Theorem 6.1, we may take $L_0 = [\infty]$ and $L_1 = [0,0,0]$. We may then coordinatize $\mathbf{PG}(3,q)$ as follows:

in $\mathcal{S}$	in $\mathbf{PG}(3,q)$
(∞)	$(1,0,0,0)$
(0)	$(0,1,0,0)$
(1)	$(1,1,0,0)$
$(0,0)$	$(0,0,0,1)$
$(0,1)$	$(1,0,0,1)$
$(0,0,0)$	$(0,0,1,0)$
$(0,0,1)$	$(0,1,1,0)$
$(1,0,0)$	$(0,0,1,1)$
$(1,0,1)$	$(1,1,1,1)$

There are exactly three subquadrangles of order $(2,2)$ containing L_0 and L_1. Every such subquadrangle $\mathcal{S}'$ is laxly embedded in $\mathbf{PG}(3,q)$ and hence it is represented as in the proof of Theorem 8.6. Note however that in the latter proof we started by assigning the point $(1,0)$ the coordinates (A,B,C,D) assuming that the point $(1,0)$ was not contained in the plane spanned by $[\infty]$ and $[0,0]$. If $(1,0)$ is a point of the plane $\langle[\infty],[0,0]\rangle$, then $(1,0)$ is not a point of the plane $\langle[0],[0,0,0]\rangle$. In such a case put $(1,0) = (A,B,C,D)$ and $(1,1) = (A',B,C,D)$, with $A,A',B,C,D \in \mathbf{GF}(q)$ and $A \neq A'$, $B \neq 0$. Putting $E = A - A'$, we obtain exactly the same coordinates for the points of the subquadrangle $\mathcal{S}'$ containing $(1,0)$ as in the proof of Theorem 8.6. The same is true for the subquadrangle of order $(2,2)$ containing L_0, L_1 and $(\epsilon,0)$. We assign to $(\epsilon,0)$ the coordinates (X,Y,Z,U), to $(\epsilon,1)$ the coordinates (X',Y,Z,U), and we put $W = X - X'$. Similarly we give the point $(\epsilon^2,0)$ respectively $(\epsilon^2,1)$ the coordinates (K,L,M,N) respectively (K',L,M,N), and we put $P = K - K'$. This way, we easily obtain the coordinates of all points of $\mathcal{S}$:

in $\mathcal{S}$	in $\mathbf{PG}(3,q)$
$(1,0)$	(A,B,C,D)
$(1,1)$	(A',B,C,D)
$(1,1,1)$	$(A,B,C-E,D)$
$(1,1,0)$	$(A',B-E,C-E,D)$
$(0,1,0)$	$(A',B-E,C-E,D-E)$
$(0,1,1)$	$(A',B,C-E,D-E)$
$(\epsilon,0)$	(X,Y,Z,U)
$(\epsilon,1)$	(X',Y,Z,U)
$(1,\epsilon,1)$	$(X,Y,Z-W,U)$
$(1,\epsilon,0)$	$(X',Y-W,Z-W,U)$
$(0,\epsilon,0)$	$(X',Y-W,Z-W,U-W)$
$(0,\epsilon,1)$	$(X',Y,Z-W,U-W)$
$(\epsilon^2,0)$	(K,L,M,N)
$(\epsilon^2,1)$	(K',L,M,N)
$(1,\epsilon^2,1)$	$(K,L,M-P,N)$
$(1,\epsilon^2,0)$	$(K',L-P,M-P,N)$
$(0,\epsilon^2,0)$	$(K',L-P,M-P,N-P)$
$(0,\epsilon^2,1)$	$(K',L,M-P,N-P)$

