Sharply 2-transitive groups of projectivities in generalized polygons

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Abstract

The group of projectivities of (a line of) a projective plane is always 3-transitive. It is well known that the projective planes with a sharply 3-transitive group of projectivities are classified: they are precisely the Pappian projective planes. It is also well known that the group of projectivities of a generalized polygon is 2-transitive. Here, we classify all generalized quadrangles, all finite generalized hexagons, and the parameter sets of all finite generalized octagons with a sharply 2-transitive group of projectivities.

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1. Introduction and statement of the main result

A generalized polygon $\Gamma$ of order $(s,t)$ is a rank 2 point-line geometry whose incidence graph has diameter $n$ and girth $2n$, for some $n \in \mathbb{N}\setminus\{0,1\}$ (in which case the generalized polygon is also called a generalized $n$-gon), each vertex corresponding to a point has valency $t+1$ and each vertex corresponding to a line has valency $s+1$. If $s,t > 1$, then the geometry is usually called thick. Each non-thick generalized polygon can be obtained from a thick one, and so one usually only considers thick generalized polygons. These objects were introduced by Tits [12]. More information is gathered in my monograph [13], to which we refer for a general introduction and basic properties. Here, we recall some notation. For an element $x$ of $\Gamma$, and a natural number $i$, we denote by $\Gamma_i(x)$, the set of elements of $\Gamma$ at distance $i$ from $x$ in the incidence graph of $\Gamma$. The distance function in that incidence graph is denoted by $\delta$. If two elements $x$ and $y$ are not at distance $n$, then there exists a unique element $\text{proj}_y x$ incident with $y$ and at distance $\delta(x,y) - 1$ from $x$. We call that element the projection of $x$ onto $y$. 

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y. Also recall that the dual of $\Gamma$ is obtained by interchanging the words ‘point’ and ‘line’. The dual of a generalized $n$-gon is obviously again a generalized $n$-gon.

Let $\Gamma$ be a generalized $n$-gon of order $(s,t)$, and let $x$ and $y$ be two elements of $\Gamma$ at distance $n$ in the incidence graph (elements of $\Gamma$ at distance $n$ in the incidence graph of $\Gamma$ are called opposite). Let $I_1(x)$ denote the set of elements of $\Gamma$ incident with $x$, and similarly for $I_1(y)$. It is well known that the relation ‘is not opposite’ is a bijection from $I_1(x)$ to $I_1(y)$. This bijection is called a perspectivity and denoted by $[x,y]$. For a collection $\{x_0,x_1,\ldots,x_\ell\}$ of points and lines, with $x_{i-1}$ opposite $x_i$, $1 \leq i \leq \ell$, we define the composition

$$[x_0,x_1,\ldots,x_\ell]:=[x_0,x_1][x_1,x_2]\cdots[x_{\ell-1},x_\ell]$$

and call this bijection from $I_1(x_0)$ to $I_1(x_\ell)$ a projectivity. The set of all projectivities $I_1(L) \to I_1(L)$, for some line $L$ of $\Gamma$, forms a group $\Pi(\Gamma)$, which is abstractly and as a permutation group, independent of $L$. It is called the group of projectivities of $\Gamma$. The ‘Fundamental Theorem of Projective Plane Geometry’ says that, for $n=3$ (a generalized 3-gon is nothing other than a projective plane), the (permutation) group of projectivities always acts 3-transitively, and it acts sharply 3-transitively if and only if the plane is Pappian (or equivalently, if and only if the projective plane arises from a three-dimensional vector space over a commutative field by taking the vector lines as points and the vector planes as lines, and inclusion as incidence). Now it is well known (for an explicit proof, see [8]) that in general, the group $\Pi(\Gamma)$ acts 2-transitively, and there are many examples of (finite and infinite) generalized 4-gons and generalized 8-gons with a group of projectivities which does not act 3-transitively (see e.g. [8] again, or Section 8.4 of the monograph [13]). In the present paper, we deal with the question (**): ‘what can be said about the generalized polygon $\Gamma$ when $\Pi(\Gamma)$ acts sharply 2-transitively?’

Question (**) has been suggested to me by Katrin Tent who, herself, classified in [11] all generalized quadrangles $\Gamma$ with a sharply 2-transitive group of projectivities under the additional assumption that the one-point stabilizers of $\Pi(\Gamma)$ are abelian.

Note that for $n$ even, the group $\Pi(\Gamma)$ has a subgroup (denoted by $\Pi^+(\Gamma)$) of index at most 2 consisting of all elements of $\Pi(\Gamma)$ associated to projectivities which are the composite of an even number of perspectivities (so-called even projectivities). Also, this group always acts 2-transitively, and hence, if $\Pi(\Gamma)$ acts sharply 2-transitively, then so does $\Pi^+(\Gamma)$. Consequently, the question: ‘When exactly does $\Pi^+(\Gamma)$ act sharply 2-transitively?’, is more general than the question (**).

A few remarks should put this question in a better perspective.

(i) Characterizations of certain classes of projective and affine planes by properties of their groups of projectivities exist in abundance, see [10] for a survey. For generalized $n$-gons with $n > 3$ (the case $n=2$ is trivial: the group of projectivities is in this case always the identity), only the results for $n=4$ of Brouns et al. [1] are available. Basically, the configurational properties induced by specific properties of the group of projectivities become too messy for $n > 3$, and hence, they do not
lead to anywhere. No classification result using groups of projectivities is known to me for generalized \( n \)-gons, with \( n > 4 \). The one we present here may not be very general (only a finite number of small polygons are characterized), but it can serve as a start for more results in this direction.

