On irreducible (B,N)-pairs of rank 2

Katrin Tent* Hendrik Van Maldeghem†

Abstract

Let G be a group with an irreducible spherical (B,N)-pair of rank 2 where B has a normal subgroup U with $B = UT$ for $T = B \cap N$. Let $\mathfrak{P}$ be the generalized n-gon associated to this (B,N)-pair and let W be the associated Weyl group. So T stabilizes an ordinary n-gon in $\mathfrak{P}$, and $|W| = 2n$. We prove that, if either U is nilpotent or G acts effectively on $\mathfrak{P}$ and $Z(U) \neq 1$, then $|W| = 2n$ with $n = 3, 4, 6, 8$ or 12. If G acts effectively and $n \neq 4, 6$, then (up to duality) $Z(U)$ consists of central elations. Also, if $n = 3$ and U is nilpotent, then $\mathfrak{P}$ is a Moufang projective plane and if, moreover, G acts effectively on $\mathfrak{P}$, then it contains its little projective group. Finally, we show that, if G acts effectively on $\mathfrak{P}$, if $Z(U) \neq 1$, and if T satisfies a certain strong transitivity assumption, then $\mathfrak{P}$ is a Moufang n-gon with $n = 3, 4$ or 6 and G contains its little projective group.

1 Introduction

For the purpose of this paper, a *thick generalized polygon* $\mathfrak{P}$ (or *thick generalized n-gon*, $n \geq 3$), or briefly a *polygon* (or *n-gon*), is a bipartite graph (the two corresponding classes are called *types*) of diameter n and girth $2n$ (the *girth* of a graph is the length of a minimal circuit) containing a proper circuit of length $2n + 2$ (the latter is equivalent with saying that all vertices have valency > 2, see [15]). If the last condition is not (necessarily) satisfied, then the polygon is called *weak*. The vertices are called the *elements* of $\mathfrak{P}$. A pair of elements $\{x, y\}$ is called a *flag* if x and y are adjacent. The set of neighbors of an element x is denoted by $D_1(x)$, and, more generally, the set of elements at distance i from x, $0 \leq i \leq n$, is denoted by $D_i(x)$. The diameter of the edge graph of $\mathfrak{P}$ is also equal to n and two flags at distance n from each other are called *opposite*. Also two elements of $\mathfrak{P}$ at distance n from each other are called *opposite*. A circuit of length $2n$ in $\mathfrak{P}$ is called an *apartment*. Two opposite flags are contained in exactly one apartment. These, and many more properties, can be found in [15]. A sequence $(x_0, x_1, \ldots, x_k)$ of elements of $\mathfrak{P}$ is called a *simple path of length k*, or a (simple) *k-path*, if x_{i-1} is incident with x_i, for all $i \in \{1, 2, \ldots, k\}$, and if $x_{i-1} \neq x_{i+1}$, for all $i \in \{1, 2, \ldots, k-1\}$.

Generalized polygons were introduced by Tits [11]. The standard examples arise from irreducible spherical (B,N)-pairs of rank 2. For this paper, we will content ourselves with a geometric definition of these.

*The first author is supported by the Bayerischer Habilitationsförderpreis

†The second author is a Research Director of the Fund for Scientific Research – Flanders (Belgium)

Therefore, let $\mathfrak{P}$ be an n-gon, and let G be a group acting (not necessarily effectively) on $\mathfrak{P}$ such that each element of G acts as a type preserving graph automorphism. If G acts transitively on the set of apartments of $\mathfrak{P}$, and if the stabilizer in G of an apartment A acts as the dihedral group of order $2n$ on A, then we say that G is a group with an *irreducible spherical* (B,N)-*pair of rank* 2, or briefly, with a (B,N)-*pair*. If we fix an apartment A and a flag f contained in A, then we call the stabilizer B in G of f a Borel subgroup of G. Also, there exists a subgroup N of B stabilizing A such that $B \cap N$ is normal in N and the corresponding quotient W has order $2n$ and is isomorphic to a dihedral group. The group W is called the *Weyl group* of G. The group N is not unique; in particular one can take the full stabilizer of A in G. If $\mathfrak{P}$ is a weak polygon, then we call G a *weak* (B,N)-pair. Groups with a (B,N)-pair were introduced by Tits; see e.g. [13].