Any point $p \in \{(1,0),(1,1),(\epsilon,0),(\epsilon,1),(\epsilon^2,0),(\epsilon^2,1)\}$ is on exactly two lines L_p, M_p of $\mathcal{S}$ not concurrent with any of $L_0, L_1, [0,1,0]$. It is clear that these twelve lines, together with (0) and $(0,0)$, are not in a common plane. It follows that there is a point p for which $\langle L_p, M_p\rangle$ does not contain both points (0) and $(0,0)$. We may choose coordinates in such a way that $\langle L_{(\epsilon^2,0)}, M_{(\epsilon^2,0)}\rangle$ does not contain both (0) and $(0,0)$. As $(\epsilon^2,0)$ is the intersection of the lines through the respective points $(1,1,0)$, $(0,\epsilon,1)$, and $(0,1,1)$, $(1,\epsilon,0)$, we have

$$\begin{vmatrix} A-E & C-E \\ X-W & Z-W \end{vmatrix} \neq 0.$$

Expressing now that $\langle (1,1,0),(0,\epsilon,1)\rangle$ and $\langle (0,1,1),(1,\epsilon,0)\rangle$ intersect in $(\epsilon^2,0)$, we see that we may put

$$\left\{ \begin{array}{rcl} K &=& WA+EX-2EW, \\ L &=& WB+EY-EW, \\ M &=& WC+EZ-2EW, \\ N &=& WD+EU-EW. \end{array} \right.$$

Expressing that $(\epsilon^2,1)$ is the intersection of the lines through respectively $(1,1,1)$, $(0,\epsilon,0)$ and $(0,1,0)$, $(1,\epsilon,1)$, we obtain after a calculation and taking account of the foregoing equalities that $K' = WA+EX-EW$ and so $P = K-K' = -EW$. Now we embed $\mathbf{PG}(3,q)$ in $\mathbf{PG}(5,q)$ as the subspace with equations $X_4 = X_5 = 0$ and we see that

$\mathcal{S}$ is the projection onto $\mathbf{PG}(3, q)$ from the line through the points with coordinates $(A, B, C, D, E, 0)$ and $(X, Y, Z, U, 0, W)$ of the following structure $\mathcal{S}'$ (on the same line we put the coordinates in $\mathbf{PG}(5, q)$ of any point of $\mathcal{S}'$ (at the right) and the Hanssens and Van Maldeghem coordinates of the corresponding point of $\mathcal{S}$ (at the left)):

in $\mathcal{S}$	in $\mathbf{PG}(5, q)$		
(∞)	$(1,0,0,0,0,0)$	$(0,1,0)$	$(1,1,1,1,1,0)$
(0)	$(0,1,0,0,0,0)$	$(0,1,1)$	$(1,0,1,1,1,0)$
(1)	$(1,1,0,0,0,0)$	$(\epsilon,0)$	$(0,0,0,0,0,1)$
$(0,0)$	$(0,0,0,1,0,0)$	$(\epsilon,1)$	$(1,0,0,0,0,1)$
$(0,1)$	$(1,0,0,1,0,0)$	$(1,\epsilon,1)$	$(0,0,1,0,0,1)$
$(0,0,0)$	$(0,0,1,0,0,0)$	$(1,\epsilon,0)$	$(1,1,1,0,0,1)$
$(0,0,1)$	$(0,1,1,0,0,0)$	$(0,\epsilon,0)$	$(1,1,1,1,0,1)$
$(1,0,0)$	$(0,0,1,1,0,0)$	$(0,\epsilon,1)$	$(1,0,1,1,0,1)$
$(1,0,1)$	$(1,1,1,1,0,0)$	$(\epsilon^2,0)$	$(2,1,2,1,1,1)$
$(1,0)$	$(0,0,0,0,1,0)$	$(\epsilon^2,1)$	$(1,1,2,1,1,1)$
$(1,1)$	$(1,0,0,0,1,0)$	$(1,\epsilon^2,1)$	$(2,1,1,1,1,1)$
$(1,1,1)$	$(0,0,1,0,1,0)$	$(1,\epsilon^2,0)$	$(1,0,1,1,1,1)$
$(1,1,0)$	$(1,1,1,0,1,0)$	$(0,\epsilon^2,0)$	$(1,0,1,0,1,1)$
		$(0,\epsilon^2,1)$	$(1,1,1,0,1,1)$

Now one can check that this is a lax embedding of $Q(5, 2)$ in $\mathbf{PG}(5, q)$. Since there is only one such lax embedding by Theorem 6.1, the theorem is proved.