(ii) If \( s = 2 \), then \( \Pi(\Gamma) = \Pi^+(\Gamma) \) is automatically sharply 2-transitive (in fact, at the same time sharply 3-transitive). A classification of all generalized polygons \( \Gamma \) with \( \Pi(\Gamma) \) or \( \Pi^+(\Gamma) \) sharply 2-transitive would imply a classification of all generalized polygons of order \( (2,t) \). The latter one is at the moment not a reasonable problem, since it would in particular settle the question whether \( t \) has necessarily to be finite for \( n \) even (and this is an open problem solved only for \( n = 4 \); see Appendix 5 of [13]). We will restrict ourselves here to the values \( n = 3, 4, 6, 8 \), which appear to be the most interesting ones by the existence of ‘classical examples’ related to simple groups.

(iii) If we consider for a moment only the finite case, then we see that a complete classification of polygons \( \Gamma \) with \( \Pi(\Gamma) \) sharply 2-transitive requires, as above, the classification of generalized octagons of order \( (2,4) \). This is a long-standing problem that we will not try to solve in the present paper.

Our Main Result reads as follows.

**Main Result.** Let \( \Gamma \) be a projective plane, a generalized quadrangle, a finite generalized hexagon, or a finite generalized octagon. Suppose that \( \Pi^+(\Gamma) \) acts sharply 2-transitively. Then \( \Pi(\Gamma) = \Pi^+(\Gamma) \) and one of the following holds:

1. \( \Gamma \) is the unique projective plane of order \( (2,2) \),
2. \( \Gamma \) is the unique generalized quadrangle of order \( (2,2) \),
3. \( \Gamma \) is the unique generalized quadrangle of order \( (2,4) \),
4. \( \Gamma \) is isomorphic to the generalized quadrangle \( Q(4,3) \) of order \( (3,3) \) arising from a non-singular quadric in the four-dimensional projective space \( PG(4,3) \) over the Galois field \( GF(3) \) of order \( 3(3) \) (see also [9]),
5. \( \Gamma \) is a generalized hexagon of order \( (2,2) \) (and there are exactly 2 such; each one the dual of the other),
6. \( \Gamma \) is the unique generalized hexagon of order \( (2,8) \) and
7. \( \Gamma \) is a generalized octagon of order \( (2,4) \) or \( (4,2) \).

Concerning Cases 5 and 6, we remark that the finite generalized hexagons of order \( (2,t) \) are classified by Cohen and Tits [3]. As for Case 7 of the Main result, we remark that for the known generalized octagons \( \Gamma \) of order \( (2,4) \) and \( (4,2) \) we actually have that \( \Pi^+(\Gamma) \) acts sharply 2-transitively (this is proved in [8]).

Concerning our proof, we note that our argument for \( n = 6, 8 \) is typically a finite one, because we heavily use Lemma 2 of the next section. We could also use it for the case \( n = 4 \) to get rid of some small examples, but here there is a better geometric way, which also immediately gives us the examples without having to refer to the explicit calculation of the groups \( \Pi^+(\Gamma) \) for some small finite generalized quadrangles \( \Gamma \).
We subdivide our proof into the following parts. After two rather general lemmas (proving in particular that $\Pi^+(\Gamma) = \Pi(\Gamma)$ under the assumptions of the Main Result), we first deal with $n=4$ (the case $n=3$ follows from the ‘Fundamental Theorem’ stated above). Then we reduce the cases $n=6,8$ to a finite set of possible counterexamples. In the last part, we get rid of those.

2. Two useful lemmas

Lemma 1. Let $\Gamma$ be any generalized $n$-gon of order $(s,t)$, $s,t > 1$ (and possibly infinite), $n > 3$. Suppose that $\Pi^+(\Gamma)$ acts sharply 2-transitively. Then $\Pi^+(\Gamma) = \Pi(\Gamma)$. Moreover, if $s$ is finite, and $n$ is not congruent to 2 modulo 3, then $s$ is not congruent to 1 modulo 3.

Proof. In this proof, we use the following observation, partly due to Norbert Knarr (private communication). Let $L$ be any line of $\Gamma$. Pick any three points $x,y,z$ incident with $L$. It is easy to see that there is an ordinary $(n+1)$-gon with sides $x_0:=L,x_2,x_4,\ldots,x_{2n},x_2$ meeting $x_{2i+2}$, but not $x_{2i+4}$ (subscripts to be taken modulo $2n+2$), such that $x$ is incident with $x_{2n}$, $y$ is incident with $x_2$ and $z$ is the projection onto $L$ of $x_{n+1}$ (if $n$ is odd) or of the intersection of $x_n$ and $x_{n+2}$ (if $n$ is even). Let $x_{2i+1}$ be the intersection of $x_2$ and $x_{2i+2}$ (subscripts again modulo $2n+2$). Let $\theta : \Gamma_1(L) \rightarrow \Gamma_1(L)$ be the even projectivity defined by $\theta := [x_0,x_n,x_{2n},x_{3n},\ldots,x_{(2n+2)n}]$ (subscripts modulo $2n+2$, and note that $x_{(2n+2)n}=x_0=L$). It was observed by Norbert Knarr that $\theta$ stabilizes $\{x,y,z\}$ and that $\theta^3$ fixes $x,y$ and $z$. In fact, it is not difficult to see that $\theta: x \mapsto y \mapsto z \mapsto x$ if $n \equiv 0 \mod 3$, that $\theta : x \mapsto z \mapsto y \mapsto x$ if $n \equiv 1 \mod 3$, and that $\theta$ fixes $x,y,z$ if $n \equiv 2 \mod 3$. If $n$ is even, then $\theta' : \Gamma_1(L) \rightarrow \Gamma_1(L)$ defined by $\theta':=[x_0,x_n,x_{2n},\ldots,x_{(n+1)n}]$ does not possibly belong to $\Pi^+(\Gamma)$ (because it is composed of an odd number of perspectivities), and one checks that $\theta' : x \mapsto y \mapsto z \mapsto x$ if $n \equiv 1 \mod 3$, that $\theta' : x \mapsto z \mapsto y \mapsto x$ if $n \equiv 0 \mod 3$, and that $\theta'$ fixes $x,y,z$ if $n \equiv 2 \mod 3$. Note that $\theta^2=\theta$.

