AUTOMORPHISMS AND OPPOSITION IN SPHERICAL BUILDINGS OF EXCEPTIONAL TYPE, III. METASYMPLECTIC SPACES

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Abstract. We classify all domestic collineations, that is, collineations mapping no chamber to an opposite one, of all spherical buildings of type F_4. Besides obvious cases like central elations and products of two perpendicular such elations, we find collineations that pointwise fix certain subspaces, also of type F_4, but over a smaller algebra, or even non-thick as a building. We also find examples that pointwise fix Moufang quadrangles, and these inclusions are new: Moufang quadrangles of absolute type D_5 are contained in buildings of type F_4 of absolute type E_6, and exceptional Moufang quadrangles of type E_6 are found inside buildings of relative type F_4 and absolute type E_7 (the so-called quaternion metasymplectic spaces). Together with the already established Moufang quadrangles of mixed type inside mixed buildings of type F_4, our results imply that domestic collineations give rise to inclusions of the three different types of Moufang quadrangles inside metasymplectic spaces: Moufang quadrangles of classical, exceptional and mixed type.

Contents

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1. Introduction

This paper fits into a series of papers classifying so-called *domestic* automorphisms of spherical buildings. Before we sketch the situation, let us recall some motivation. A domestic automorphism of a spherical building is an automorphism that does not map any chamber onto an opposite chamber—hence this is very specific to spherical buildings. As soon as there are no rank 2 residues defined over the smallest field $\mathbb{F}_2$, every domestic automorphism comes with an *opposition diagram*, which encodes the types of simplices that are mapped onto an opposite. These diagrams are classified in [16] and the result is—very roughly— that ignoring the arrows these diagrams coincide with the Tits indices [30] for which the Galois group is an involution (the exceptions occur in rank 2 and for type F_4; in the latter case, however, we can appeal to the *mixed Galois descent* introduced in [14]). Tits indices generalise to *fix diagrams*—encoding the types of simplices that are fixed by the automorphism. The initial crucial observation is that the fix diagram and opposition diagram of each domestic duality of any spherical buildings of the second half of the second row of the Freudenthal-Tits Magic Square coincide with the Tits indices of the corresponding cells in the relative Magic Square and those of the cells lying symmetric with respect to the main diagonal. This led to the conjecture that the nonsplit Magic Square encodes all domestic automorphisms of the buildings of exceptional type in the split Magic Square that do not fix a chamber, see [35]. This conjecture did not turn out to be be correct, but only a slight adaptation is necessary, see

Figure 1. The Tits indices ${}^2\mathsf{D}_{5,2}^{(2)}$ and ${}^2\mathsf{E}_{6,2}^{16'}$.

[15, 20]. In any case, domestic automorphisms seem to be related to automorphisms fixing a large substructure, in particular linearisations of Galois descent, called *linear descent* in [35]. This linear descent provides ways to see certain buildings inside others as a larger fix point structure than is the case in the corresponding Galois descent. For example, the quaternion buildings of type F_4 (with Tits index $\mathsf{E}_{7,4}^9$), which are (Galois) forms of E_7, arise as fixed point structures of groups of domestic automorphisms in buildings of type E_8.

The situation in buildings of type F_4 is particularly interesting. Not in the least because it is the unique type of exceptional buildings of rank at least 3 admitting non-split examples. But on the level of Tits indices and fix diagrams: On the one hand, there are not many Tits indices; on the other hand, there are fix diagrams that are not Tits indices, and they correspond to the mixed Galois descent introduced and explained in [14], giving rise to the exceptional Moufang quadrangles of type F_4. In the same paper [14], the linearization of this mixed Galois descent is presented, and the full fix group is determined in [23]. We will show that an automorphism of a mixed building of type F_4, fixing no chamber, is domestic if, and only if, it belongs to such a linear descent group. We also pin down the domestic collineations that do fix a chamber. The situation in non-mixed buildings of type F_4 is also very intriguing. Besides an explicit list of unipotent and torus elements, we obtain two new classes of linear descent groups. One is related to the Tits index ${}^2\mathsf{D}_{5,2}^{(2)}$ (see Fig. 1), which we disclose in the buildings of type F_4 having Tits index ${}^2\mathsf{E}_{6,4}^2$ and the other is related to the Tits index ${}^2\mathsf{E}_{6,2}^{16'}$ (an exceptional Moufang quadrangle of type E_6, see Fig. 1 again), which we find in buildings of type F_4 having Tits index $\mathsf{E}_{7,4}^9$. Both correspond to the opposition diagrams $\mathsf{F}_{4;2}$. Note that it was generally believed among experts that Moufang quadrangles arising like this in metasymplectic spaces were a characteristic 2 phenomenon, see Remark 2.2 of [26]. The new examples in the present paper refute this conjecture.

We mention in passing an interesting consequence of our construction: Since the exceptional Moufang quadrangles of type E_6 appear now in quaternionic metasymplectic spaces, their automorphism group can be written with quaternionic 27×27 matrices, see [8] and [36].

More exactly, with the notation and conventions of Section 2, we will show the classification of the Main Result below, where we use the following terminology. By [31, Theorem 10.2], thick buildings of type F_4 are classified by the pairs $(\mathbb{K}, \mathbb{A})$, where $\mathbb{K}$ is a field and $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$, and we denote the corresponding building by $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$. Recall from Theorem 20.3 in [33] that $\mathbb{A}$ is either

Class (K) equal to $\mathbb{K}$ and $\mathrm{char}\,(\mathbb{K}) \neq 2$,
Class (L) a separable quadratic extension of $\mathbb{K}$,
Class (H) a quaternion division algebra over $\mathbb{K}$,
Class (O) a Cayley algebra (octonionic division algebra) over $\mathbb{K}$, or
Class (M) a (purely inseparable but possibly trivial) extension of $\mathbb{K}$, $\mathrm{char}\,(\mathbb{L}) = 2$, with $\mathbb{A}^2 \subseteq \mathbb{K}$ (where $\mathbb{A}^2$ denotes the field of all squares of $\mathbb{A}$).

We number the vertices of the building as in Fig. 3. This numbering allows to consider the Lie incidence geometries $\Gamma_1 := \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ and $\Gamma_4 := \mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ (see Section 2 for more details), and

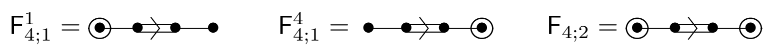


FIGURE 2. The possible opposition diagrams for nontrivial domestic collneations of $F_4(\mathbb{K},\mathbb{A})$.

also gives a precise meaning to the opposition diagrams $F^1_{4;1}$ and $F^4_{4;1}$ (see Fig. 2 for the list of opposition diagrams of type F_4 of nontrivial domestic collineations), and also to long and short root elations (a long root elation being a central elations with centre a vertex of type 1, and a short root elation being an elation where the corresponding root has a type 4 vertex as central vertex). To understand the Main Result, it suffices to know for now that in the opposition diagram (only) the types of the elements that are mapped onto opposites are encircled (see Section 2.11 for more details). Note that the Main Result for the split case (class (K)) has already been proved in [18].

Main Result. *Let, with the above notation, θ be a nontrivial automorphism of $F_4(\mathbb{K},\mathbb{A})$, $|\mathbb{K}|>2$. Then θ is domestic if, and only if, it has opposition diagram either $F^1_{4;1}$, or $F^4_{4;1}$, or $F_{4;2}$. More exactly,*

(Dom1) *θ has opposition diagram $F^1_{4;1}$ if, and only if, θ is a long root elation;*

(Dom4) *θ has opposition diagram $F^4_{4;1}$ if, and only if, one of the following occurs in the corresponding class:*
- (K) *θ is an involution with fix structure the weak subbuilding corresponding to an extended equator geometry and its tropics geometry in $F_{4,4}(\mathbb{K},\mathbb{K})$;*
- (L) *θ is an involution pointwise fixing a subbuilding canonically isomorphic to $F_4(\mathbb{K},\mathbb{K})$;*
- (M) *θ is a (central) short root elation;*

(Dom14) *θ has opposition diagram $F_{4;2}$ if, and only if, either*
- *(i) θ is the product of two perpendicular long root elations, or*
- *(i′) θ is the product of two perpendicular central short root elations in Class* (M)*, or*
- *(ii) θ pointwise fixes some apartment and one of the following occurs in the corresponding class:*
  - (L) *the fix structure of θ is the weak subbuilding corresponding to an extended equator geometry and its tropics geometry;*
  - (H) *the fix structure is a thick subbuilding of class* (L) *(isomorphic to $F_4(\mathbb{K},\mathbb{L})$ for some quadratic field extension of $\mathbb{K}$) canonically embedded in $F_4(\mathbb{K},\mathbb{A})$, and $\mathbb{L}$ is a subalgebra of $\mathbb{A}$ of dimension 2 fixed under some automorphism of $\mathbb{A}$, or*
- *(iii) the fix structure of θ consists of vertices of types 1 and 4 only, naturally defining a Moufang generalised quadrangle Γ in such a way that the fixed vertices of type i incident with a fixed vertex of type j, $\{i,j\}=\{1,4\}$, forms an ovoid in the corresponding symplecton of $F_{4,i}(\mathbb{K},\mathbb{A})$ and we have the following cases:*
  - (L) *Γ is a classical Moufang quadrangle with Tits index ${}^2D^{(2)}_{5,2}$;*
  - (H) *Γ is an exceptional Moufang quadrangle with Tits index ${}^2E^{16'}_{6,2}$;*
  - (M) *Γ is a mixed Moufang quadrangle and θ is an involution.*

In particular, the only domestic collineations of $F_4(\mathbb{K},\mathbb{O})$, with $\mathbb{O}$ a Cayley division algebra over $\mathbb{K}$, are the central elations and the products of two perpendicular central elations. Also, there do not exist domestic dualities of any building of type F_4.