Remark 8.9 The only thick finite Moufang generalized quadrangles whose lax embeddings in $\mathbf{PG}(3, q)$ are not yet classified are the generalized quadrangles $W(s)$ with s odd.

APPENDIX A: Regularity in generalized quadrangles of order $(s, s-2)$

If each point of a GQ of order (s, t), $s > 1$, is regular, then, by 1.3.6 and 1.5.1 of Payne & Thas [10] we have that $s \geq t$ and that $t + 1$ divides $(s^2 - 1)s^2$. Hence, a priori, a GQ of order $(s, s-2)$ could have all its points regular. This problem, which is important for the theory of the lax embeddings of GQ (see Theorem 5.3), will be considered in this first Appendix.

Theorem 8.10 *All the points of a generalized quadrangle $\mathcal{S}$ of order $(s, s-2)$, $s \geq 4$, are regular if and only if $s = 4$ and $\mathcal{S} \cong H(3, 4)$.*

Proof. First let $s=4$. By (the dual of) PAYNE & THAS [10](5.3.2), any GQ of order (4,2) is isomorphic to the classical GQ $H(3,4)$ and consequently all its points are regular.

Now let $s>4$ and assume, by way of contradiction, that $\mathcal{S}=(P,B,\mathrm{I})$ is any GQ of order $(s,s-2)$ having all its points regular. Let $L=\{y,z\}^{\perp\perp}$ with $y\not\sim z$. Then $|L|=s-1$. The set of all points of $\mathcal{S}$ collinear with no point of L will be denoted by V_L. Then $V_L=V_{L^\perp}$. Also, $|V_L|=(s+1)(s-1)^2-(s-1)^2(s-1)-2(s-1)=2(s-1)(s-2)$.

Let $x\in V_L$, and consider

$$\bigcup_{u\in L}\{x,u\}^{\perp\perp}\setminus L=:W.$$

Then $|W|=(s-2)^2$. Let $v\in W, v\neq x$. Assume that $v\sim u'$, with $u'\in L$. If $v\in\{u,x\}^{\perp\perp}$, $u\in L$, and if w is the point of vu' collinear with u, then $w\sim x$ and wx intersects L, a contradiction as $x\in V_L$. Consequently $W\subseteq V_L$. As $V_L=V_{L^\perp}$, also

$$\cup_{u\in L^\perp}\{x,u\}^{\perp\perp}\setminus L^\perp=:\widetilde{W}$$

is a subset of V_L. If $v'\in W\cap\widetilde{W}$, with $v'\neq x$, then the point $\{x,v'\}^{\perp\perp}\cap L$ is collinear with the point $\{x,v'\}^{\perp\perp}\cap L^\perp$, a contradiction as no two points of $\{x,v'\}^{\perp\perp}$ are collinear. Hence $W\cup\widetilde{W}$ is a subset of size $2(s-2)^2-1$ of V_L.

Now we show that no two distinct points of $W\cup L$ are collinear, that is, we show that $W\cup L$ is a (s^2-3s+3)-cap of $\mathcal{S}$. No two distinct points of L are collinear. Also, if v_1 and v_2 are distinct points of $\{u,x\}^{\perp\perp}$, $u\in L$, then $v_1\not\sim v_2$. If $u\in L$ and $v\in W$, then $u\not\sim v$ as $v\in V_L$. Finally, let v,v' be distinct points of W, with $v\neq x\neq v'$ and $\{x,v\}^{\perp\perp}\neq\{x,v'\}^{\perp\perp}$. Assume that $v\sim v'$ and let v'' be the point of vv' collinear with x. Further, let $L\cap\{x,v\}^{\perp\perp}=\{u\}$ and $L\cap\{x,v'\}^{\perp\perp}=u'$. Then $v''\sim u$ and $v''\sim u'$. Hence $v''\in L^\perp$. It follows that xv'' intersects L, a contradiction as $x\in V_L$.