Now, if $n$ is odd, then automatically $\Pi^+(\Gamma) = \Pi(\Gamma)$ (because a composition of an odd number of perspectivities always maps $\Gamma_1(\text{line})$ to $\Gamma_1(\text{point})$, and vice versa). Suppose now that $n$ is even. Assume that $\Pi(\Gamma) \neq \Pi^+(\Gamma)$. Then $\theta^3$ of the previous paragraph fixes $x,y,z$ and belongs to $\Pi(\Gamma) \setminus \Pi^+(\Gamma)$ (hence $\theta^3 \neq \text{id}$). Let $u$ be a point incident with $L$ and not fixed by $\theta^3$. Noting that $x,y,z$ were chosen arbitrarily, we can consider an element $\sigma : \Gamma_1(L) \rightarrow \Gamma_1(L)$ of $\Pi(\Gamma) \setminus \Pi^+(\Gamma)$ fixing $x,y,u$. Clearly, the composition $\sigma \theta^3$ fixes $x$ and $y$, but not $u$. But $\sigma \theta^3 \in \Pi^+(\Gamma)$, a contradiction. Hence, $\theta^3$ is the identity and $\Pi^+(\Gamma) = \Pi(\Gamma)$.

Now suppose that $n \not= 2 \mod 3$, and let $s \equiv 1 \mod 3$ be finite. Then the map $\theta$ above belongs to $\Pi^+(\Gamma)$ and is not trivial. Clearly, $\theta^3$ is trivial, so $\theta$ defines a number of 3-cycles in $\Gamma_1(L)$. Since $s \equiv 1 \mod 3$, there are at least two points on $L$ fixed by $\theta$, hence $\theta$ is trivial by the sharp 2-transitivity, a contradiction.

The lemma is proved. □
Lemma 2. With the above notation, all lines of \( \Gamma^{(L,M)} \) (respectively \( \Gamma^{(p)} \)) are the points of \( \Gamma \) opposite \( p \); the lines of \( \Gamma^{(L,M)} \) (respectively \( \Gamma^{(p)} \)) are the lines of \( \Gamma \) opposite both \( L \) and \( M \) (respectively at a distance \( n-1 \) from \( p \)); incidence is inherited from \( \Gamma \).

For the next lemma, we introduce some notation. Let \( \Gamma \) be a finite generalized \( n \)-gon, \( n = 4, 6, 8 \). Let \( p \) be any point of \( \Gamma \), and fix two lines \( L \) and \( M \) through \( p \). Now we consider the following subgeometry \( \Gamma^{(L,M)} \) (respectively \( \Gamma^{(p)} \)) of \( \Gamma \). The points of \( \Gamma^{(L,M)} \) (respectively \( \Gamma^{(p)} \)) are the points of \( \Gamma \) opposite \( p \); the lines of \( \Gamma^{(L,M)} \) (respectively \( \Gamma^{(p)} \)) are the lines of \( \Gamma \) opposite both \( L \) and \( M \) (respectively at a distance \( n-1 \) from \( p \)); incidence is inherited from \( \Gamma \).

**Lemma 2.** With the above notation, the geometry \( \Gamma^{(L,M)} \) is connected except possibly in the following cases:

(a) \( \Gamma \) is a quadrangle and \( (s,t) \in \{ (2,2), (2,4), (3,3), (4,2) \} \),

(b) \( \Gamma \) is a hexagon and \( (s,t) \in \{ (2,2), (2,8), (3,3), (4,4), (8,2) \} \),

(c) \( \Gamma \) is an octagon and \( (s,t) \in \{ (2,4), (3,6), (4,2), (6,3) \} \).