Let $\mathfrak{P}$ be an n-gon. An *elation* g of $\mathfrak{P}$ is an automorphism of $\mathfrak{P}$ fixing $D_1(x_i)$, $1 \leq i \leq n-1$, for some simple path $(x_1, x_2, \ldots, x_{n-1})$ of $\mathfrak{P}$. The group of elations fixing $D_1(x_i)$, $1 \leq i \leq n-1$, for the simple path $(x_1, x_2, \ldots, x_{n-1})$ acts freely on $D_1(x_0) \setminus \{x_1\}$, for every element $x_0 \in D_1(x_1) \setminus \{x_2\}$. If this action is transitive for all such x_0, then we say that the path $(x_1, x_2, \ldots, x_{n-1})$ is a *Moufang path*. If all simple paths of length $n-2$ are Moufang, then we say that $\mathfrak{P}$ is a *Moufang polygon*. If n is even, and if all simple paths of length $n-2$ starting with an element of fixed type are Moufang, then we say that $\mathfrak{P}$ is half Moufang. All Moufang polygons are classified by Tits and Weiss [14]. An elation is called *central* if it fixes $D_i(x)$, for some element x, and for all positive $i \leq n/2$ (in which case x is called a *center* of the elation). The *little projective group* of a Moufang polygon is the group generated by all elations. It is a group with a natural (B,N)-pair and it always contains central elations. For the notions introduced in this paragraph, see [14] and [15].

Let G be a group with an irreducible spherical (B,N)-pair of rank 2, let $\mathfrak{P}$ be the corresponding polygon and let A be any apartment of $\mathfrak{P}$. If for any element x of A, the pointwise stabilizer in G of A acts transitively on the set of elements of $D_1(x)$ which are not contained in A, then we call G *highly transitive*. It is equivalent to require this for two adjacent elements x of A.

If an n-gon $\mathfrak{P}$ admits a type preserving automorphism group G acting transitively on the set of proper circuits of length $2n+2$, and such that the stabilizer of such a circuit acts as the dihedral group of order $2n+2$ on that circuit, then G is a group with a (B,N)-pair (and corresponding n-gon $\mathfrak{P}$), and we call this (B,N)-pair *strong*. A group G with a strong (B,N)-pair is automatically highly transitive.

Granted the classification of finite simple groups, all finite groups with an irreducible spherical (B,N)-pair of rank 2 can be classified, see [1]. The finiteness condition can not be dispensed with as is shown by the 'free' and 'universal' examples of Tits [12] and Tent [8]. Hence, one must have additional hypotheses in order to classify. Therefore, let us have a look at some results in the finite case the proofs of which do not use the classification of finite simple groups.

(i) A fundamental result of Feit and Higman [2] states that the Weyl group W of a weak finite (B,N)-pair must have order $|W| = 2n$ for $n = 2, 3, 4, 6, 8$ or 12. In fact, this is a consequence of their theorem that thick finite generalized n-gons exist only for $n = 3, 4, 6$ and 8. This result does not hold in the infinite case: for any n, there are infinite groups with a (B,N)-pair whose Weyl group has order $2n$ (see [8, 12]).

(ii) Consider the following condition for a group G with a (B,N)-pair:

   (*) there exists a normal nilpotent subgroup U of B such that $B = UT$, for $T = B \cap N$.

   Fong and Seitz [3] classified all finite irreducible spherical (B,N)-pairs of rank 2 satisfying (*). They showed that such groups are all of Lie type equipped with a natural (B,N)-pair structure, and hence the corresponding polygon is known.

(iii) The finite n-gons with a strong (B,N)-pair, and the corresponding groups (acting faithfully on the n-gon) are classified in [6, 10, 16]. However, in the infinite case, strong (B,N)-pairs exist for each $n \geq 3$, see [8] (and so, in particular, there are infinite generalized n-gons with a highly transitive group, for all n), and the construction shows that a classification is out of reach.

So in the infinite case, possibly except for the second result above, one needs additional hypotheses. In this paper, we will show in a purely geometrical way the following results, which are respective infinite analogs of the finite theorems mentioned above. Before stating these results, we introduce the following condition for a group G with a (B,N)-pair:

(**) there is a normal subgroup U of B such that $B = UT$, with $T = B \cap N$, and $Z(UR/R) \neq 1$, where R is the kernel of the action of G on the corresponding polygon $\mathfrak{P}$.