Let $\{M_i:i=1,2,\ldots,s-1\}$ be the set of lines of $\mathcal{S}$ incident with x and fix any $i\in\{1,2,\ldots,s-1\}$. As $x\in V_L$ no two distinct points of L are collinear with the same point of M_i. It immediately follows that M_i contains exactly two distinct points of V_L, say x and t_i. Clearly $t_i\in V_L\setminus(W\cup\widetilde{W})$. Put $T=\{t_1,t_2,\ldots,t_{s-1}\}$ and $X=V_L\setminus(W\cup\widetilde{W}\cup T)$. Then $|X|=s-2$. Further, let x' be a point of V_L collinear with t_i, but not incident with M_i. Clearly $x'\notin T$. If $x'\in W$, where $\{x',x\}^{\perp\perp}\cap L=\{u\}$, then $t_i\sim u$, a contradiction as $t_i\in V_L$. Analogously, $x'\notin\widetilde{W}$. Hence $x'\in X$. So the $s-2$ points of $V_L\setminus\{x\}$ collinear with t_i belong to X, for all $i\in\{1,2,\ldots,s-1\}$. It follows that $X\cup\{x\}$ and T are hyperbolic lines, and that $(X\cup\{x\})^\perp=T$. Also, $V_L=W\cup\widetilde{W}\cup T\cup X$.

Now we show that the set $O_{L,x}=W\cup L\cup X$ is a $(s-1)^2$-cap, that is, an ovoid, of $\mathcal{S}$. Let $x'\in X$. Clearly $x'\not\sim x$. If $x'\sim v$, with $v\neq x$, $v\in W\cup L$ and $\{v,x\}^{\perp\perp}\cap L=\{u\}$, then $x'v$ contains one of the points of T, say t_i, so $t_i\sim u$, a contradiction as $t_i\in V_L$. Consequently $W\cup L\cup X$ is a (s^2-1)-cap of the GQ $\mathcal{S}$. Analogously, $\widetilde{W}\cup L^\perp\cup X$ is an ovoid $O_{L^\perp,x}$ of $\mathcal{S}$.

Let $V_L \cup L \cup L^\perp = U_L$. Then $|U_L| = 2(s-1)^2$. In fact $U_L = O_{L,x} \cup \widetilde{O}_{L,x}$, with $\widetilde{O}_{L,x} = (\widetilde{W} \cup L^\perp \cup T) \setminus \{x\}$. As $\widetilde{O}_{L,x} = (O_{L^\perp,x} \setminus (X \cup \{x\})) \cup (X \cup \{x\})^\perp$, it is also an ovoid of $\mathcal{S}$.