**Proof.** The lemma will be proved by the method introduced by Brouwer [2], which he attributes to Willem Haemers. In fact, we can more or less copy Section 4 of Brouwer [2] (and we explicitly do so because we will need a slight modification later on). So, suppose that \( \Gamma^{(L,M)} \) is disconnected. Let \( A \) be the adjacency matrix of the collinearity graph of \( \Gamma^{(p)} \). Let \( U, V \) be two disjoint components whose union is \( \Gamma^{(L,M)} \). Consider the corresponding partition of \( A \) and let \( B \) be the condensed form of average row sums of the blocks of \( A \). Putting \( r = (s-1)(t+1) \), which is the valency of the collinearity graph of \( \Gamma^{(p)} \), \( u = |U| \) and \( v = |V| \), we find

\[
B = \begin{pmatrix}
    r - \varepsilon & \varepsilon \\
    \varepsilon u/v & r - \varepsilon u/v
\end{pmatrix},
\]

where \( \varepsilon \) is the average number of points in \( V \) collinear (in \( \Gamma^{(p)} \)) with a point of \( U \). The eigenvalues of \( B \) are \( r \) and \( r - \varepsilon - \varepsilon u/v \), and they must interlace the eigenvalues of \( A \). So, as in [2], we must have

\[
(s-1)(t+1) - \varepsilon(1+u/v) \leq s - 1 + \sqrt{ast},
\]

with \( a = n/2 - 2 \). Similarly as in [2], the expression \( \varepsilon(1+u/v) \) is maximized by having all lines of \( \Gamma^{(p)} \) which do not belong to \( \Gamma^{(L,M)} \) meet \( U \) in the same number of points, in which case \( \varepsilon(1+u/v) = 2s \). Hence

\[
(s-1)(t+1) - 2s \leq s - 1 + \sqrt{ast}.
\]

For \( n = 4 \), this reduces to \( st \leq 2s + t \). We easily obtain \( (s,t) \in \{ (2,2), (2,4), (3,3), (4,2) \} \). For \( n = 6 \), this means that \( st \leq 2s + t + \sqrt{st} \). Since \( st \) is a perfect square (see [4]) and since \( s \leq r^3 \) (see [6]), this implies that \( (s,t) \in \{ (2,2), (2,8), (3,3), (4,4), (8,2) \} \).
Similarly, for \( n = 8 \), we have \( st \leq 2s + t + \sqrt{2st} \). As \( 2st \) is a perfect square [4] and \( s \leq t \leq s^2 \) [7], we obtain \( (s,t) \in \{(2,4),(3,6),(4,2),(6,3)\} \).

The lemma is proved. \( \square \)

3. Generalized quadrangles

In this section, we assume that \( \Gamma \) is a generalized quadrangle (4-gon) with \( II^+(\Gamma) \) sharply 2-transitive. All generalized quadrangles of order \((2,t)\) are classified, see for instance the monograph [13, 1.7.9]. Hence, we may assume that the order of \( \Gamma \) is \((s,t)\) with \( s > 2 \). We show that in this case \( t \leq 3 \). Let \( z \) be any point of \( \Gamma \) and let \( p, a, b \) be three mutually opposite points collinear with \( z \), chosen in such a way that there exists a point \( x \) opposite \( p \) and collinear with both \( a, b \) (one easily checks that this is always possible). Let \( a' \) (respectively \( b' \)) be the projection of \( p \) onto \( ax \) (respectively \( bx \)). Let \( L \) be any line through \( p \) distinct from \( pa', pb' \) and \( pz \) (if such a line \( L \) does not exist, then \( t = 2 \) and we are done). Consider the even projectivity \( \theta = [L, ax, pz, bx, L] \). It is clear that \( \theta \) maps \( p \) onto itself, and that it also fixes the point \( proj_x x \). Hence \( \theta \) also fixes \( proj_x a \), which is mapped onto \( proj_x b \). We conclude that \( proj_x a = proj_x b \) and hence \( |I_2(p) \cap I_2(a) \cap I_2(b)| = t - 1 \). Now let \( b^* \) be a point incident with \( bz \) but distinct from \( b \), from \( z \) and from \( proj_{bz} a' \) (since \( s > 2 \), we can find such a point \( b^* \)). Interchanging the roles of \( x \) and \( proj_{ax} b^* \), and of \( b \) and \( b^* \), we see that \( |I_2(p) \cap I_2(a) \cap I_2(b^*)| = t - 1 \). But no element of \( I_2(p) \cap I_2(a) \cap I_2(b) \) is collinear with \( b^* \), except for \( z \). Moreover, also \( a' \) does not belong to \( I_2(b^*) \). Hence \( I_2(p) \cap I_2(a) \cap I_2(b^*) \) contains at most 2 elements (namely \( z \) and possibly a point incident with \( pb' \)). This implies \( t - 1 \leq 2 \).

So we have shown that \( t \leq 3 \). But now \( \Gamma \) is finite and is known (see 1.7 of the monograph [13], cp. 6.1 and 6.2 of [9]). The result now follows from the explicit determination of \( II^+(\Gamma) \), with \( \Gamma \) a quadrangle of order \((s,2)\) or \((s,3)\). This is done in [8] for the orders \((4,2),(3,3)\) and \((9,3)\), and in [5] for the quadrangle of order \((5,3)\).

Alternatively, we may argue as follows. Let \( L \) and \( M \) be two opposite lines of \( \Gamma \). Let \( L' \) and \( M' \) be two opposite lines each meeting both \( L \) and \( M \). Finally, let \( N \) be opposite both \( L \) and \( M \), and meeting both \( L' \) and \( M' \). Since \( II^+(\Gamma) = II(\Gamma) \) by Lemma 1, the projectivity \([L,M,N,L]\) is trivial, and this readily implies that, in the terminology of Payne and Thas [9], the pair \( \{L,M\} \) is regular, and hence that each line of \( \Gamma \) is regular. Hence, by 2.2.2(i) of [9], we have \( t \geq s \). Hence, only the quadrangles of order \((2,2)\) and \((3,3)\) must be considered (this argument also works for \( s \) infinite!). Moreover, for order \((3,3)\), all lines are regular, and hence we have the generalized quadrangle \( Q(4,3) \) arising from a non-degenerate quadric in the four-dimensional projective space \( \text{PG}(4,3) \) over the Galois field \( \text{GF}(3) \) of order 3. Now Knarr [8] tells us that \( II^+(\text{Q}(4,3)) \cong \text{PSL}_2(4) \) and so Case 4 of the Main Result follows.