We will also construct collineations in each of the cases displayed in the Main Result. Our tool to do so will be Tits' extension Theorem 4.16 of [31], together with the construction of some specific subgeometries in Γ_i, $i=1,4$, taking advantage of the duality between those two. In particular we will explicitly show that Γ_1 admits all central elations and Γ_4 does not admit any central elation except if it is in Class (M).

The case $|\mathbb{K}| = 2$ is a true exception but we are allowed to disregard it since all domestic collineations in this case are classified in [17].

Structure of the paper—In Section 2 we define the metasymplectic spaces that we will work with, define equator and extended equator geometries, the corresponding tropics geometry and derive from this fully embedded polar spaces of a certain type in Γ_1. This allows us to define *imaginary lines*, which play a prominent role in the proof of our main result. All results are new and interesting in their own right, although certain versions in the split case (and sometimes under the additional hypothesis of the underlying field being algebraically closed) of some of the results obtained exist in the literature (for instance in [25]). In Section 3 we prove some lemmas for polar spaces, which appear in the metasymplectic spaces as symplecta, point residuals, equator geometries and extended equator geometries. In Section 4 we prove the converse of our Main Result, namely, that all automorphisms in the conclusion of the Main Result are really domestic. Section 5 then contains the full proof of our Main Result. In Section 6 we construct all the examples, showing that all cases do exist. This section is completely independent of the others and could be read first.

2. Preliminaries

In this section we review the basic notation and terminology that we will use in this paper. Many proofs are geometrical, using the Lie incidence geometries Γ_1 and Γ_4 mentioned in the introduction. A crucial notion in this approach is that of an *equator geometry*, an *extended equator geometry* and the corresponding *tropics geometry*. These have been defined in $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$, see [10, 7] and in $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ for general $\mathbb{A}$ in [22]. However, the proofs in *loc. cit.* are rather sketchy and incomplete concerning the tropics geometry, so we provide full proofs here. We also define equator geometries in $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ with corresponding proofs (which is also missing in the literature).

Concerning buildings of type F_4, we refer to the literature, e.g. [31], for a formal definition. In this paper, we content ourselves with defining the Lie incidence geometries $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ and $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ in an axiomatic way, so that we are able to provide full and precise proofs, based on these axioms.

We also review all relevant notions on domesticity and opposition. The split case was already treated in [18], but we also include it here as it not only goes without any additional effort, but it would generate artificial arguments in trying to avoid this case. We start, however, with some necessary basics of incidence geometry.

2.1. **A crash course on point-line geometries.**

Definition 2.1.1. A *point-line geometry* is a pair $\Delta = (\mathscr{P},\mathscr{L})$ with $\mathscr{P}$ a set and $\mathscr{L}$ a set of subsets of $\mathscr{P}$. The elements of $\mathscr{P}$ are called *points*, the members of $\mathscr{L}$ are called *lines*. If $p \in \mathscr{P}$ and $L \in \mathscr{L}$ with $p \in L$, we say that the point p *lies on* the line L, and the line L *contains* the point p, or *goes through* p. If two (not necessarily distinct) points p and q are contained in a common line, they are called *collinear*, denoted $p \perp q$. If they are not contained in a common line, we say that they are *noncollinear*. For any point p and any subset $P \subseteq \mathscr{P}$, we denote

$$p^\perp := \{q \in \mathscr{P} \mid q \perp p\} \text{ and } P^\perp := \bigcap_{p \in P} p^\perp.$$

A *partial linear space* is a point-line geometry in which every line contains at least three points, and where there is a unique line through every pair of distinct collinear points p and q. That line is then denoted with pq.

Example 2.1.2. Let V be a vector space of dimension at least 3. Let $\mathscr{P}$ be the set of 1-spaces of V, and let $\mathscr{L}$ be the set of 2-spaces of V, each of them regarded as the set of 1-spaces it contains. Then $(\mathscr{P},\mathscr{L})$ is called a *projective space (of dimension* $\dim V - 1$*) and denoted by* $\mathsf{PG}(V)$*, or* $\mathsf{PG}(n,\mathbb{K})$ *if* V *is defined over the field* $\mathbb{K}$ *and had dimension* $n+1$.

Definition 2.1.3. Let $\Delta = (\mathscr{P},\mathscr{L})$ be a partial linear space.

- (i) A *path of length* n in Δ from point x to point y is a sequence $(p_0, p_1, \ldots, p_{n-1}, p_n)$, with $(p_0, p_n) = (x, y)$, of points of Δ such that $p_{i-1} \perp p_i$ for all $i \in \{1, \ldots, n-1\}$. If n is minimal, then it is called *the distance* between x and y in Δ.
- (ii) The partial linear space Δ is called *connected* when for any two points x and y, there is a path (of finite length) from x to y. If moreover the set of distances between points has a supremum in $\mathbb{N}$, this supremum is called the *diameter* of Δ.
- (iii) A subset S of $\mathscr{P}$ is called a *subspace* of Δ when every line $L \in \mathscr{L}$ that contains at least two points of S, is contained in S. A subspace that intersects every line in at least a point, is called a *(geometric) hyperplane*; it is *proper* if it does not coincide with $\mathscr{P}$. A subspace is called *convex* if it contains all points on every path of minimal length that connects any two points in S. We usually regard subspaces of Δ in the obvious way as subgeometries of Δ.
- (iv) A subspace S in which all points are collinear, or equivalently, for which $S \subseteq S^\perp$, is called a *singular* subspace. If S is moreover not contained in any other singular subspace, it is called a *maximal* singular subspace. If it is contained in at least one other singular subspace, but al such singular subspaces are maximal, then we call it *submaximal*. A singular subspace is called *projective* if, as a subgeometry, it is a projective space (cf. Example 2.1.2). Note that every singular subspace is trivially convex.
- (v) For a subset P of $\mathscr{P}$, the *subspace generated by* P is denoted $\langle P \rangle_\Delta$ and is defined to be the intersection of all subspaces containing P. The *convex hull of* P is defined to be the intersection of all convex subspaces that contain P. A subspace generated by three mutually collinear points, not on a common line, is called a *plane*. Note that, in general, this is not necessarily a singular subspace; however we will only deal with geometries satisfying Axiom (GS) (see below), which implies that subspaces generated by pairwise collinear points are singular; in particular planes will be singular subspaces.

Polar and parapolar spaces—We recall the definition of a polar space, mainly to fix notation and vocabulary. We take the viewpoint of Buekenhout–Shult [2]. All results in this section are well known. Since we are only interested in polar spaces of finite rank, we include this in our definition.

Definition 2.1.4. A *polar space* is a point-line geometry Γ in which for every point p the set $p^\perp$ is a proper hyperplane, and each maximal nested family of singular subspaces is finite and had size $r+1$ at least 3. The integer r is the *rank* of the polar space.

One shows that a polar space Γ is partial linear, and that each singular subspace is a projective space, see [2]. The maximal singular subspaces of a polar space of rank r have dimension $r-1$. Two singular subspaces are called Γ-*opposite* if no point of either of them is collinear to all points of the other. This coincides with the building theoretic notion of opposition, see chapter 3 of [31].

Example 2.1.5. Let $\mathbb{K}$ be a field, n an integer at least 2, V_0 a vector space over $\mathbb{K}$ and let $q : V \to \mathbb{K}$ be an anisotropic quadratic form, that is, a quadratic form without nontrivial isotropic vectors. Let V be a vector space of dimension $2n$. Then, with respect to any reference system, the set of points

p of $\mathsf{PG}(V \oplus V_0)$ with $p = \langle (v, v_0) \rangle$ having coordinates satisfying $X_{-1}X_1 + X_{-2}X_2 + \cdots + X_{-n}X_n = f(v_0)$, forms a nondegenerate quadric the points and lines of which form a polar space of rank n. The singular subspaces are precisely the projective subspaces of $\mathsf{PG}(V \oplus V_0)$ entirely contained in the quadric.

We also recall the definition of a parapolar space—for more details (and unproved claims in tis section) see Chapter 13 of the book of Shult [24].

Definition 2.1.6. A *parapolar space* Δ is a connected point-line geometry, which is not a polar space, and for which every pair $\{p, q\}$ of points with $|p^\perp \cap q^\perp| \geq 2$ is contained in a convex subspace isomorphic to a nondegenerate polar space. Any such convex subspace is called a *symplecton* of Δ (which is short for *symplecton*).

A pair of points p and q is called *special* if $|p^\perp \cap q^\perp| = 1$. A pair of noncollinear points p and q is called *symplectic* if $|p^\perp \cap q^\perp| \geq 2$. In this case, the convex hull of p and q is a nondegenerate polar space.

If all symplecta have the same rank r, then we say that Δ has *uniform (symplectic) rank* r. If this is the case, and if $r \geq 3$, then automatically all singular subspaces are projective spaces.

Example 2.1.7. If Γ is a polar space of rank at least 3, then the corresponding *dual polar space* is the point-line geometry with point set the set of singular subspaces of dimension $r-1$ and set of lines the sets of singular subspaces of dimension $r-1$ containing an arbitrary but fixed singular subspace of dimension $r-2$. If this geometry has thick lines, that is, each line contains at least three points, then it is a parapolar space of uniform rank 2.