If $m \in U_L$, then from the preceding it is clear that all points of $U_L \setminus \{m\}$ collinear with m form a hyperbolic line. Let m, m' be distinct points of $O_{L,x}$ or $\widetilde{O}_{L,x}$, and let respectively N, N' be the corresponding hyperbolic lines contained in U_L. Then $N \cup N' \subseteq O_{L,x}$ or $N \cup N' \subseteq \widetilde{O}_{L,x}$. Assume that $n \in N \cap N'$. If $n \in L$, then $m, m' \in L^\perp$, so $N = N' = L$; if $n \in L^\perp$, then $m, m' \in L$, so $N = N' = L^\perp$. Now let $n \notin L \cup L^\perp$. Interchanging roles of n and x, we see that U_L is the union of all hyperbolic lines containing n and a point of $L \cup L^\perp$, the hyperbolic line S consisting of all the points of $U_L \setminus \{n\}$ collinear with n, and one hyperbolic line containing n and disjoint from $L \cup L^\perp$ (this last hyperbolic line is $S^\perp$). Clearly $(N \cup N') \cap S = \emptyset$. Also, if $N \cup N'$ contains a point of $L \cup L^\perp$, say $N \cup N'$ contains a point of L, then at least one of m, m' is collinear with a point of L, so at least one of N, N' coincides with L, from which $n \in L$, a contradiction. It follows that $(N \cup N') \cap (L \cup L^\perp) = \emptyset$. Hence N respectively N' is the unique hyperbolic line contained in U_L, containing n, and disjoint from $L \cup L^\perp$. Consequently, $N = N'$. The ovoid $O_{L,x}$ is the union of the hyperbolic lines in $O_{L,x}$ which correspond with the points of $\widetilde{O}_{L,x}$, and so with the $(s-1)^2$ points of $\widetilde{O}_{L,x}$ there correspond exactly $s-1$ mutually disjoint hyperbolic lines $L, L_1, \ldots, L_{s-2}$ in $O_{L,x}$; analogously, with the $(s-1)^2$ points of $O_{L,x}$ there correspond exactly $s-1$ mutually disjoint hyperbolic lines $L^\perp, \widetilde{L}_1, \ldots, \widetilde{L}_{s-2}$ in $\widetilde{O}_{L,x}$. Now it is also immediate that $L_i^\perp$ coincides with some $\widetilde{L}_j$, $i = 1, 2, \ldots, s-2$. Consequently, indices can be chosen in such a way that $L_i^\perp = \widetilde{L}_i$, $i = 1, 2, \ldots, s-2$.

Consider now a point $v \in O_{L,x} \setminus (L \cup \{x\})$. Then $O_{L,v} = L \cup L_1^* \cup L_2^* \cup \ldots \cup L_{s-2}^*$, where $L_i^* \in \{L_i, L_i^\perp\}, i = 1, 2, \ldots, s-2$. First, suppose that $\{x, v\}^{\perp\perp} \notin \{L_1, \ldots, L_{s-2}\}$. Then $\{x, v\}^{\perp\perp}$ has one point in common with each of L, L_1, $L_2, \ldots, L_{s-2}$. As $O_{L,x}$ is the union of the $s-1$ hyperbolic lines in the set $\mathcal{L} = \{L, L_1, \ldots, L_{s-2}, L^\perp, L_1^\perp, \ldots, L_{s-2}^\perp\}$ which have exactly one point in common with $\{x, u\}^{\perp\perp}$, with u any point in L, the ovoid $O_{L,v}$ is the union of the $s-1$ elements of $\mathcal{L}$ which have exactly one point in common with $\{x, v\}^{\perp\perp}$ (as $\{x, v\}^{\perp\perp} = \{v, u\}^{\perp\perp}$ for some point $u \in L$). Hence $O_{L,v} = L \cup L_1 \cup \ldots \cup L_{s-2} = O_{L,x}$. Next, assume that $\{x, v\}^{\perp\perp} \in \{L_1, L_2, \ldots, L_{s-2}\}$, say $\{x, v\}^{\perp\perp} = L_1$. Let $v' \in L_2$. Then, by the foregoing, $O_{L,x} = O_{L,v'}$ and $O_{L,v'} = O_{L,v}$, so $O_{L,x} = O_{L,v}$. We conclude that for any point $v \in O_{L,x} \setminus L$, we have $O_{L,x} = O_{L,v}$.

Indices can be chosen in such a way that $x \in L_1$. The ovoid $O_{L^\perp,x}$ is the union of the hyperbolic lines $L^\perp, L_1, L_2^\perp, L_3^\perp, \ldots, L_{s-2}^\perp$. As $s > 4$, we can choose a point $w \in O_{L^\perp,x}$ not in $L^\perp, L_2^\perp, L_1$. Then $O_{L^\perp,w} = O_{L^\perp,x}$. Let u be a point of L_2. Then the ovoid $O_{L^\perp,u}$ is the union of the hyperbolic lines $L^\perp, L_1^\perp, L_2, L_3^\perp, \ldots, L_{s-2}^\perp$. Also, $O_{L^\perp,u} = O_{L^\perp,w}$. Consequently, $O_{L^\perp,x} = O_{L^\perp,u}$. It follows that $L_2 \subseteq O_{L^\perp,x}$, clearly a contradiction. Now the theorem is completely proved.