Theorem 1. *The Weyl group of the group G with an irreducible spherical (B,N)-pair of rank 2 satisfying (**) must have order $2n$ with $n = 3, 4, 6, 8$ or 12. If, moreover, $n \in \{3, 8, 12\}$, then the center of UR/R (with R defined as in (**)) consists of central elations. In particular, if G is a group with an irreducible spherical (B,N)-pair of rank 2 satisfying (*) and corresponding n-gon $\mathfrak{P}$, then $n \in \{3, 4, 6, 8, 12\}$.*

Theorem 2. *If G is a group with a (B,N)-pair satisfying (*) and with Weyl group W of order 6, then the associated projective plane $\mathfrak{P}$ is a Moufang plane and G/R contains its little projective group, where R denotes the kernel of the action of G on $\mathfrak{P}$.*

Theorem 3. *If G is a highly transitive group with an irreducible spherical (B,N)-pair of rank 2 satisfying (**), then the associated polygon $\mathfrak{P}$ is a Moufang polygon and G/R contains the little projective group of $\mathfrak{P}$, where R is the kernel of the action of G on $\mathfrak{P}$.*

2 A general lemma

2.1 Standing Hypotheses. Throughout, let G be a group with an irreducible spherical (B,N)-pair of rank 2 and let $\mathfrak{P}$ be the associated n-gon. Let A be some apartment in $\mathfrak{P}$ and let $\{p, q\}$ be a flag in A. Let B be the stabilizer of $\{p, q\}$, and let $N \leq G$ be such that it stabilizes A and such that $T := B \cap N \trianglelefteq B$ with $W := B/T$ isomorphic to the dihedral group of order $2n$. Finally, let R be the kernel of the action of G on $\mathfrak{P}$. Then

G/R is a group with a (B,N)-pair and with corresponding polygon $\mathfrak{P}$. The stabilizer of $\{p,q\}$ in G/R is B/R. The group N/R stabilizes A and $T/R = B/R \cap N/R \trianglelefteq B/R$, with $W \equiv (B/R)/(T/R)$. If G satisfies (*) or (**), respectively, then so does G/R. Hence, in order to show Theorems 1, 2 and 3, we may assume that R is trivial and hence that G acts effectively (faithfully) on $\mathfrak{P}$.

Since in this case, (*) implies (**), we assume throughout that U is a normal subgroup of B satisfying $B = UT$ with $Z(U) \neq 1$.

We observe that U acts transitively on the set of flags opposite $\{p,q\}$. Also, we will use the following well known observation frequently:

2.2 Lemma *Let G be a group acting on a set X, and let g and h be commuting elements of G. If g fixes some $x \in X$, then it also fixes $h(x)$.* □

Now, it is an immediate consequence of Lemma 2.2 and the transitivity of U on flags opposite $\{p,q\}$ that, if an element in $Z(U)$ fixes an element in $D_i(p)$ for $i < n$, then it fixes all elements in $D_i(p)$. This implies in particular that, if $Z(U)$ fixes a path $(x_0, \ldots x_k)$, then $Z(U)$ fixes all elements in $D_1(x_1) \cup \ldots \cup D_1(x_{k-1})$ and acts semi-regularly (freely) on $D_1(x_0)$ and $D_1(x_k)$.

The following result uses a small modification of Lemma 5 of [17].

2.3 Lemma *The group $Z(U)$ fixes the set $D_k(p) \cup D_k(q)$ elementwise, for all $k < n/2$. In particular, if n is odd, then for any flag $\{x,y\}$ of $\mathfrak{P}$, there exists a non-trivial central elation with two centers x and y.*

Proof. Suppose not. Without loss of generality, let $v \in Z(U)$ be an element of the center not fixing all of $D_k(q)$ with $k < n/2$ minimal, and hence not fixing any element in $D_k(q)$. Choose a simple path $\gamma = (p, q, x_2, \ldots, x_n)$ of length n, put $q = x_1$ and let U_γ denote the subgroup of U fixing γ. Then U_γ acts transitively on $D_1(p) \setminus \{q\}$ and on $D_1(x_n) \setminus \{x_{n-1}\}$.

Now let $u \in U_\gamma$. Since u and v commute, we conclude by Lemma 2.2 that u also fixes $v(\gamma)$. But since v does not fix x_{k+1}, the sequence $(x_n, \ldots x_{k+1}, x_k, v(x_{k+1}), \ldots v(x_n))$ is a path. It is fixed by u and has length $2n - 2k > n$. Now, the flag $\{v(x_{2k}), v(x_{2k+1})\}$ is opposite the flag $\{x_n, x_{x-1}\}$, and hence u fixes the unique apartment determined by these two flags; this implies that u fixes the unique element $y \in D_1 x_n \setminus \{x_{n-1}\}$ of that apartment. Thus, any element of U which fixes γ fixes y. But U acts transitively on $D_1(x_n) \setminus \{x_{n-1}\}$, a contradiction.