Remark 2.1.8. The definition of parapolar space immediately implies that it is a partial linear space. Also, parapolar spaces are so-called *gamma spaces*, that is, they satisfy the following axiom, which is sometimes superfluously added in the definition.

(GS) Every point is collinear to zero, one or all points of any line.

Definition 2.1.9. Let Γ be a polar or parapolar space of (uniform) rank r and let U be a singular subspace of Γ of dimension at most $r-3$. We define $\mathrm{Res}_\Gamma(U)$ to be the point-line geometry $(\mathscr{P}, \mathscr{L})$ with

$$\mathscr{P} := \{\text{singular subspaces } K \text{ of } \Gamma \text{ with } U \subset K \text{ and } \mathrm{codim}_K(U) = 1\},$$
$$\mathscr{L} := \{\text{singular subspaces } L \text{ of } \Gamma \text{ with } U \subset L \text{ and } \mathrm{codim}_L(U) = 2\},$$

where any element of $\mathscr{L}$ is identified with the set of elements of $\mathscr{P}$ contained in it.

If U is a point, then we say that $\mathrm{Res}_\Gamma(U)$ is a *point residual.*

Point residuals of polar and parapolar spaces of (uniform) rank $r \geq 3$ are polar and parapolar spaces, respectively, of (uniform) rank $r-1$.

2.2. **Families of buildings of type F_4.** As noted in the introduction, due to Chapter 10 of [31], a building of type F_4 is completely determined by a pair $(\mathbb{K}, \mathbb{A})$, where $\mathbb{K}$ is a field and $\mathbb{A}$ is a quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$. We label the diagram as explained in the introduction and denote the corresponding building by $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$.

We list the properties of the different classes of quadratic alternative division algebras in Table 1, introducing the notation we will adopt for these algebras. Then we fetch the first four rows—Classes (K), (L) , (H) and (O)—under the name *separable* and Class (M) is referred to as the *inseparable* case ("M" stands for *Mixed*). Note that the latter includes the case $\mathbb{K} = \mathbb{A}$ with

Notation	$\mathsf{dim}_{\mathbb{K}}(\mathbb{A})$	$\mathsf{char}(\mathbb{K})$	Class	Properties
$\mathbb{K}$	1	$\neq 2$	(K)	commutative, associative
$\mathbb{L}$	2		(L)	commutative, associative
$\mathbb{H}$	4		(H)	non-comm, associative
$\mathbb{O}$	8		(O)	non-comm, non-ass, alternative
$\mathbb{K}'$	$2^h, \infty$	2	(M)	commutative, associative

TABLE 1. Quadratic alternative division algebras over $\mathbb{K}$

$\operatorname{char}\mathbb{K} = 2$. Also, we refer to the case $\mathbb{A} = \mathbb{K}$ in either characteristic as the *split case*; the other cases then are *nonspilt*. We also use these notions for the symplecta.

Cayley-Dickson process—Let, with the notation of Table 1, $(\mathbb{A}, \mathbb{B})$ be one of $(\mathbb{K}, \mathbb{L})$, $\operatorname{char}\mathbb{K} \neq 2$, $(\mathbb{L}, \mathbb{H})$, $(\mathbb{H}, \mathbb{O})$. So $\mathbb{L}$ is a quadratic (Galois) extension of $\mathbb{K}$, $\mathbb{H}$ is a quaternion division algebra over $\mathbb{K}$ and $\mathbb{O}$ is an octonion division algebra over $\mathbb{K}$. Then $\mathbb{A}$ can be obtained from $\mathbb{B}$ by the so-called *Cayley-Dickson process*, see [12], as follows. Let $x \mapsto \overline{x}$ be the standard involution in $\mathbb{B}$ (for $\mathbb{B} = \mathbb{K}$ this is just the identity), and let $b \in \mathbb{K}$ be such that it cannot be written as $x\overline{x}$, for any $x \in \mathbb{B}$. Then $\mathbb{A}$ consists of all pairs $(u, v) \in \mathbb{B} \times \mathbb{B}$ with standard addition and multiplication given by the rule

$$(u, v) \cdot (u', v') = (uu' + bv'\overline{v}, \overline{u}v' + u'v),$$

for all $u, v, u', v' \in \mathbb{B}$. The new standard involution is given by $(u, v) \mapsto (\overline{u}, -v)$.

Standig hypothesis. From now on we denote by $\mathbb{K}$ an arbitrary field, and $\mathbb{A}$ is a quadratic alternative division algebra over $\mathbb{K}$.

2.3. **Two families of polar spaces.** Now we define the two families of polar spaces which we will need in the definition of the metasymplectic spaces we are concerned with.

Definition 2.3.1. The polar space $\mathsf{B}_{\mathsf{r},1}(\mathbb{K}, \mathbb{A})$ is the quadric in $\mathsf{PG}(n, \mathbb{K}) = \mathsf{PG}(\mathsf{V})$, with $n = 2r - 1 + \mathsf{dim}_{\mathbb{K}}(\mathbb{A})$ and $V = \mathbb{K}^{2r} \oplus \mathbb{A}$, with equation

$$x_{-r}x_r + \cdots + x_{-2}x_2 + x_{-1}x_1 = \mathsf{N}(x_0),$$

where $x_{-r}, x_r, \ldots, x_{-2}, x_2, x_{-1}, x_1 \in \mathbb{K}$, $x_0 \in \mathbb{A}$ and N the natural norm form of $\mathbb{A}$.

Definition 2.3.2. The polar space $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$, with $\mathbb{A}$ not equal to $\mathbb{K}$ and not an octonion division algebra, is the hermitian polar space in $\mathsf{PG}(5, \mathbb{A})$ with point set the points the coordinates of which satisfy

$$\overline{x}_{-3}x_3 + \overline{x}_{-2}x_2 + \overline{x}_{-1}x_1 \in \mathbb{K},$$

where $x_{-3}, x_3, x_{-2}, x_2, x_{-1}, x_1 \in \mathbb{A}$ and $x \mapsto \overline{x}$ the standard involution of $\mathbb{A}$. If $\mathbb{A} = \mathbb{K}$, then $\mathsf{C}_{3,1}(\mathbb{K}, \mathbb{K})$ is the symplectic polar space of rank 3 corresponding to the standard alternating form $x_{-3}y_3 + x_{-2}y_2 + x_{-1}y_1 - x_1y_{-1} - x_2y_{-2} - x_3y_{-3}$. If $\mathbb{A} = \mathbb{O}$ is an octonion division algebra, then $\mathsf{C}_{3,1}(\mathbb{O}, \mathbb{K})$ is the non-embeddable polar space with planes over $\mathbb{O}$, see chapter 9 of [31].

We will not need a precise definition of the non-embeddable case. An explicit construction with coordinates is provided in [6].

2.4. **Metasymplectic spaces.** Now we can finally define the metasympletic spaces. Sometimes, for example in [7] and [22], the axioms used in the following definition are referred to as facts which can be proven from the building-theoretic definition, as stated in [34] p. 80 or proved in [4].

Definition 2.4.1 (Metasymplectic space)**.** A *metasymplectic space* $\Gamma_i = \mathsf{F}_{\mathsf{4,i}}(\mathbb{K}, \mathbb{A})$ ($i \in \{1, 4\}$) is a parapolar space of uniform rank 3 whose points, lines, planes and symplecta satisfy axioms 2.4.2, 2.4.3, 2.4.4, 2.4.5 and 2.4.6, where $\mathbb{A}$ is a quadratic alternative division algebra over $\mathbb{K}$.

Axiom 2.4.2 (Symp residue)**.** *The points, lines and planes of Γ_i contained in a given symplecton ξ, endowed with the natural inherited incidence relation, are the points, lines and planes, respectively, of a polar space $\mathrm{Res}_{\Gamma_i}(\xi)$ isomorphic to $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$ if $i = 1$, and $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$ if $i = 4$.*

Axiom 2.4.3 (Point residue)**.** *The symplecta, planes and lines of Γ_i through a given point p, endowed with the natural incidence relation, form a polar space $\mathrm{Res}_{\Gamma_i}(p)$ isomorphic to $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$ if $i = 1$, and $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$ if $i = 4$, where the points of that polar space are the symplecta through p, the lines are the planes through p, and the planes are the lines through p.*

In particular, it follows that the isomorphism class of the geometry $\mathrm{Res}_{\Gamma_i}(p)$ does not depend on p. It also follows that the point residual at p as defined earlier is the dual polar space corresponding to $\mathrm{Res}_{\Gamma_i}(p)$.

Axiom 2.4.4 (Point-point relation)**.** *Let x and y be two points of Γ_i. Then exactly one of the following situations occurs:*

(0) *$x = y$;*
(1) *there is a unique line incident with both x and y;*
(2) *there is a unique symplecton incident with both x and y. In this case, there is no line incident with both x and y, and we call x and y* symplectic. *We denote the unique symplecton by $\xi(x, y)$ and write $x \perp\!\!\!\perp y$;*
(3) *there is a unique point z collinear to both x and y. In this case, x and y are special. We denote $x \bowtie y$ and $z = \mathfrak{c}(x, y)$;*
(4) *there is no point collinear to both x and y. In this case, x and y are at distance 3 and we say that they are* opposite.