APPENDIX B: A characterization of the classical generalized quadrangle $H(4, s^2)$

Theorem. *A generalized quadrangle $\mathcal{S} = (P, B, I)$ of order $(s^2, s^3), s \neq 1$, is isomorphic to $H(4, s^2)$ if and only if any two non-concurrent lines are contained in a proper subquadrangle of order (s^2, t), with $t \neq 1$.*

Proof. Let $\mathcal{S} \cong H(4, s^2)$. Then any two non-concurrent lines are contained in a subquadrangle $\mathcal{S}' \cong H(3, s^2)$ of order (s^2, s).

Conversely, assume that $\mathcal{S} = (P, B, \mathrm{I})$ is a GQ of order $(s^2, s^3), s \neq 1$, for which any two non-concurrent lines are contained in a proper subquadrangle of order $(s^2, t), t \neq 1$. Then by Payne & Thas [10](2.2.2), we have $t = s$.

Fix a subquadrangle $\mathcal{S}'$ of order (s^2, s) of $\mathcal{S}$, and let L be a line of $\mathcal{S}'$. Now let $\mathcal{S}''$ be a subquadrangle of order (s^2, s), distinct from $\mathcal{S}'$, containing the line L. By Payne & Thas [10](2.2.1), each line of $\mathcal{S}''$ respectively $\mathcal{S}'$ has a point in common with the set of all common points of $\mathcal{S}'$ and $\mathcal{S}''$. As by Payne & Thas [10](2.2.2) the GQ $\mathcal{S}'$ respectively $\mathcal{S}''$ has no proper subquadrangle of order (s^2, t') it now easily follows that $\mathcal{S}' \cap \mathcal{S}''$ consists of $s + 1$ concurrent lines together with the points incident with these lines. As a GQ of order $(s^2, s), s \neq 1$, has no proper subquadrangle of order (s^2, t'), the lines L and M of $\mathcal{S}$, with M any line of $\mathcal{S}$ not in $\mathcal{S}'$ and not concurrent with L, are contained in exactly one subquadrangle $\mathcal{S}'' \neq \mathcal{S}'$ of order (s^2, s).

Let x, y, z be points of $\mathcal{S}'$ with $x \sim z \sim y$, and $x \not\sim y$. Further, let $z' \in \{x, y\}^\perp$, with $z \neq z'$ and z' in $\mathcal{S}'$. If $\mathcal{S}''' \neq \mathcal{S}'$ is a subquadrangle of order (s^2, s) containing z', x, y, then $\mathcal{S}'''$ does not contain the point z. Now let u be a point of $\mathcal{S}'''$ collinear with x and y, where $u \neq z'$. Further, let $\widetilde{\mathcal{S}}$ be the subquadrangle of order (s^2, s) containing x, y, z, u. Then the $s + 1$ lines of $\mathcal{S}'$ through z are the $s + 1$ lines of $\widetilde{\mathcal{S}}$ through z, and the $s + 1$ lines of $\widetilde{\mathcal{S}}$ through u are the $s + 1$ lines of $\mathcal{S}'''$ through u. Hence $\mathcal{S}' \cap \mathcal{S}''' \cap \widetilde{\mathcal{S}}$ is the set V consisting of the $s + 1$ points collinear in $\widetilde{\mathcal{S}}$ with u and z. The set V is the set of all points of $\mathcal{S}'''$ belonging to the lines of $\mathcal{S}'$ through z.