So, $Z(U)$ fixes $D_k(p) \cup D_k(q)$ for all $k < n/2$. For odd n, this implies immediately that $Z(U)$ consists of elations having two centers p and q. □

3 Proof of Theorem 1

In this section, we prove Theorem 1. So under the assumptions of our standing hypotheses, we have to show that $n \in \{3,4,6,8,12\}$, and if $n \neq 4, 6$, then $Z(U)$ consists of central elations. The proof is almost identical to parts of [17], except that we make some additional explicit observations (and that the general assumptions are different).

So, as in [17], the idea of the proof to rule out the values $n \notin \{3, 4, 6, 8, 12\}$ is roughly speaking as follows. We consider the commutator of two central elements with respect to flags at a certain distance and find that it (1) fixes too much to be non-trivial, but (2) does not fix everything, yielding a contradiction.

Case 1: n is odd.

First assume that n is odd. Let (x, y) and (x', y') be flags where, say, $d(x, x') = \frac{n+3}{2}$ and $d(y, y') = \frac{n-1}{2}$. By the Lemma 2.3 we know that there exist elations α with centers x, y and β with centers x', y'. Since α fixes y', and β fixes y, it is easy to see that the commutator $\theta := [\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ fixes all elements at distance $\leq \frac{n-1}{2}$ from y and all elements at distance $\leq \frac{n-1}{2}$ from y'. Hence θ fixes $D_1(z)$ pointwise for all z belonging to any simple path $(z_1, z_2, \dots, z_{\frac{3n-5}{2}})$, with $y = z_{\frac{n-1}{2}}$ and $y' = z_{n-1}$. Hence θ is the identity whenever the length $\frac{3n-7}{2}$ of that path exceeds $n-2$. But now consider $z \in D_{\frac{n-1}{2}}(x') \cap D_{\frac{n+1}{2}}(y')$, and suppose that θ fixes z. Then $\alpha^{-1}(z) = \beta^{-1}\alpha^{-1}(z)$ and so β fixes $\alpha^{-1}(z)$. Since α does not fix x', $\alpha^{-1}(z)$ belongs to $D_{\frac{n+1}{2}}(y') \cap D_{\frac{n+3}{2}}(x')$. Hence β would be the identity, a contradiction. So θ is not the identity, implying $\frac{3n-7}{2} \leq n-2$. This reduces to $n \leq 3$.

Case 2: $n = 2m$ and $Z(U)$ contains an automorphism which is not a central elation.

In this case, for any flag $\{x, y\}$, there exists a non-trivial automorphism $\alpha_{x,y}$ fixing $D_k(x) \cup D_k(y)$, for $0 \leq k \leq \frac{n}{2} - 1$, and acting freely on the sets $D_{n/2}(x) \cap D_{n/2+1}(y)$ and $D_{n/2}(y) \cap D_{n/2+1}(x)$ (by Lemma 2.2 and Lemma 2.3). Let (x, y) and (x', y') be flags with $d(x, x') = n/2 + 1$ and $d(y, y') = n/2 - 1$. Choose $\alpha_{x,y} =: \alpha$ and $\alpha_{x',y'} =: \beta$. Since α fixes y', and β fixes y, we see as before that the commutator $\theta := [\alpha, \beta]$ fixes all elements at distance $\leq n/2 - 1$ from y and all elements at distance $\leq n/2 - 1$ from y'. Hence θ fixes $D_1(z)$ pointwise for all z belonging to any simple path $(z_1, z_2, \dots, z_{3n/2-4})$, with $y = z_{n/2-1}$ and $y' = z_{n-2}$. Hence θ is the identity whenever the length $3n/2 - 5$ of that path exceeds $n-2$. But now consider $z \in D_{n/2-1}(x') \cap D_{n/2}(y')$. As in Case 1 one easily shows that θ does not fix z. So θ is not the identity, implying $3n/2 - 5 \leq n - 2$. This reduces to $n \leq 6$.

Remark that, if $n = 6$, then the length of the path $(z_1, \dots, z_5)$ is equal to $n - 2 = 4$, hence θ is a non-trivial elation fixing $D_2(z_2)$ and $D_2(z_4)$ pointwise. By choosing the flags $\{x, y\}$ and $\{x', y'\}$ appropriately, we thus obtain in this case such non-trivial elations for all simple paths of length 4.

Case 3a: $n = 2m$ with m odd where $Z(U)$ consists of central elations.