Remark 2. Completely similar as in the beginning of this section, one shows the following more general fact. If \( \Gamma \) is a generalized \( n \)-gon, \( n \geq 4 \) even, of order \((s,t)\), with \( II^+(\Gamma) \) sharply 2-transitive, \( p \) is some point of \( \Gamma \), and \( x, y, z \) are points opposite \( p \) with
x and y collinear with z, but x not collinear with y, then \(|\Gamma_2(p) \cap \Gamma_{n-2}(x) \cap \Gamma_{n-2}(y)| \in \{0, t - 1\}\).

4. Finite generalized hexagons and octagons

In this section, we suppose that \(\Gamma\) is a finite generalized hexagon or octagon of order \((s, t)\), and that \(\Pi^+(\Gamma)\) acts sharply 2-transitively. Let \(n\) be the diameter of the incidence graph of \(\Gamma\) (so \(n = 6\) or \(8\)).

Let \(p\) be any point of \(\Gamma\), and let \(x_0\) be a point of \(\Gamma\) opposite \(p\). If \(L\) is some line through \(p\), then we label the point \(\text{proj}_L x_0\) by \((L; 0 \mod 3)\). We now choose an arbitrary order \((L_1; L_2; L_3; L_4; L_5; L_6; L_7)\) of the lines through \(p\), and we label the two points on \(L_1\) distinct from \(p\) and from \(\text{proj}_L x_0\) arbitrarily by \((L_1; 1 \mod 3)\) and \((L_1; 2 \mod 3)\). For convenience, we usually omit ‘\(\mod 3\)’ when it is clear it should be there. Let \(\theta_i\),
$2 \leq i \leq 7$ be any even projectivity from $L_1$ to $L_i$ which maps $p$ to $p$ and $(L_i, 0)$ to $(L_i, 0)$ \((0, \cdot , 0)\), exists by the 2-transitivity of $\Pi^+(\Gamma')$. Then we label the image of $(L_i, \ell)$, \(\ell \in \{1, 2\}$, by $(L_i, \ell)$. This labeling is independent of the choice of $\ell$, by the sharp 2-transitivity of $\Pi^+(\Gamma)$. Now with every point $x$ opposite $p$, we can associate a unique 7-tuple $7(x):=(i_1, i_2, \ldots , i_7) \in \{0, 1, 2\}^3$ defined by $\text{proj}_x x = (L_i, i_j), 1 \leq j \leq 7$. Now let $y$ be any point opposite $p$ collinear with $x$. Without loss of generality we may assume that the line $xy$ is not opposite $L_1$. Hence $7(y)$ is of the form $(i_1, i_2, \ldots , i_7)$. Consider the even projectivity $\sigma_\ell:=(L_2, xy, L_\ell), 3 \leq \ell \leq 7$. Clearly it maps $(L_2, i_2)$ to $(L_\ell, i_\ell)$. We now claim that it maps $(L_2, j_2)$ to $(L_\ell, i_\ell + j_2 - i_2)$. First, remark that every even projectivity from $L_\ell$ to $L_2$ which maps $p$ to $p$ and $(L_\ell, 0)$ to $(L_2, 0)$ maps $(L_\ell, 1)$ to $(L_2, 1)$. Now let $\sigma$ be any projectivity from $L_2$ to $L_\ell$ mapping $p$ to $p$ and $(L_2, 0)$ to $(L_\ell, 0)$ to $(L_\ell, 1)$. Suppose $\sigma$ maps $(L_2, 1)$ to $(L_\ell, 0)$. Then we may compose $\sigma$ with an even projectivity $\sigma'$ from $L_\ell$ to $L_2$, where $\sigma'$ fixes $p$ and maps $(L_\ell, k)$ to $(L_2, k)$, $k = 0, 1, 2$, and we obtain an even projectivity $\sigma\sigma'$ from $L_2$ onto itself fixing $p$ and $(L_2, 2)$ and swapping $(L_2, 0)$ with $(L_2, 1)$. This contradicts the sharp 2-transitivity of $\Pi^+(\Gamma)$. Hence $\sigma$ maps $(L_2, 1)$ to $(L_2, 2)$ and $(L_2, 2)$ to $(L_\ell, 0)$. Similarly, every even projectivity from $L_2$ to $L_\ell$ mapping $p$ to $p$ and $(L_2, 0)$ to $(L_\ell, 2)$, maps $(L_2, 1)$ to $(L_\ell, 0)$ and $(L_2, 2)$ to $(L_\ell, 1)$. Consequently, we have shown that the even projectivities from $L_2$ to $L_\ell$ fixing $p$ are of the form $(L_2, k) \leftrightarrow (L_\ell, k + \varepsilon)$, with $\varepsilon \in \{0, 1, 2\}$ (modulo 3). Our claim now follows easily. Putting $\varepsilon = j_2 - i_2$, we now have that $7(y) = (i_1, i_2 + \varepsilon, i_3 + \varepsilon, \ldots , i_7 + \varepsilon, \varepsilon)$. Since $\varepsilon$ appears 6 times, we deduce that the sum of all entries of $7(y)$ is congruent modulo 3 to the sum of all entries of $7(x)$. We can draw two conclusion out of this.