Axiom 2.4.5 (Point-symp relation)**.** *Let x be a point and let ξ be a symplecton of Γ_i. Then exactly one of the following situations occurs:*

(0) *$x \in \xi$;*
(1) *the set of points of ξ collinear to x is a line L. Every point y of $\xi \setminus L$ which is collinear to each point of L is symplectic to x and $\xi(x, y)$ contains L. Every other point z of ξ (i.e., every point z of ξ collinear to a unique point z' of L) is special to x and $\mathfrak{c}(x, z) = z' \in L$. We say that x and ξ are* close*;*
(2) *there is a unique point u of ξ symplectic to x and $\xi \cap \xi(x, u) = \{u\}$. All points v of ξ collinear to u are special to x and $\mathfrak{c}(x, v) \notin \xi$. All points of ξ not collinear to u are opposite x. We say that x and ξ are* far.

Axiom 2.4.6 (Symp-symp relation)**.** *The intersection of two symplecta is either empty, or a point, or a plane.*

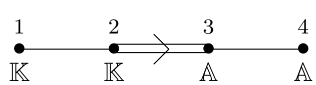
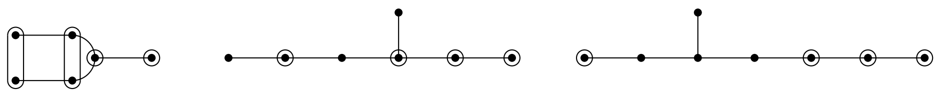


FIGURE 3. The Dynkin diagram of type F_4 with Bourbaki labeling



${}^2\mathsf{E}^2_{6,4}$ $\mathsf{E}^9_{7,4}$ $\mathsf{E}^{28}_{8,4}$

FIGURE 4. The Tits indices corresponding to Class (L), (H) and (O), respectively

Remark 2.4.7. Defining the dual point-line geometry to Γ_i as the geometry with point set the set of symplecta of Γ_i and line set the set of sets of symplecta sharing a given plane, we deduce from the diagram that the dual of Γ_1 is Γ_4 and vice versa. We will refer to this correspondence as the *natural duality.* Lemma 2.8.4 is the key ingredient to deduce this natural duality from the axioms above, but we will not do so explicitly.

Remark 2.4.8. The split buildings of type F_4 have trivial Tits index—every node of the F_4 diagram is encircled; those of Class (M) are of mixed type (and have no Tits index if $\mathbb{K} \neq \mathbb{A}$). The Tits indices of those of Classes (L), (H) and (O) are gathered in Fig. 4. This is purely informative and shall not be used in this paper; hence we do not define Tits indices in a formal way, but refer to [30].

2.5. Some properties of metasymplectic spaces.

The axioms in the previous section have some immediate corollaries, which are stated in e.g. [7] and [22].

Corollary 2.5.1. *Every singular subspace of Γ_i is contained in some symplecton, and hence is either empty, a point, a line or a projective plane.*

Corollary 2.5.2 (Point-line relation)**.** *Let x be a point and let L be a line in a metasymplectic space. Then precisely one of the following situations occurs:*

(0) *$x \in L$;*
(1) *$x \perp L$;*
(2) *$x \perp p \in L$ for exactly one point p, and $x \perp\!\!\!\perp q$ for all $q \in L \setminus \{p\}$;*
(3) *$x \bowtie p \in L$ for exactly one point p, and x is opposite q for all $q \in L \setminus \{p\}$;*
(4) *$x \perp p \in L$ for exactly one point p, and $x \bowtie q$ for all $q \in L \setminus \{p\}$, with evidently $\mathfrak{c}(x,q) = p$;*
(5) *$x \perp\!\!\!\perp p \in L$ for exactly one point p, and $x \bowtie q$ for all $q \in L \setminus \{p\}$, with $\mathfrak{c}(x,q) = a$ with $a \perp L$ independent of q;*
(6) *$x \bowtie L$, with $M = \{z | z = \mathfrak{c}(x,p), p \in L\}$ a line.*

Corollary 2.5.3. *If $a \perp b \perp c \perp d$ is a path in Γ_i, then $a \bowtie c$ and $b \bowtie d$ if, and only if, a is opposite d.*

We can also prove the following.

Corollary 2.5.4. *Let ξ be a symplecton of Γ_i and let p, q be two points close to ξ. Then p and q are opposite if, and only if, the lines $L := p^\perp \cap \xi$ and $M := q^\perp \cap \xi$ are opposite in the polar space ξ.*

Proof. Suppose first that L and M are not opposite. Let x be a point of L collinear to all points of M. Then $q \perp\!\!\!\perp x$ by (1) of Axiom 2.4.5 (point-symp relation). Now p must be close to $\xi(q,x)$, which implies that p is not opposite q.

Suppose now that L and M are opposite. Let a be a point of L and denote by b the unique point of M collinear to a. Then by the point-symp relations, $p \bowtie b$ and $q \bowtie a$. With Lemma 2.5.3 we find that p and q are indeed opposite. □

2.6. **The equator and extended equator geometries.** In this section, we will define some geometries which are included in a metasymplectic space. Among these are the equator geometries. As remarked in [22], these have been treated in the split case in [7]. This was generalised in [22] to all metasymplectic spaces Γ_4. In the present paper, we define the equator geometry for both metasymplectic spaces Γ_1 and Γ_4, and this requires a sightly different approach. For the extended equator geometries, we have to restrict ourselves to metasymplectic spaces Γ_4, which we will motivate below. Concerning the tropics geometries, the authors of [22] claim that the proof of Lemma 2.6.17 remains the same as in the split case. However, this does not seem to be entirely true, and so we provide a detailed, different proof. Along the way, we also prove some more properties of the interaction of the equator geometry with hyperbolic lines.

Definition 2.6.1 (Equator geometry)**.** Let p, q be two opposite points of Γ_i ($i \in \{1, 4\}$). The *equator geometry* $E(p, q)$ is the point-line geometry with point set the points symplectic to p and q and line set the sets of points corresponding to symplecta through a fixed plane through p.

Note that this definition differs from the one in [7] since we also want to include Γ_1. Also, the definition (of the line set of $E(p, q)$) is not symmetric in p and q; see however Lemma 2.6.4 below.

Proposition 2.6.2. *Let p, q be two opposite points of Γ_i. The equator geometry, $E(p, q)$, is isomorphic to the point residue $\mathrm{Res}_{\Gamma_i}(p)$ and is consequently a polar space of rank 3. If $i = 1$, then $E(p, q) \cong \mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$ and if $i = 4$, then $E(p, q) \cong \mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$.*

Proof. Define the map
$$\phi : E(p, q) \to \mathrm{Res}_{\Gamma_i}(p) \colon x \mapsto \xi(x, p).$$
We prove that ϕ is an isomorphism of point-line geometries. The injectivity follows from the possible point-symp relations (Axiom 2.4.5). Suppose $x, y \in E(p, q)$ and $\xi = \xi(x, p) = \xi(y, p)$, then q is far from ξ, because ξ contains a point opposite q. But x and y are symplectic to q, so $x = y$. Also the surjectivity follows from this axiom. Let ξ be a symplecton through p. Then ξ is far from q and there exists a unique point a of ξ symplectic to q, so $\xi = \xi(a, p) = \phi(a)$ with $a \in E(p, q)$. It is clear that lines are preserved, because they are defined in the same way in $E(p, q)$ and $\mathrm{Res}_{\Gamma_i}(p)$. □

The lines in a equator geometry will also briefly be called *lines* and it should be clear from the context which kind of lines is meant. However, we will frequently write the word "line" within quotation marks when we mean a line in the equator geometry. Similarly we will refer to a plane of an equator geometry writing "plane".

Lemma 2.6.3. *Let p, q be opposite points of Γ_i, and let $x \neq y$ be two points in $E(p, q)$. Then x is collinear to y in $E(p, q)$ if, and only if, $x \perp\!\!\!\perp y$ in Γ_i. Also, if $x \perp\!\!\!\perp y$, then $x^\perp \cap y^\perp \cap p^\perp$ is a line in the plane $\alpha := \xi(x, p) \cap \xi(y, p)$, and that line coincides with $q^{\bowtie} \cap \alpha$.*

Proof. If x is collinear to y in $E(p, q)$, then the symplecta $\xi(x, p)$ and $\xi(y, p)$ intersect in a plane α. Since a symplecton is a polar space of rank 3, x is collinear to a line $L \subseteq \alpha$ and y is collinear to a line $M \subseteq \alpha$. If $L = M$, then x and y are symplectic (and then $L = M = x^\perp \cap y^\perp \cap p^\perp$) or collinear. If they were collinear, we would have a singular subspace of dimension 3, which contradicts Corollary 2.5.1. If $L \neq M$, then x and y are special, with $\mathfrak{c}(x, y) = L \cap M$ which is in particular collinear to p. By the point-symp relations, y has to be close to $\xi(x, q)$, because y

is symplectic to q and special to x, but these two are not collinear. Let M' be the unique line of $\xi(x,q)$ collinear to y. But then the plane containing q and M' is contained in $\xi(x,q)\cap\xi(y,q)$ and similarly to the first part of this paragraph we get $\mathfrak{c}(x,y)\perp q$, which contradicts the opposition between p and q.