By the transitivity of G on elements of a given type, every element of one type of $\mathfrak{P}$ is center of a non-trivial elation. Let p and p' be such elements at distance $m + 1$ from each other and choose non-trivial elations α and β with center p and p', respectively. Then, as before, one easily shows that the commutator $\theta = [\alpha, \beta]$ is non-trivial. Also, if $\{q\} = D_1(p) \cap D_m(p')$ and $\{q'\} = D_1(p') \cap D_m(p)$, then θ fixes $D_{m-1}(q) \cup D_{m-1}(q')$. As before, this implies that $3m - 5 \leq n - 2$, hence $n \leq 6$.

Case 3b: $n = 2m$ with m even where $Z(U)$ consists of central elations.

Here, we argue similarly as in Case 3a, except that we have to choose elements p and p' at distance $m+2$ from each other. So we obtain the condition $3m-8 \leq n-2$, implying $n \leq 12$, so $n \in \{4, 8, 12\}$.

Thus, either $Z(U)$ consists of elations and $n \in \{3, 4, 6, 8, 12\}$ or $Z(U)$ does not consist entirely of elations and $n = 4$ or 6. □

The remark in Case 2 of the proof of the previous theorem shows:

3.1 Proposition *If, under the standing hypotheses, $n = 6$, then either $Z(U)$ consists of central elations or U contains elations for any simple path $(x_1, \ldots x_5)$ fixing $D_2(x_2) \cup D_2(x_4)$ pointwise.* □

4 Proof of Theorem 2

In this section we prove Theorem 2, as a corollary of a more general proposition.

4.1 Proposition *Let G be a group with an irreducible spherical (B,N)-pair of rank 2 satisfying (*), i.e., in terms of our standing hypotheses, U is nilpotent. Then — up to duality, i.e., up to interchanging p and q — for all $x \in D_1(p) \setminus \{q\}$, and for all k, $0 < k < n/2$, the subgroup of U fixing the set $D_1(p) \cup D_1(q)$ pointwise acts transitively on all elements in $D_k(x) \cap D_{k+1}(p)$. Also, for all $y \in D_1(q) \setminus \{p\}$, and for all k, $0 < k < n/2$, the subgroup of U fixing the set $D_1(q)$ elementwise, acts transitively on all elements in $D_k(y) \cap D_{k+1}(q)$.*

Proof. Clearly, we may assume that k is maximal with respect to the property $k < n/2$. Let $\{1\} \trianglelefteq Z(U) = Z_1(U) \trianglelefteq Z_2(U) \trianglelefteq \cdots \trianglelefteq Z_{m-1}(U) \trianglelefteq Z_m(U) = U$ be the ascending central series of U and let $i > 0$ be minimal with the property that $Z_{i+1}(U)$ does not fix all of $D_1(p) \cup D_1(q)$. Note that such an i exists because $Z(U)$ fixes $D_1(p) \cup D_1(q)$ by Lemma 2.3. Without loss of generality, there is some $v \in Z_{i+1}(U)$ not fixing $D_1(p)$ pointwise. Since $Z_i(U)$ fixes all elements of $D_1(p)$ and U acts transitively on $D_1(p) \setminus \{q\}$, this implies that v does not fix any element of $D_1(p) \setminus \{q\}$. Let $x \in D_1(p) \setminus \{q\}$ be arbitrary, and let $\gamma = (x, x_1, \ldots x_k)$ and $\gamma' = (x, y_1, \ldots y_k)$ be two simple paths of length k with $x_1, y_1 \neq p$.

Then the simple path $(x_k, \ldots x_1, x, p, v(x), v(x_1), \ldots v(x_k))$ has length $2k+2 \in \{n, n+1\}$ and hence it is contained in some ordinary n-gon Γ. Similarly there is an ordinary n-gon Γ' containing the simple path $(y_k, \ldots y_1, x, p, v(x), v(x_1), \ldots v(x_k))$.

Let (p_1, q_1) be the unique flag in Γ opposite $(p, v(x))$ and let (p_2, q_2) be the unique flag of Γ' opposite $(p, v(x))$. Then there exists $u \in U$ mapping the flag (p_1, q_1) onto the flag (p_2, q_2). Clearly, u fixes x and since the commutator $[u, v]$ fixes all elements of $D_1(p)$, we conclude that u also fixes $v(x)$. By choice of Γ and Γ', then u also fixes the path $(v(x_1), \ldots, v(x_k))$ and maps the path γ to γ'.