First. With the usual subtraction, we have that $7(x) - 7(y)$ contains a unique zero entry and either six 1’s or six 2’s when $x$ and $y$ are distinct collinear points opposite $p$. The zero entry is at position $i$ if and only if $xy$ is not opposite $L_i$, $i \in \{1, 2, \ldots , 7\}$.

Second. Since we can reach every point opposite $p$ by a sequence of collinear points (because $\Gamma^{(p)}$ is connected, see [2]), we have exactly $3^6$ 7-tuples which are actually equal to $7(z)$, for some point $z$ of $\Gamma$ opposite $p$. Since there are $3^4 \cdot 3^3$ points in $\Gamma$ opposite $p$, this means that on the average, every admissible 7-tuple appears as $7(x)$ for 24 points $x$ (an admissible 7-tuple is one which is equal to $7(u)$, for some point $u$ opposite $p$).

Now we consider any admissible 7-tuple, and without loss of generality we may take $7(x_0) = (0, 0, \ldots , 0)$. Let $x_1$ be any point opposite $p$ collinear with $x_0$ and such that the line $x_0x_1$ is not opposite $L_1$ (there are 2 choices for $x_1$). Without loss of generality we may assume that $7(x_1) = (0, 1, 1, \ldots , 1)$. Now we consider any point $x_2$ opposite $p$, collinear with $x_1$ and not on the line $x_0x_1$ (fixing $x_1$, there are 12 choices for $x_2$; hence in total we have 24 choices). Without loss of generality, we may assume that $x_1x_2$ is not opposite $L_7$. Then, since $7(x_1) - 7(x_2)$ contains either six 1’s or six 2’s (and the zero entry appears at the last position because $\text{proj}_{L_7} x_1 = \text{proj}_{L_7} x_2$) we have two possibilities.

1. $7(x_2) = (1, 2, 2, 2, 2, 1)$. In this case there is a unique point $x_3$ collinear with $x_2$, opposite $p$, such that $x_2x_3$ is not opposite $L_1$, and with $7(x_3) = (1, 1, 1, 1, 1, 1, 0)$. It
is now easily seen that a point \( x_4 \) opposite \( p \) and collinear with \( x_3 \) exists such that \( 7(x_4) = 7(x_0) \).

2. \( 7(x_2) = (2,0,0,0,0,1) \). In this case we can take for \( x_3 \) the unique point opposite \( p \), collinear with \( x_2 \), such that \( x_2 x_3 \) is not opposite \( L_1 \), and with \( 7(x_3) = (2,2,2,2,2,0) \). Also in this case, there is now a point \( x_4 \) collinear with \( x_3 \) opposite \( p \) with \( 7(x_4) = 7(x_0) \).

Hence, each of the 24 choices for \( x_2 \) gives rise to a point \( x_4 \) at distance 7 from \( x_0 x_1 \) with \( 7(x_4) = 7(x_0) \). If two such points coincide, then there unique paths to \( x_0 x_1 \) must coincide, a contradiction (they are all different by construction). Hence, we have a set of 25 points (all points \( x_4 \) and in addition the point \( x_0 \)) giving rise to the same prechosen 7-tuple. Hence, the average of points \( x \) with \( 7(x) \) prechosen must be at least 25, a contradiction to our previous paragraph.

Hence \( \Gamma \) cannot exist.

5.2. The case \((n,s,t) = (8,6,3)\)

Let \( \Gamma \) be a generalized octagon of order \((6,3)\) with \( \Pi^+(\Gamma) \) sharply 2-transitive. Let \( p \) be any point of \( \Gamma \), and let \( x_0 \) be a point of \( \Gamma \) opposite \( p \). As in the previous case, we can associate a 4-tuple \((0,0,0,0)\) to \( x_0 \) by taking an order \((L_1, L_2, L_3, L_4)\) of the lines through \( p \), and by labeling the point \( \text{proj}_{L_i} x_0 \) as \((L_i, 0 \text{ mod } 6), 1 \leq i \leq 4 \) (and we will omit ‘mod 6’ again in the sequel).

We now choose a point on \( L_1 \) distinct from \( p \) and from \((L_1,0)\) and label it \((L_1,1)\). There is a unique element \( \theta \) of \( \Pi^+(\Gamma) \) mapping \( L_1 \) to itself, fixing \( p \) and mapping \((L_1,0)\) to \((L_1,1)\). We define \((L_1,j)^\theta = (L_1,j+1) \) inductively, for all \( j \) (modulo 6). As before, this induces a unique labeling on the lines \( L_i, i = 2,3,4 \), and we can associate a 4-tuple \( 4(x) \) with every point \( x \) opposite \( p \), in exactly the same way as before. One also shows similarly that the sum of the labels is congruent 3 modulo 6, and that for collinear points \( x \) and \( y \), the 4-tuples \( 4(x) \) and \( 4(y) \) have the same entry at a certain position, and the entries in the other positions have a constant difference.