If $x \perp\!\!\!\perp y$ in Γ_i, then x is close to $\xi(p,y)$, because of the possible point-symp relations and the fact that x is symplectic to at least two points of $\xi(p,y)$, but not contained in $\xi(p,y)$ (that would contradict the opposition of p and q). Let L be the line of $\xi(p,y)$ collinear to x. Since $x \perp\!\!\!\perp p$ and $x \perp\!\!\!\perp y$, we get that $p \perp L$ and $x \perp L$ by the point-symp relations and therefore $L \subseteq \xi(x,p)\cap\xi(y,p)$. By the symp-symp relations the intersection is a plane α containing L. Corollary 2.5.3 yields $L \subseteq q^{\bowtie}$ and the lemma follows. □

Lemma 2.6.4. *Let p,q be opposite points of Γ_i. Then $E(p,q)$ coincides with $E(q,p)$.*

Proof. Let α be a plane through p and let x,y be points in $E(p,q)$ corresponding to symplecta through α. Then, Lemma 2.6.3, x and y are symplectic and $x^\perp\cap\alpha = y^\perp\cap\alpha =: L$. By applying the same lemma to $E(q,p)$, the symplecta $\xi(x,q)$ and $\xi(y,q)$ intersect in a plane β and $x^\perp\cap\beta = y^\perp\cap\beta =: M$. Let now ζ be an arbitrary symplecton through α corresponding to some point z in $E(p,q)$. Then x,y and z are pairwise symplectic and again by Lemma 2.6.3 the corresponding symplecta through q pairwise intersect in planes which, by Axiom 2.4.3, have at least one common line K (through q). Lemma 2.6.3 yields a point $r\in K\cap x^\perp\cap y^\perp\cap z^\perp \subseteq \beta\cap x^\perp\cap y^\perp = M$. Hence $\xi(x,y)$, which clearly contains L and M, also contains z. Since $z \perp\!\!\!\perp q$, the point-symp relations applied to q and $\xi(x,y)$ imply that $z\perp M$ and so $\xi(z,q)$ contains β. Similarly a symplecton through β corresponds to a symplecton through α. □

Lemma 2.6.5. *Let p,q be opposite points of Γ_i and let x,y be points in $E(p,q)$. Then either $x=y$, or $x \perp\!\!\!\perp y$, or x is opposite y.*

Proof. Suppose $x\perp y$, then x is close to $\xi(y,p)$ and x is collinear to a line L of $\xi(y,p)$ through y. Since $x \perp\!\!\!\perp p$, p has to be collinear to L by the point-symp relations and this contradicts $p \perp\!\!\!\perp y$.

Suppose for a contradiction that $x \bowtie y$. Then x has to be close to $\xi(p,y)$ by the point-symp relations, because it is special to y and symplectic to p, but p is not collinear to y. Then $z=\mathfrak{c}(x,y)$ must lie in $\xi(x,p)$ and similarly in $\xi(x,q)$, so $z=x$, a contradiction. □

Lemma 2.6.6. *Let p,q be opposite points of Γ_i. Then every line in $E(p,q)$ is contained in a unique symplecton in Γ_i. In particular, if x and y are contained in a "line", then the symplecton is $\xi(x,y)$.*

Proof. Let α be a plane through p corresponding to a line h in $E(p,q)$ and let β be the corresponding plane through q, according to Lemma 2.6.4. Lemma 2.6.3 implies that every point of h is collinear to both lines $q^{\bowtie}\cap\alpha$ and $p^{\bowtie}\cap\beta$, which then are contained in $\xi(x,y)$ for distinct $x,y\in h$. Then clearly $h\subseteq\xi(x,y)$. □

Lemma 2.6.7. *Let p,q be two opposite points of Γ_i and let ξ be a symplecton. Then the intersection $\xi\cap E(p,q)$ is either empty, or a point, or a line of $E(p,q)$.*

Proof. If the intersection contains two points x,y, these have to be symplectic by the possible relations between two points in a symplecton and the possible relations between two points in $E(p,q)$ (Lemma 2.6.5). By Lemma 2.6.6 the symplecton contains then every point $z\in E(p,q)$ with $\xi(x,p)\cap\xi(y,p) =: \alpha \subseteq \xi(z,p)$, which is by definition the "line" containing x and y.

Now we prove that the intersection cannot contain more. When the intersection contains the line in $E(p,q)$ through x and y, then p and q must be close to ξ; denote by L and M the unique line of ξ collinear to p and q, respectively. Since p and q are opposite in Γ_i, by Corollary 2.5.4 M and L are opposite in the polar space ξ. Now every point z in $\xi \cap E(p,q)$ must be collinear to L and M and so the symplecton $\xi(x,z)$ contains the plane $\langle L,p\rangle$ which defines the line through x and y in $E(p,q)$. So z is contained in the "line" through x and y. □

Now we will see that we can define the lines in $E(p,q)$ in a different way, if we are in a metasymplectic space Γ_4. This is because the symplecta are then polar spaces isomorphic to $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$. We will see that in this case the so-called hyperbolic lines (Definition 2.6.8) will correspond to sets of points given as the common perp of two opposite lines in a symplecton (Lemma 2.6.9), which will allow us to identify these lines with the lines in $E(p,q)$ (Proposition 2.6.11). This will then also show that the defintion of the equator geometry in Γ_4 in the present paper is equivalent with that in [22].

Definition 2.6.8 (Hyperbolic line)**.** Let ξ be a polar space and let x,y be two opposite points in ξ. The *hyperbolic line* $h(x,y)$ is the set of points $(x^\perp \cap y^\perp)^\perp$.

Lemma 2.6.9. *Let ξ be the polar space $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$. If x,y are two opposite points in ξ and L,M are two opposite lines in $x^\perp \cap y^\perp$, then $h(x,y) = L^\perp \cap M^\perp$ and the number of points on $h(x,y)$ is $|\mathbb{K}|+1$.*

Proof. It is clear that $h(x,y) \subseteq L^\perp \cap M^\perp$, so it suffices to prove that $L^\perp \cap M^\perp \subseteq h(x,y)$. We provide two proofs: First we will give a proof that is applicable to the embeddable polar spaces, i.e. the polar spaces $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$, with $\mathbb{A}$ not an octonion division algebra. Then we will give a prove that can be applied to the separable case only, i.e. $\mathbb{A}$ is not an inseparable field extension in characteristic 2.

Suppose first that ξ is an embeddable polar space $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$. Then by Definition 2.3.2 and the general theory of polar spaces (see for example Chapter 8 in [31]), ξ is a polar space embeddable in (the absolute elements of) a nondegenerate polarity ρ in $\mathsf{PG}(5,\mathbb{A})$. Let $z \in L^\perp \cap M^\perp$ be a point. The polar space $x^\perp \cap y^\perp$ is embeddable in dimension 3 and is consequently spanned by two opposite lines. In other words every point of $x^\perp \cap y^\perp$ lies on a line of the underlying projective space that intersects L and M. Since z is collinear to these points and z^ρ is a subspace in the underlying projective space, z is collinear to each point of $x^\perp \cap y^\perp$. The number of points on $h(x,y)$ is now equal to the number of planes through a line in the polar space, because each plane through L contains exactly one point collinear to M. This, in turn, is equal to the number of lines through a point in any point residual, and equals $|\mathbb{K}|+1$ by Proposition 2.3.5 of [34]..

Suppose now that ξ is not isomorphic to $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ with $\mathbb{A}$ an inseparable field extension in characteristic 2. By Theorem 5.9.4 of [34], the quadrangle $x^\perp \cap y^\perp$ has no proper thick subquadrangles with full lines. So the quadrangle spanned by L and M must be the whole quadrangle or a grid. The latter is impossible by Lemma 5.5.8 of [34] and our assumption on ξ. So the quadrangle spanned by L and M is the whole quadrangle $x^\perp \cap y^\perp$. If a point z is now collinear to both L and M, it is collinear to $x^\perp \cap y^\perp$. The number of points on $h(x,y)$ in the octonion case is now also the number of planes through a line, which equals $|\mathbb{K}|+1$, as follows from the construction in [6]. □

Lemma 2.6.10. *Let ξ be the polar space $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ and assume that the latter is separable. If x,y are two opposite points in ξ, then $h(x,y) = \{x,y\}$.*

Proof. By the definition of $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$, we may look at the underling projective space $\mathsf{PG}(n,\mathbb{K})$ of this polar space. By Proposition 3.20 of [22], $\perp$ defines a nondegenerate polarity ρ in $\mathsf{PG}(n,\mathbb{K})$.

Hence $(x^\perp \cap y^\perp)^\rho$ is a line intersecting the quadric in the two points x and y, implying $(x^\perp \cap y^\perp)^\perp = \{x, y\}$, which proves the statement. □

Proposition 2.6.11. *Let p, q be two opposite points of Γ_4, and let x, y be collinear points in $E(p, q)$. Then the line through x and y in $E(p, q)$ is exactly the hyperbolic line $h(x, y)$. In particular $E(p, q)$ is closed under taking hyperbolic lines of pairs of symplectic points.*

Proof. Let z be a point on the line through x and y in $E(p, q)$. Then z is contained in the symplecton $\xi(x, y)$ by Lemma 2.6.6. By the point-symp relations p is collinear to some line L of this symplecton and L is collinear to the points x, y, z. Similarly q is collinear to such a line M and because p and q are opposite in Γ_4, L and M are opposite in $\xi(x, y)$. With Lemma 2.6.9 we now have that $z \in h(x, y)$.

Let z' now be a point on $h(x, y)$. Then z' is collinear to the unique line L of $\xi(x, y)$ collinear to p, because this line is contained in $x^\perp \cap y^\perp$. So z' is symplectic to p and similarly to q, so $z' \in E(p, q)$. The symplecton $\xi(p, z)$ also contains the plane $\langle p, L\rangle = \xi(x, p) \cap \xi(y, p)$, and z' is consequently contained in the line through x and y in $E(p, q)$. □

Now we can define the extended equator geometry in the case of metasymplectic spaces Γ_4. The reason that we are not able to do this in general for Γ_1, is that hyperbolic lines are no longer determined by the common perp of two distinct lines, like in Lemma 2.6.9.