Now consider the commutator $uv^{-1}u^{-1}v \in Z_i(U)$. It is easy to see that it maps γ to γ'; moreover it fixes $D_1(p) \cap D_1(q)$ by our assumption on i, proving the first part of the proposition.

The second part is proved in a completely similar way. □

Now Theorem 2 follows since for $n = 3$, Proposition 4.1 immediately implies that the flag (p, q) is a Moufang path. Hence all flags are Moufang paths and the projective plane is a Moufang plane.

5 Proof of Theorem 3

The following theorem generalizes Theorem 6.4.9 of [15].

5.1 Proposition *Suppose $\mathfrak{Q}$ is a half Moufang generalized n-gon with $n = 2m$ even, and such that all the corresponding elations are central elations. Then $\mathfrak{Q}$ is a generalized quadrangle or a Moufang generalized hexagon.*

Proof. Let $(x_1, \ldots, x_{n-1})$ be a Moufang path and suppose all corresponding elations are central elations with center x_m. Choose $x_0 \in D_1(x_1)\setminus\{x_2\}$ and $x_n \in D_1(x_{n-1})\setminus\{x_{n-2}\}$. Let $(x_n, x_{n+1}, \ldots, x_{2n-1}, x_0)$ be an arbitrary path of length n joining x_n with x_0 such that $x_1 \neq x_{2n-1}$. Let y be either an arbitrary element of $D_m(x_m) \cap D_m(x_{3m})$ or an arbitrary element of $D_{m+1}(x_{3m+1}) \cap D_{m-1}(x_{m+1})$, with $y \neq x_0$. Applying the group of central elations with center x_m, we easily see that $D_{m-1}(x_0) \cap D_{m+1}(x_n) = D_{m-1}(x_0) \cap D_{m+1}(y)$. It follows from Lemma 1 of [4] and the symmetry between x_0 and x_n that the pair (x_0, x_n) is *distance-$(m-1)$-regular*, see 6.4.1 of [15] for a precise definition. Now, Theorem 6.4.5(i) of [15] implies that $m - 1 \leq \frac{n+2}{4}$, hence $n \leq 6$. For $n = 6$, the result follows from [7]. □

In the case of a half Moufang quadrangle, we will use the following result.

5.2 Proposition *Suppose $\mathfrak{Q}$ is a half Moufang generalized quadrangle and suppose that all simple paths of length 2 starting with an element of type 1 are Moufang paths. Moreover, suppose that for every flag $\{p, q\}$, with p an element of type 1, the action on $D_1(q) \setminus \{p\}$ of the elation group corresponding with any simple path (p', r, p), $r \neq q$, is independent of (p', r). Then $\mathfrak{Q}$ is a Moufang quadrangle and all elations are generated by the elations corresponding with simple paths of length 2 starting with an element of type 1.*

Proof. See Proposition 3.6 of [9]. □

Throughout the rest of this section, we consider the standing hypotheses, and we assume that G is highly transitive. We now embark on the proof of Theorem 3.

By Theorem 1, we know that $n \in \{3, 4, 6, 8, 12\}$. If $n = 3, 8$ or 12, then $Z(U)$ consists of central elations, which by the transitivity assumption on T are transitive for one type of $(n-2)$-paths. Thus the cases $n = 8$ and $n = 12$ are excluded by Proposition 5.1. If $n = 3$, then, clearly, $\mathfrak{P}$ is a Moufang projective plane.

If $n = 6$, then either $Z(U)$ consists of central elations and we are done by Proposition 5.1, or, by Proposition 3.1, we have root elations of both types. By the transitivity assumption on T we then see that $\mathfrak{P}$ is Moufang.

Now consider the case $n = 4$. Assume first that $Z(U)$ contains central elations, with center q, say. Choose an element $r \in D_1(p)\setminus\{q\}$. Let U_0 be the group of all central elations with center q. This group acts transitively as a regular abelian group on $D_1(r) \setminus \{p\}$. Let q' be arbitrary in $D_1(p) \setminus \{q, r\}$. Let $g \in G$ be such that $g(q) = q'$ and put $U'_0 = gU_0g^{-1}$. Also U'_0 acts as a regular abelian group on $D_1(r) \setminus \{p\}$. Every element of the commutator $[U_0, U'_0]$ fixes $D_2(q) \cup D_2(q')$ pointwise, hence must be the identity. It follows that the actions on $D_1(r)$ of both U_0 and U'_0 are the same (see e.g. [5] 4.2.A(v)). Thus we can apply Lemma 5.2 to see that $\mathfrak{P}$ is in fact a Moufang quadrangle and G contains the little projective group.