Now let w be a point of $\{x, y\}^\perp$, with w not in $\widetilde{\mathcal{S}}$. Let $\widetilde{\mathcal{S}}'$ be the subquadrangle of order (s^2, s) containing x, y, w, u, and let $\widetilde{\mathcal{S}}''$ be the subquadrangle of order (s^2, s) containing x, y, z, w. Then $\widetilde{\mathcal{S}} \cap \widetilde{\mathcal{S}}' \cap \widetilde{\mathcal{S}}''$ is the set V' consisting of the $s + 1$ points collinear in $\widetilde{\mathcal{S}}''$ with z and w, and also the set of the $s + 1$ points of $\widetilde{\mathcal{S}}$ collinear with u and z. Hence $V' = V$, and all points of V are collinear with w. If we choose w in $\mathcal{S}'$, then it follows that the pair $\{x, y\}$ is regular in $\mathcal{S}'$. Now it is clear that any point of any subquadrangle of order (s^2, s) is regular. Hence any point of V is collinear with any point $\widetilde{w}$ of $\{x, y\}^\perp$ in $\widetilde{\mathcal{S}}$. Consequently $|\{x, y\}^{\perp\perp}| \geq |V| = s + 1$. Now by Payne & Thas [10](5.5.1), we have $\mathcal{S} \cong H(4, s^2)$.

The theorem is proved.

References

[1] F. Buekenhout and C. Lefèvre, 'Generalized quadrangles in projective spaces', *Arch. Math.* **25** (1974) 540 – 552.

[2] L. E. Dickson, *Linear Groups, with an exposition of the Galois field theory*, Teubner, Leibzig, 1901.

[3] K. J. Dienst, 'Verallgemeinerte Vierecke in Pappusschen projektiven Räumen', *Geom. Dedicata* **9** (1980) 199 – 206.

[4] K. J. Dienst, 'Verallgemeinerte Vierecke in projektiven Räumen', *Arch. Math.* **35** (1980) 177 – 186.

[5] G. Hanssens and H. Van Maldeghem, 'A new look at the classical generalized quadrangles', *Ars Combin.* **24** (1987) 199 – 210.

[6] N. Knarr, 'Projectivities of generalized polygons', *Ars Combin.* **25B** (1988) 265 – 275.

[7] C. Lefèvre-Percsy, 'Quadrilatères généralisés faiblement plongés dans $\mathbf{PG}(3, q)$', *European J. Combin.* **2** (1981) 249 – 255.

[8] C. Lefèvre-Percsy, 'Projectivités conservant un espace polaire faiblement plongé', *Acad. Roy. Belg. Bull. Cl. Sci.* **67** (1981) 45 – 50.

[9] M. Limbos, *Plongements et arcs projectifs*, Ph. D. Thesis, Université Libre de Bruxelles, Belgium, 1981.

[10] S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, Pitman, Boston, London, Melbourne, 1984.

[11] J. A. Thas, 'Partial geometries in finite affine spaces', *Math. Z.* **158** (1978) 1 – 13.

[12] J. A. Thas, 'Generalized polygons', **in** *Handbook of Incidence Geometry, Buildings and Foundations*, (ed. F. Buekenhout), Chapter 9, North-Holland (1995) 383 – 431.

[13] J. A. Thas and H. Van Maldeghem, 'Generalized quadrangles weakly embedded in finite projective spaces', *J. Stat. Plann. Inference* **73** (1998) 353 – 361.

[14] J. Tits, 'Sur la trialité et certains groupes qui s'en déduisent', *Inst. Hautes Etudes Sci. Publ. Math.* **2** (1959) 14 – 60.

[15] H. Van Maldeghem, *Generalized Polygons*, Birkhäuser Verlag, *Monographs in Mathematics* **93**, Basel, Boston, Berlin, 1998.

Address of the Authors:

J. A. Thas & H. Van Maldeghem
University of Ghent
Department of Pure Mathematics and Computer Algebra,
Galglaan 2,
9000 Gent,
BELGIUM