It is now a little elementary exercise to show that, if \((a,b,c,d)\) is an admissible 4-tuple (as before, this means that there exists a point \( x \) opposite \( p \) with \( 4(x) = (a,b,c,d) \)), then

\[
(c - a, d - b) \in \{(0,0),(2,4),(4,2),(3,3),(1,5),(5,1),(0,3),(2,1),(4,5),
(3,0),(1,2),(5,4)\} =: \mathcal{A}.
\]

For \((i,j) \in \mathcal{A} \), we put \( \mathcal{I}(i,j) = \{x \in \Gamma_5(p) | 4(x) = (a,b,a + i,b + j)\} \), for some \( a,b \). Suppose now two points \( x \) and \( y \) are collinear in \( \Gamma^{(L_3,L_4)} \). Then \( xy \) is opposite both \( L_3 \) and \( L_4 \), hence we may assume it is not opposite \( L_1 \). So, \( 4(x) = 4(y) + (0,e,e,e) \), and we see that \( x \) and \( y \) belong to the same set \( \mathcal{I}(i,j) \) for some suitable \((i,j)\). This means that each \( \mathcal{I}(i,j) \) is the union of connected components of \( \Gamma^{(L_1,L_4)} \), and hence there are at least 12 connected components. Now we set \( \mathcal{I}_1 = \mathcal{I}(0,0) \cup \mathcal{I}(2,4) \cup \mathcal{I}(4,2), \mathcal{I}_2 = \mathcal{I}(3,3) \cup \mathcal{I}(1,5) \cup \mathcal{I}(5,1), \mathcal{I}_3 = \mathcal{I}(0,3) \cup \mathcal{I}(2,1) \cup \mathcal{I}(4,5) \) and \( \mathcal{I}_4 = \mathcal{I}(3,0) \cup \mathcal{I}(5,4) \cup \mathcal{I}(1,2) \cup \mathcal{I}(4,5) \).
$\mathcal{I}(1, 2) \cup \mathcal{I}(5, 4)$. It is easy to check that an arbitrary member of \( \mathcal{I}_1 \) (respectively \( \mathcal{I}_2 \), \( \mathcal{I}_3 \), \( \mathcal{I}_4 \)) is collinear (in \( \Gamma^{(p)} \)) with exactly 14 members of \( \mathcal{I}_1 \) (respectively \( \mathcal{I}_2 \), \( \mathcal{I}_3 \), \( \mathcal{I}_4 \)), with no members of \( \mathcal{I}_2 \) (respectively \( \mathcal{I}_1 \), \( \mathcal{I}_4 \), \( \mathcal{I}_3 \)), with exactly three members of both \( \mathcal{I}_3 \) and \( \mathcal{I}_4 \) (respectively \( \mathcal{I}_3 \) and \( \mathcal{I}_4 \), \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \), \( \mathcal{I}_1 \), \( \mathcal{I}_2 \)). Indeed, let us check this for instance for a point \( x \) with \( 4(x) = (0, 0, 0, 0) \in \mathcal{I}_1 \). The neighbors of \( x \) have corresponding 4-tuple (and \( \cdot \rightarrow \cdot \) means ‘gives rise to members of’)

\[
(0, \ell, \ell, \ell), (\ell, 0, \ell, \ell) \rightarrow \mathcal{I}(0, 0) \subseteq \mathcal{I}_1, \quad \ell \in \{1, 2, 3, 4, 5\},
\]

\[
(2, 2, 0, 2), (4, 4, 4, 0) \rightarrow \mathcal{I}(2, 4) \subseteq \mathcal{I}_1,
\]

\[
(2, 2, 2, 0), (4, 4, 0, 4) \rightarrow \mathcal{I}(4, 2) \subseteq \mathcal{I}_1,
\]

\[
(1, 1, 0, 1), (3, 3, 0, 3), (5, 5, 0, 5) \rightarrow \mathcal{I}_3,
\]

\[
(1, 1, 1, 0), (3, 3, 3, 0), (5, 5, 5, 0) \rightarrow \mathcal{I}_4.
\]

The condensed form of the adjacency matrix with corresponding partition is thus

\[
\begin{pmatrix}
14 & 0 & 3 & 3 \\
0 & 14 & 3 & 3 \\
3 & 3 & 14 & 0 \\
3 & 3 & 0 & 14
\end{pmatrix}
\]

and this has eigenvalues 20 (multiplicity 1), 14 (multiplicity 2) and 8 (multiplicity 1).

As before, by interlacing, we must have \( 14 \leq s - 1 + \sqrt{2st} = 11 \), a contradiction.

### 5.3. The case \((n, s, t) = (6, 3, 3)\)

Let \( \Gamma \) be a generalized hexagon of order \((3, 3)\) such that \( \Pi^+(\Gamma) \) is sharply 2-transitive. Let \( p \) be a point of \( \Gamma \). Exactly in the same way as in the two previous subsections, we can associate a 4-tuple \( 4(x) \) with every point \( x \) opposite \( p \), and such a 4-tuple \((i_1, i_2, i_3, i_4)\) consists of 4 integers \( i \) modulo 3 which sum up to 0 modulo 3. Adjacent to \( x \) in \( \Gamma^{(p)} \) are 8 points with corresponding 4-tuples \((i_1, i_2 + e, i_3 + e, i_4 + e)\), \((i_1 + e, i_2, i_3 + e, i_4 + e)\), \((i_1 + e, i_2 + e, i_3, i_4 + e)\), \((\ldots, i_1 + e, i_4)\). We observe that no two of these 8 quadruples share in exactly one position an element. Hence, since \( \Gamma^{(p)} \) is connected (see [2]), we have 27 admissible quadruples, and if we consider the graph \( G \) with vertex set the admissible quadruples, and we call two quadruples adjacent if they share in exactly one position an element, then we obtain a (strongly regular) graph without triangles. It can also be easily seen that there are no two quadruples differing in exactly one position.