Hence it remains to deal with the case $n = 4$ where $Z(U)$ does not contain central elations. We will exclude this situation by a series of lemmas. Note that we do not know whether or not $\mathfrak{P}$ admits central elations (necessarily not belonging to any conjugate of U).

5.3 Lemma *Let $\mathfrak{P}$ be as before, and let x be any element of $\mathfrak{P}$. Then the pointwise stabilizer of $D_1(x)$ in G acts freely on the set $D_4(x)$.*

Proof. In order to use our standard notation, we may without loss of generality suppose that x is the unique element in $D_1(p)$ different from q and contained in the apartment A. Suppose the lemma is false, then there exists some element $u \in G \setminus \{1\}$ fixing $D_1(x) \cup A$ pointwise. So $u \in B$ (recall that B is the pointwise stabilizer of the flag $\{p, q\}$), hence u normalizes $Z(U)$. Let p' be the element of A incident with q and distinct from p and let q' be incident with p', different from q and contained in A. If u fixes $D_1(p')$ pointwise, then u is the identity by 4.4.2(v) of [15]. Hence there exists $y \in D_1(p')$ with $u(y) \neq y$. Let $v \in Z(U)$ be such that $v(q') = y$, then the commutator $\theta := vuv^{-1}u^{-1}$ belongs to $Z(U)$ and fixes $D_1(x)$ pointwise. Hence it is a central elation with center p, and consequently it must, by assumption, be the identity. But it clearly does not fix y, a contradiction. $\square$

5.4 Lemma *Let $\mathfrak{P}$ be as before, and let $\gamma = (q'', p, q)$ be a simple path of length 2. If α is any elation for γ, then α is in fact a central elation.*

Proof. Since $Z(U)$ acts transitively on $D_1(q'') \setminus \{p\}$, every element of U fixing at least one element of $D_1(q'') \setminus \{p\}$ fixes $D_1(q'')$ pointwise. So, by assumption, we obtain a subgroup H^* of U acting transitively on $D_4(q'') \cap D_2(q)$ and fixing $D_1(q'')$ pointwise.

Now if α is a root elation for γ, then by Lemma 5.3 (putting $x = q''$), α is in $H^* \leq U$ and hence must commute with all $\beta \in Z(U)$. But this says that α is a central elation. $\square$

5.5 Lemma *Let $\mathfrak{P}$ be as before, and let $\gamma = (q'', p, q, p', q')$ be a simple path of length 4. If α is any elation for (p, q, p'), then $\alpha \in Z(U)$.*

Proof. By Lemma 5.4, α is a central elation. By similar arguments as in Lemma 5.4, the subgroup H' of U fixing $\{q'', q'\}$ fixes $D_1(p')$ pointwise and acts transitively on $D_1(q'') \setminus \{p\}$. Thus, by Lemma 5.3, $\alpha \in H' \leq U$. If $\alpha \notin Z(U)$, then there is some $u \in U$ such that the commutator $[\alpha, u]$ is non-trivial. Clearly, the action of H' on $D_1(q'')$ commutes with the action of $Z(U)$, and since $Z(U)$ is regular and abelian, these actions agree. Hence there is some $v \in Z(U)$ which induces the same action on $D_1(q'')$ as α does. Then v and α agree on $D_2(p)$ because otherwise αv^{-1} is an elation for (q'', p, q) which is not a central elation. But this is impossible. Thus $[\alpha, u]$ is the identity on $D_1(q'')$, but since $[\alpha, u]$ is clearly also a central elation, this is a contradiction. Consequently $\alpha \in Z(U)$. $\square$

5.6 Lemma *Let $\mathfrak{P}$ be as before, and let $\gamma = (p, q, p', q')$ be a simple path of length 3 contained in A. Suppose that the group H fixing $D_1(p) \cup D_1(q) \cup \{q'\}$ pointwise is non-trivial. Then the path (p, q, p') is a Moufang path.*

Proof. By the transitivity of T, the group H acts transitively on $D_1(q') \setminus \{p'\}$. Let (q'', p, q, p', q') be a simple path of length 4 contained in A, then, by symmetry, the group H' fixing $D_1(p') \cup D_1(q) \cup \{q''\}$ pointwise acts transitively on $D_1(q'') \setminus \{p\}$. Hence for every element $h \in H$, there exists $h' \in H'$ such that $h'h$ fixes A pointwise. Since it also fixes $D_1(q)$ pointwise, it must be identity by Lemma 5.3. Hence $h = h'^{-1}$ fixes $D_1(p) \cup D_1(q) \cup D_1(p')$ pointwise. Applying the transitivity of T, the result now follows. □

We can now finish the proof of Theorem 3.