Definition 2.6.12 (Extended equator geometry)**.** Let p, q be two opposite points of Γ_4. Then define the *extended equator geometry* $\widehat{E}(p, q)$ as the point-line geometry with point set

$$\bigcup\{E(x, y) | x, y \in E(p, q), x \text{ opposite } y\},$$

and line set all the hyperbolic lines contained in this point set.

Note that, by Lemma 2.6.5 and Proposition 2.6.2, $E(p, q)$ contains pairs of opposite points, so $\widehat{E}(p, q)$ is nonempty. We also get directly that p, q and $E(p, q)$ are contained in $\widehat{E}(p, q)$.

The following three results come from [22].

Lemma 2.6.13. *Let p, q be two opposite points in Γ_4 and let x be a point in $\widehat{E}(p, q)$. Then the set of points of $E(p, q)$ symplectic to or equal to x is a geometric hyperplane of $E(p, q)$, viewed as a polar space, or coincides with it.*

Proof. This is Corollary 3.16 of [22]. □

Lemma 2.6.14. *Let p, q be two opposite points in Γ_4 and let x, y be two points in $\widehat{E}(p, q)$. Then either $x = y$, or $x \perp\!\!\!\perp y$, or x is opposite y. If, moreover, $x \perp\!\!\!\perp y$, then there exist opposite $a, b \in E(p, q)$ so that $h(x, y) \subseteq E(a, b)$ and $h(x, y)$ is consequently completely contained in $\widehat{E}(p, q)$.*

Proof. This is Lemma 3.17 of [22]. □

Proposition 2.6.15. *Let p, q be two opposite points in Γ_4. The extended equator geometry $\widehat{E}(p, q)$ is a polar space isomorphic to $\mathsf{B}_{4,1}(\mathbb{K}, \mathbb{A})$.*

Proof. This is Proposition 3.18 of [22]. □

We now provide some additional properties of the extended equator geometries, either only proved in the split case (in [7]) and stated without proof in [22], or new.

Lemma 2.6.16. *Let p, q be two opposite points in Γ_4 and let ξ be a symplecton intersecting $\widehat{E}(p,q)$ in at least a point. Then $\xi \cap \widehat{E}(p,q)$ contains a hyperbolic line.*

Proof. Suppose that ξ intersects $\widehat{E}(p,q)$ in a point x. If $x = p$ or $x = q$ it is clear that $\xi \cap \widehat{E}(p,q)$ contains at least the hyperbolic line $h(x,y)$ with $y = \xi \cap E(p,q)$ by the fact that p, q and $E(p,q)$ are contained in $\widehat{E}(p,q)$ and Lemma 2.6.14. If $x \in E(p,q)$, we can find a point y opposite x in $E(p,q)$ and then the hyperbolic line $h(x,z)$ with $z = \xi \cap E(x,y)$ is similarly as the previous case, using Lemma 2.6.14, contained in $\widehat{E}(p,q)$. So we can assume without loss of generality that x is opposite p. Then p is far from ξ and we denote by y the unique point of ξ symplectic to p. By the previous case, the symplecton $\xi(p,y)$ intersects $\widehat{E}(p,q)$ at least in a hyperbolic line h. Because $\widehat{E}$ is a polar space, this hyperbolic line has at least one point symplectic to x. But because x is far from the symplecton $\xi(p,y)$, with y the unique point symplectic to x, the point y has to be contained in $h \subseteq \widehat{E}(p,q)$. Now again by Lemma 2.6.14, $h(x,y)$ has to be contained in $\widehat{E}(p,q)$ and consequently in the intersection of $\widehat{E}(p,q) \cap \xi$. □

Lemma 2.6.17. *Let p, q be two opposite points in Γ_4 and let $a, b \in \widehat{E}(p,q)$ be opposite points. Then, $\widehat{E}(a,b) = \widehat{E}(p,q)$.*

Proof. We start by showing that $E(a,b) \subseteq \widehat{E}(p,q)$. Let x be an arbitrary point of $E(a,b)$. Consider the symplecton $\xi(a,x)$. By Lemma 2.6.16, this has a hyperbolic line h in common with $\widehat{E}(p,q)$. By Proposition 2.6.15 b must be symplectic to some point of this line h. It is however clear that b is far from $\xi(a,x)$ and the only point of that symplecton symplectic to b is x. So x has to be contained in $h \subseteq \widehat{E}(p,q)$. By the arbitrariness of x, we get that $E(a,b) \subseteq \widehat{E}(p,q)$.

Now an arbitrary point w of $\widehat{E}(a,b)$ is by definition contained in $E(x,y)$ for some opposite points $x, y \in E(a,b)$. Applying the previous paragraph to x, y as opposite points in $\widehat{E}(p,q)$, one gets that $w \in E(x,y) \subseteq \widehat{E}(p,q)$. By the arbitrariness of w, we get that $\widehat{E}(a,b) \subseteq \widehat{E}(p,q)$.

Now note that $E(a,b) \cap E(p,q)$ is a geometric hyperplane of $b^{\perp\!\!\!\perp} \cap E(p,q)$, a geometric hyperplane of $E(p,q)$ by Lemma 2.6.13. Now $E(a,b) \cap E(p,q)$ contains two opposite points x, y (cf. Lemma 4.2.3 of [7]). But then $p, q \in E(x,y) \subseteq \widehat{E}(a,b)$ and we can apply the previous two paragraphs switching the roles of a, b and p, q to obtain $\widehat{E}(p,q) \subseteq \widehat{E}(a,b)$. □

Lemma 2.6.18. *Let p, q be two opposite points in Γ_4 and let ξ be a symplecton. Then either ξ is disjoint from $\widehat{E}(p,q)$ or $\xi \cap \widehat{E}(p,q)$ is a hyperbolic line. Hence every symplecton that has a point x in common with $\widehat{E}(p,q)$ intersects it in a hyperbolic line through x. In particular, any hyperbolic line in $\widehat{E}(p,q)$ appears as the intersection of $\widehat{E}(p,q)$ and a unique symplecton.*

Proof. By Lemma 2.6.16 it suffices to prove that ξ does not intersect $\widehat{E}(p,q)$ in more than a hyperbolic line. As a hyperbolic line defines a unique symplecton containing it, the rest of the lemma follows then immediately.

Suppose now for a contradiction that the said intersection is more than a "line", namely at least a hyperbolic line $h(u,v)$ and a point $w \notin h(u,v)$. By Lemma 2.6.14 we find some opposite points a, b in $E(p,q)$ with $h(u,v) \subseteq E(a,b)$. By Lemma 2.6.17, we get that $\widehat{E}(p,q) = \widehat{E}(a,b)$. By the definition of the extended equator geometry we now find some opposite points $x, y \in E(a,b)$ such that w is symplectic to x and y. In $E(a,b)$, x is symplectic to a point of $h(u,v)$ and so x is symplectic to two points of the symplecton ξ. Consequently x is close to ξ and similarly also y is close to ξ. But by case (1) of Axiom 2.4.5 (the point-symp relations) and Lemma 2.6.14 (the point-point relations in $\widehat{E}$), x and y must be symplectic to every point of $\widehat{E}(p,q)\cap\xi$. Now $\widehat{E}(p,q)\cap\xi\subseteq E(x,y)$ and so $E(x,y)\cap\xi$ contains more than a hyperbolic line, contradicting Lemma 2.6.7. □

2.7. **The tropics geometries.** Another geometry living in the metasymplectic spaces, the so called tropics geometry, is defined starting from the extended equator geometry. This section is strongly based on Section 5.3 in [7]. As noted in [22], most of the results stay valid in the non-split case. There are however some subtleties that are no longer valid as hyperbolic lines are no longer always lines in a underlying projective space, and which were overlooked in [22]. Therefor we display the full proofs.

Before defining those tropics geometries, the next lemma is very useful. A so called hyperbolic solid in the next lemma is just a solid (a singular subspace of projective dimension 3) in the polar space $\widehat{E}(p,q)$, where the lines are the so called hyperbolic lines. Similarly, one can define a hyperbolic plane.

Lemma 2.7.1. *Let p,q be two opposite points of Γ_4. Let x be a point of Γ_4 which is collinear to at least two points of $\widehat{E}(p,q)$. Then $x^\perp\cap\widehat{E}(p,q)$ is a hyperbolic solid.*

Proof. By Lemma 2.6.17 and Proposition 2.6.15, we may assume that $p\perp x$. Let a be a second point of $\widehat{E}$ collinear with x. By the possible relations between points in $\widehat{E}(p,q)$ (Lemma 2.6.14), we get that $p \perp\!\!\!\perp a$. Hence, by Lemma 2.6.17 and Proposition 2.6.15, we can choose q opposite p and symplectic to a. So we have that $a\in E(p,q)$ and $x\in\xi(a,p)$. By Proposition 2.6.11, the set of intersections with $E(p,q)$ of the symplecta through the line px is a hyperbolic plane π of $E(p,q)$. Let $b\in\pi$ be a point different from a. Since a is collinear with x and $x\in\xi(b,p)$, the point a is close to $\xi(b,p)$. Since $b \perp\!\!\!\perp a$, the possible point-symp relations (Axiom 2.4.5) imply that $x\perp b$.