Since there are 27 admissible quadruples, and \( 3^3 \) points opposite \( p \), there must be at least one admissible quadruple equal to \( 4(x) \), for at least 9 points \( x \) opposite \( p \). Now suppose, without loss of generality, that \((0, 0, 0, 0)\) is such a quadruple, and let \( 4(x) = 4(y) = (0, 0, 0, 0) \) for two distinct points \( x \) and \( y \). We now determine the mutual position of \( x \) and \( y \) by ruling out some possibilities.

Suppose that \(|I_1(p) \cap I_3(x) \cap I_3(y)| = 0\). Let \( M \) be any line through \( y \) and put \( N = \text{proj}_y M \). The point \( \text{proj}_N y \) is opposite \( p \) since otherwise it would coincide with \( \text{proj}_N p \), and the latter is opposite \( p \) (because, if \( U = \text{proj}_p N \) and \( u = \text{proj}_U N \), we
have by assumption that proj$_y$y $\neq$ proj$_x$x. Similarly, the point proj$_M$ x is opposite $p$. By Remark 2, the sets $I_2(p) \cap I_2(x)$ and $I_2(p) \cap I_2(\text{proj}_M x)$ have exactly two elements in common. But since $y$ and proj$_M$ x are collinear, the sets $I_2(p) \cap I_2(y)$ and $I_2(p) \cap I_2(\text{proj}_M x)$ have exactly one element in common, a contradiction (because $I_2(p) \cap I_2(x) = I_2(p) \cap I_2(y)$ by assumption).

Suppose now $I_3(p) \cap I_3(x) \cap I_3(y) = \{L\}$. Suppose, moreover, that proj$_L$ x $\neq$ proj$_L$ y. Then $\delta(x, y) = 6$ and considering a line $M \neq \text{proj}_y L$ through $y$, we can copy the argument in the previous paragraph to reach a contradiction.

Similarly, we can rule out the case $I_3(p) \cap I_3(x) \cap I_3(y) = \{L, L'\}$, $L \neq L'$ (proj$_L$ x $\neq$ proj$_L$ y is automatic since $x \neq y$). Note that an analogous argument shows that $|I_3(p) \cap I_3(x) \cap I_3(y)| \neq 3$.

Suppose now $|I_3(p) \cap I_3(x) \cap I_3(y)| = 4$. There is at most one further point $z$ opposite $p$ with $|I_3(p) \cap I_3(x) \cap I_3(z)| = 4$. Since there are at least nine points $u$ with $4(u) = 4(x)$, there is at least one point $w$ opposite $p$ with $4(w) = 4(x)$ and $I_3(p) \cap I_3(x) \cap I_3(w) = \{L\}$, for some line $L$, and proj$_L$ x $=$ proj$_L$ w. But then $I_3(p) \cap I_3(y) \cap I_3(w) = \{L\}$ with proj$_L$ y $\neq$ proj$_L$ w. So $4(w) \neq 4(y)$, a contradiction.

Hence we have shown that $I_3(p) \cap I_3(x) \cap I_3(y)$ consists of a unique line $L$ with proj$_x$x$=$proj$_L$ y. It is clear that each such line $L$ gives rise to at most two points $y$, $y \neq x$, with $4(y) = 4(x)$, because on each line $K$ through proj$_x$x, $K \neq L$, $K$ not through $x$, the point $y$ must be equal to the projection of every element of $(I_3(p) \cap I_3(x)) \setminus \{\text{proj}_L p\}$. Since there are four lines in $I_3(p) \cap I_3(x)$, there are at most nine elements $y$ with $4(y) = 4(x)$. Our assumption now implies that there are exactly nine such elements. We can do the same with a second admissible quadruple, and continuing this way, we finally have that every admissible quadruple arises from exactly nine points opposite $p$. We can show that such a set of nine points is contained in a subhexagon of order $(1, 3)$, but we will not need this fact.

Now put $I_3(p) \cap I_3(x) = \{L_0, L_1, L_2, L_3\}$. Let $u$ be a point on $L_0$ distinct from proj$_L_0$ x. Let $(u, uv_1, w_1, w_1u_1, L_i)$, $i = 1, 2$ be path from $u$ to $L_i$. Then $w_1 \neq w_2$ (otherwise $4(w_1)$ and $4(x)$ differ in at most one position, a contradiction). Since $II'(G) = II^+(G)$, the projectivity $[L_2, L_0, L_1, L_2]$ is the identity. Hence $\delta(u_1, u_2) = 4$, and there is a path $(u_1, u_1u_12, u_12, u_12u_2, u_2)$ from $U_1$ to $U_2$. By an argument in the previous paragraph, we know that on the line $uw_2$, there is a unique point $v$ with $4(v) = 4(u_1)$. Hence $4(w_1)$ and $4(w_2)$ differ in exactly three positions (because if they were equal, then they would have to be equal to $4(x)$, a contradiction). Similarly, $4(w_1)$ (respectively $4(w_2)$) and $4(w_12)$ differ in exactly three positions. But this induces a triangle in the graph $G$ (see above), a contradiction.

This completes the proof of our Main Result. □

References