We keep the same notation as above, so we have the simple path (q'', p, q, p', q') and a regular and abelian subgroup H' of U fixing $\{q'', q'\} \cup D_1(p')$ pointwise and acting transitively on $D_1(q'') \setminus \{p\}$. Similarly, we obtain a group $H \leq B$ fixing $D_1(q) \cup \{q', q''\}$ pointwise and acting transitively on $D_1(q'') \setminus \{p\}$.

Now consider the commutator group $[H, H'] \leq H \cap H'$ (by Lemma 5.2). If $[H, H']$ is non-trivial, then Lemma 5.6 implies that the path (p, q, p') is Moufang, and Lemma 5.5 yields a contradiction. If on the other hand $[H, H']$ is trivial, then the action of H on $D_1(q'')$ agrees with the action of H' on $D_1(q'')$. If $H \neq H'$, then there are elements $h \in H$ and $h' \in H'$ such that hh' is non-trivial and fixes $D_1(q'') \cup \{q'\}$, contradicting Lemma 5.3. Hence $H = H'$ and Lemma 5.6 implies that the path (p, q, p') is Moufang. Now Lemma 5.5 yields a contradiction. □

ACKNOWLEDGMENT. We would like to thank Richard Weiss for some very helpful comments on a preliminary version of this paper.

References

[1] F. Buekenhout and H. Van Maldeghem, Finite distance transitive generalized polygons, *Geom. Dedicata* **52** (1994), 41 – 51.

[2] W. Feit and G. Higman, The nonexistence of certain generalized polygons *J. Algebra* **1** (1964), 114 – 131.

[3] P. Fong and G. Seitz, Groups with a (B, N)-pair of rank 2. I, II. *Invent. Math.* **21** (1973), 1 – 57; ibid. **24** (1974), 191 – 239.

[4] E. Govaert, A combinatorial characterization of some finite classical generalized hexagons, *J. Combin. Theory Ser.* A **80** (1997), 339 – 346.

[5] J. D. Dixon, B. Mortimer *Permutation groups*, Springer, *Graduate Texts in Mathematics* **163**, 1996.

[6] T. G. Ostrom and A. Wagner, On projective and affine planes with transitive collineation groups, *Math. Z.* **71** (1959), 186 – 199.

[7] M. A. Ronan, A geometric characterization of Moufang hexagons, *Invent. Math.* **57** (1980), 227 – 262.

[8] K. Tent, Very homogeneous generalized n-gons of finite Morley Rank, to appear in *J. London Math. Soc.*

[9] K. Tent and H. Van Maldeghem, The Cherlin-Zil'ber Conjecture for BN-pairs with panels of Morley rank 1, submitted.

[10] J. A. Thas and H. Van Maldeghem, The classification of all finite generalized quadrangles admitting a group acting transitively on ordered pentagons, *J. London Math. Soc.* (2) **51** (1995), 209 - 218.

[11] J. Tits, Sur la trialité et certain groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13 – 60.

[12] J. Tits, Endliche Spiegelungsgruppen, die als Weylgruppen auftreten, *Invent. Math.* **43** (1977), 283 – 295.

[13] J. Tits, *Buildings of spherical type and finite BN-pairs*, Springer Verlag, *Lecture Notes in Math.* **386**, Berlin, Heidelberg, New York, 1974.

[14] J. Tits and R. Weiss, *The classification of Moufang Polygons*, in preparation.

[15] H. Van Maldeghem, *Generalized polygons*, Birkhäuser Verlag, *Monographs in Mathematics* **93**, Basel, Boston, Berlin, 1998.

[16] H. Van Maldeghem, A finite generalized hexagon admitting a group acting transitively on ordered heptagons is classical, *J. Combin. Theory Ser.* (A) **75** (1996), 254 – 269.

[17] R. Weiss, The nonexistence of certain Moufang polygons, *Invent. Math.* **51** (1979), 261 – 266.

Katrin Tent
Mathematisches Institut
Universität Würzburg,
Am Hubland,
D–97074 Würzburg,
Germany
E-mail: `tent@@mathematik.uni-wuerzburg.de`

Hendrik Van Maldeghem
Pure Mathematics and Computer Algebra
Universiteit Gent
Galglaan 2,
B–9000 Gent,
Belgium
E-mail: `hvm@@cage.rug.ac.be`