Hence all points u of π are collinear with x. But x belongs to $\xi(u,p)$, and in the latter symplectic polar space, u and p belong to $x^\perp$; hence, by the definition of the hyperbolic line $h(u,p)$, all points of $h(u,p)$ are collinear with x, implying that all points of the maximal singular hyperbolic subspace of $\widehat{E}(p,q)$ spanned by π and p are collinear with x. Every two points in $x^\perp\cap\widehat{E}(p,q)$ lie at distance at most two, so they must be symplectic by the possible relations between points in $\widehat{E}(p,q)$. As two points are symplectic if, and only if, they are contained in a hyperbolic line, this implies that the singular hyperbolic subspace of $\widehat{E}(p,q)$ spanned by π and p is exactly $x^\perp\cap\widehat{E}(p,q)$. □

Definition 2.7.2 (Tropics Geometry)**.** Let p,q be two opposite points of Γ_4. Then define the *tropics geometry* $\widehat{T}(p,q)$ as the point-line geometry with point set

$$\{x\in\Gamma : |x^\perp\cap\widehat{E}(p,q)|\geq 2\},$$

and line set the set of all the lines of Γ_4 contained in this point set.

Remark that the big difference with the (extended) equator geometry, is that the lines in this geometry $\widehat{T}(p,q)$ are no longer hyperbolic lines, but really the lines of the metasymplectic space. Note also that this construction is only possible in the metasymplectic space Γ_4, as it relies on the extended equator geometry which is only defined there. Also remark that by the possible relations between points in $\widehat{E}(p,q)$, we see that $\widehat{E}(p,q)\cap\widehat{T}(p,q)=\emptyset$.

Lemma 2.7.1, allows us now to introduce the next notation. This is actually the core idea of the rest of this section. To track down the structure of the tropics geometry, we will define a map between this geometry and the dual of the extended equator geometry. This map is in fact the β defined here.

Remark 2.7.3. Let p, q be opposite points of Γ_4 and let x be a point of the tropics geometry $\widehat{T}(p,q)$. Then we denote by $\beta(x)$ the hyperbolic solid $\widehat{E}(p,q) \cap x^{\perp}$.

First of all we give a lemma that follows immediately from these definitions.

Lemma 2.7.4. *Let p, q be two opposite points of Γ_4. Then no point of $\widehat{T}(p,q)$ is opposite nor symplectic to any point of $\widehat{E}(p,q)$.*

Proof. Let $x \in \widehat{T}(p,q)$ and $y \in \widehat{E}(p,q)$ be arbitrary points, $y \notin \beta(x)$. Then y is symplectic to a hyperbolic plane π of $\beta(x)$ in the polar space $\widehat{E}(p,q)$. Now x is not opposite y as it lies close to the symplecton $\xi(y,z)$, for all $z \in \pi$. This also implies that x is not symplectic to y as the point-symp relation would then yield $z \perp y$, for each $z \in \pi$, contradicting Lemma 2.6.14. □

Now we will show that β is a bijection between $\widehat{T}(p,q)$ and the dual of $\widehat{E}(p,q)$ as a polar space.

Lemma 2.7.5. *Let p, q be two opposite points of Γ_4. Let U be a hyperbolic solid of $\widehat{E}(p,q)$. Then there exists exactly one $x \in \widehat{T}(p,q)$ such that $\beta(x) = U$. Moreover, this is the only point in Γ_4 collinear with U.*

Proof. By Lemma 2.6.17, we may suppose that p belongs to U. Then $U \cap E(p,q)$ is a hyperbolic plane π. The intersection of all symplecta $\xi(p,z)$ with $z \in \pi$ is by the definition of $E(p,q)$ a line L through p.

We first prove the uniqueness. Suppose there are two points $x, y \in \widehat{T}(p,q)$ with $\beta(x) = \beta(y) = U$. Then, both x and y must be contained in all the symplecta through p and a point of π, hence both are on L. Let $z \in \pi$ be arbitrary. Then in $\xi(z,p)$, the point z is collinear with exactly one point of L and this point must be $x = y$.

Now we prove the existence. Let $a, b \in \pi$ be arbitrary but distinct. Then b is not contained in $\xi(a,p)$ and hence is close to it. So b is collinear with a line $M \subseteq \xi(a,p)$ and by the point-symp relations a and p must also be collinear with M. Clearly, L is contained in the plane generated by p and M, which is the intersection of $\xi(a,p)$ and $\xi(b,p)$. So $x := L \cap M$ is collinear with both a and b. Since x is the unique point of L collinear with a, we see, by varying $b \in \pi$, that x is collinear with all points of π. Since also $x \perp p$, we see that x is collinear to the hyperbolic subspace spanned by p and π, as x is collinear to every point of a hyperbolic line that has at least two points collinear to x. This means $\beta(x) = U$.

The last assertion follows from the uniqueness combined with the fact that any point in Γ_4 collinear with U is of course collinear with at least two points of $\widehat{E}(p,q)$ and belongs consequently to $\widehat{T}(p,q)$.
□

The next proposition relates the mutual position between two hyperbolic solids on $\widehat{E}(p,q)$ and their preimages under β. In fact it checks that β preserves indeed the structure.

Proposition 2.7.6. *Let p, q be two opposite points of Γ_4 and let $\beta(a) = U$, $\beta(b) = V$ be two different hyperbolic solids in $\widehat{E}(p,q)$, with $a, b \in \widehat{T}(p,q)$. Then*

(*i*) *$U \cap V$ is a hyperbolic plane π if, and only if, $a \perp b$ in Γ_4. In this case, some point x is collinear with all points of π if, and only if, x belongs to ab;*
(*ii*) *$U \cap V$ is a hyperbolic line if, and only if, $a \perp\!\!\!\perp b$ in Γ_4. In this case, every point of $h(a,b)$ belongs to $\widehat{T}(p,q)$ and is collinear with all points of $U \cap V$;*
(*iii*) *$U \cap V$ is a singleton $\{z\}$ if, and only if, $a \bowtie b$ in Γ_4. In this case, $z = \mathfrak{c}(a,b)$;*

(*iv*) $U \cap V = \emptyset$ *if, and only if,* a *and* b *are opposite in* Γ_4.

Proof. By Lemma 2.6.17, Proposition 2.6.15 and the assumption that $U \neq V$, we may assume that $p \in U \setminus V$ and $q \in V \setminus U$. We get then that $(U \cap V) \subseteq E(p,q)$. We assume this throughout the proof.

(*i*) Suppose first that $a \perp b$. By the above assumption and Lemma 2.7.4, we infer $a \bowtie q$ and $b \bowtie p$. Let ξ be any symplecton through bq, and denote $\{x\} := E(p,q) \cap \xi$. Then $p \perp\!\!\!\perp x$ and p and ξ are far. Since p is special to b, the point-symp relations imply $b \perp x$. Now b is close to $\xi(p,x)$, hence there is a line L in $\xi(p,x)$ containing x such that L is collinear with b. As p is collinear to a point of this line, $a = \mathfrak{c}(b,p)$ is also contained in L. So a is collinear to x. Varying ξ over all symplecta through bq, the point x varies over a plane of $E(p,q)$. This plane must coincide with $U \cap V$ as $x \perp a$, $x \perp b$ and the intersection $U \cap V$ is at most a hyperbolic plane.

By Lemma 2.5.2, no point of the line ab is symplectic to or opposite p. Lemma 2.6.14 then implies that the line ab has empty intersection with $\widehat{E}$. Let z now be any point of $U \cap V$. Then $a \perp z \perp b$, and so every point of the line ab is collinear with z and hence with all points of π. Hence every point of the line ab is collinear with all points of π.

Now assume that U and V intersect in a plane π. Consider two points $x, y \in \pi$. Then both a and b are collinear with both x, y and hence both are contained in $\xi(x,y)$. It follows that a, b are either symplectic or collinear. If they were symplectic, then $\xi(a,b)$ would contain π, contradicting Lemma 2.6.18, so $a \perp b$.

Suppose now that some point c is collinear with all points of π. Then $c \in \widehat{T}(p,q)$ and we have just shown that $a \perp c \perp b$. Suppose for a contradiction that c does not belong to the line ab. Then take two points $u, v \in \pi$. It follows that $a, b, c \in \xi(u,v)$, contradicting the fact that $\xi(u,v)$ is a polar space of rank 3 and hence no plane can be contained in the intersection $u^\perp \cap v^\perp$.

(*ii*) Assume first that U and V intersect in a hyperbolic line h. We then have that $h \subseteq E(p,q)$. Consider two points $x, y \in h$. Then both a and b are collinear with both x, y and hence contained in $\xi(x,y)$. It follows that a, b are either symplectic or collinear. But they are not collinear by (*i*), so they must be symplectic.

Now assume that $\{a,b\}$ is a symplectic pair. Then by (*i*), we know that $U \cap V$ is at most a hyperbolic line. Both p and q are close to $\xi(a,b)$. Hence p is collinear with the points of a line $L \subseteq \xi(a,b)$, and q is collinear with the points of a line $M \subseteq \xi(a,b)$ and these are opposite viewed as lines of the polar space $\xi(a,b)$ by Corollary 2.5.4. With Lemma 2.6.9, this implies that $\xi(a,b)$ contains a unique hyperbolic line h all of whose points are collinear with L and M, i.e., $h = L^\perp \cap M^\perp$. In particular, h is contained in $a^\perp \cap b^\perp$. By Axiom 2.4.5 (1), all points of h are symplectic to both p and q, hence $h \subseteq E(p,q)$. So $h \subseteq U \cap V$, implying $h = U \cap V$.

Let z now be a point on the hyperbolic line $h(a,b)$. Then z is collinear to the intersection $a^\perp \cap b^\perp$, which contains $U \cap V$. Hence it follows immediately that z is collinear to $U \cap V$ and so z is also a point of $\widehat{T}(p,q)$.

(*iii*) Suppose first that U and V intersect in a point. Then a and b are collinear with a common point and hence cannot be opposite. Moreover, they are neither symplectic nor collinear by (*i*) and (*ii*). Consequently, they are special.

Now suppose that a and b are special. We show that $z = a \bowtie b$ belongs to $\widehat{E}(p,q)$, which will complete the proof of (*iii*) taking the previous two statements into account. Note that no point of $U \cup V$ can be special to z as this would give with Lemma 2.5.3 that this point is opposite a or b, contradicting Lemma 2.7.4. So z must be collinear or symplectic to the points of $U \cup V$. Suppose z is collinear to at least two points of $U \cup V$, then z is contained in $\widehat{T}(p,q)$ and by (i) the intersections $\beta(z) \cap \beta(a)$ and $\beta(z) \cap \beta(b)$ are hyperbolic planes in $\beta(z)$, contradicting the fact that the intersection $\beta(a) \cap \beta(b)$ contains at most one point by (i) and (ii). So z is collinear to at most one point of $U \cup V$. Without loss of generality, we may assume that z is symplectic to every point in U and at least one point y of V. It is easy to see that U contains a point y' not symplectic to y, as otherwise U and y would be contained in a singular hyperbolic subspace of $\widehat{E}(p,q)$ with dimension at least 4, a contradiction. But then, y and y' are opposite by Lemma 2.6.14 and $z \in E(y,y') \subseteq \widehat{E}(y,y') = \widehat{E}(p,q)$ by Lemma 2.6.17.

(iv) This follows by elimination and the previous cases. □

This proposition has an immediate corollary.

Corollary 2.7.7. *Let p,q be opposite points in Γ_4. Then $\widehat{T}(p,q)$ is a subspace of Γ_4.*

Proof. Let a,b be two collinear points in $\widehat{T}(p,q)$. By (i) of Proposition 2.7.6, we see that all the points of the line ab are contained in $\widehat{T}(p,q)$. □

However, the most important corollary is of course that we know now the structure of this tropics geometry.

Corollary 2.7.8. *Let p,q be opposite points in Γ_4. Then the tropics geometry $\widehat{T}(p,q)$ is isomorphic to the dual polar space $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$.*

Proof. This follows immediately from Proposition 2.6.15 combined with the fact that β is an isomorphism, which follows from Lemma 2.7.5 and Proposition 2.7.6. □

2.8. **Opposition and projection.** Opposition is a very important notion in the theory of spherical buildings, and it is of course also central in the idea of domesticity. Also typical in spherical buildings is the notion of projection. Opposition and projection are also intimately related, in particular by Theorem 3.28 of [31]. We review some basics here. We refer to Chapter 3, Sections 3.22–3.32 of [31] for more details.

Definition 2.8.1 (Opposition)**.** The *opposition* of singular spaces and symplecta in a polar space or a metasymplectic space Γ_i is defined as follows.

(1) Two points are opposite if they are at maximal distance from each other: not collinear in polar spaces, distance 3 in metasymplectic spaces (this agrees with Axiom 2.4.4);
(2) Two singular subspaces or symplecta are opposite if every point of one of them is opposite some point of the other.

Remark 2.8.2. We will sometimes speak about *locally opposite* spaces or symplecta in polar or metasymplectic spaces. Then there will always be a residue obvious from the context containing both (for example the intersection of these elements) and we mean that they are opposite in this residue.

For lines and planes of Γ_i, $i = 1,4$, we can be more precise.

Lemma 2.8.3. *Two lines are opposite in Γ_i if, and only if, every point is special to exactly one point of the other line and opposite all the other ones. Two planes are opposite in Γ_i if, and only if, every point is special to the point set of exactly one line of the other plane and opposite all the other points.*

Proof. The statement about the opposite lines follows immediately from the definition above and the possible point-line relations in Lemma 2.5.2. The statement about the planes follows from the same lemma: It is clear that, if α_1, α_2 satisfy the stated condition, then the planes are opposite. Suppose conversely that the planes are opposite. Let $p \in \alpha_k$ be a point and denote by p' an opposite point in α_l, $l \neq k$. Then by Lemma 2.5.2 every line M through p' has an unique point special to p and all the other points of M are opposite p. We now claim that the set of these points special to p is exactly a line. This is the case, because every line through two of these points has to be completely special to p, by Lemma 2.5.2 and the fact that α_l contains only points opposite and special to p. □

For symplecta, we could appeal to the duality between Γ_1 and Γ_4, as already mentioned, and as follows from the connection with buildings. However, for foundational reasons, we prove the following lemma merely using the axioms.

Lemma 2.8.4. *Let ξ_1, ξ_2 be two symplecta of Γ_i. Then ξ_1, ξ_2 are opposite if, and only if, the intersection of ξ_1 and ξ_2 is empty and there is no symplecton which intersects both in a plane. If ξ_1 and ξ_2 are disjoint and not opposite, this symplecton intersecting both in a plane is unique.*

Proof. First suppose that ξ_1 and ξ_2 are opposite. If there was a point in the intersection, it would not have an opposite point in one of the symplecta, so their intersection must be empty. If there was a symplecton intersecting both in a plane, every point in such a plane would be collinear to a line of the other plane and consequently be close to the other symplecton. This makes it impossible to have an opposite point in the other symplecton and contradicts our assumption.

Now suppose conversely that the intersection of ξ_1 and ξ_2 is empty and there is no symplecton which intersects each of them in a plane. Then we claim that every point of ξ_k has to be far from ξ_l, $l \neq k$, from where it follows immediately that the symplecta are opposite, by the point-symp relations. Suppose for a contradiction without loss of generality that there is a point $p \in \xi_1$ close to ξ_2. Then this point is collinear to some line L of ξ_2. Let $p_1, p_2 \in L$ be two different points which are now collinear to lines through p in ξ_1, say respectively L_1, L_2. If $L_1 = L_2$, this line is collinear to the line L and they span consequently a projective plane, which contradicts the empty intersection of ξ_1 and ξ_2. So we may suppose that $L_1 \neq L_2$. Let q be a point of L_1 different from p. Then q is clearly symplectic to p_2, as p and p_1 are collinear to both of them. The symplecton $\xi(p_2, q)$ now has the lines L and L_1 in common with the symplecta ξ_2 and ξ_1, respectively. By the symp-symp relations, this contradicts our assumption.

Suppose now that ξ_1 and ξ_2 are disjoint, but not opposite. Suppose for a contradiction that there exist two different symplecta ζ and ζ' intersecting both in a plane. Denote by $\pi_i^{(\prime)} := \zeta^{(\prime)} \cap \xi_i, i = 1, 2$. We will now take a closer look at the different possibilities for the intersections of the planes π_1 and π_1'.

- *Suppose π_1 and π_1' share at least a line.* Two distinct points of such line are collinear to two distinct respective lines, both lying in both π_2 and π_2'. By the possible point-symp relations these lines and hence these planes coincide. Now interchanging ξ_1 and ξ_2, also π_1 and π_1' coincide.
- *Suppose $\pi_1 \cap \pi_1'$ is a point.* Then that point is collinear to a line of both π_2 and π_2'; hence these intersect in at least a line and we are reduced to the previous case.
- *Suppose $\pi_1 \cap \pi_1'$ is empty.* By the previous case, we may also assume that $\pi_2 \cap \pi_2'$ is empty. Let p be a point in π_1. Then, since ξ_1 and ζ are polar spaces, p is collinear to a line L_1 of π_1' and a line L_2 of π_2. Let now p' be a point of π_2'. Then similarly p' is collinear to a line M_1 of π_1' and a line M_2 of π_2. Now the points p and p' have (at least) two points in

their common perp, namely $L_1 \cap M_1$ and $L_2 \cap M_2$ and are consequently symplectic. The symplecton $\xi(p, p')$ intersects the symplecta ξ_1, ξ_2, ζ and ζ' in respective planes. Applying the previous cases now twice (once to $\xi(p, p')$ and ζ, and once to $\xi(p, p')$ and ζ'), yields again the contradiction that $\zeta = \zeta'$. □

The concept of projection is again something that descends from building theory and which has very strong properties as proved in [31], for example Theorems 3.28 and 3.29, see also below. Let us define the projections in the metasymplectic spaces Γ_1 and Γ_4 that we will need.

Definition 2.8.5. Let p and q be two opposite points of Γ_i. The *projection from* $\mathsf{Res}_{\Gamma_i}(p)$ *onto* $\mathsf{Res}_{\Gamma_i}(q)$ is the collineation that maps each symplecton ζ through p to the unique symplecton through q that intersects ζ in a point; that maps each plane α through p to the unique plane through q containing a line that lies in a symplecton together with a line of α and that maps each line L through p to the unique line through q having a point collinear to a point of L. We denote this by proj_q^p.

Dually, one defines the projection of a symplecton ξ onto an opposite symplecton ζ. This, however, can also be defined as the isomorphism from ξ to ζ determined by mapping a point $x \in \xi$ to the unique point $y \in \zeta$ symplectic to x, that is, $x \perp\!\!\!\perp y$.

The uniqueness and existence of proj_q^p follows almost immediately from the point-line relations in Lemma 2.5.2(3) and the reasoning in the proof of Proposition 2.6.2. From this definition it is immediately clear that the types of elements are preserved, so we only have to check that inclusion is preserved to conclude that this is indeed a collineation. We leave this to the interested reader, being aware that this also follows from the general theory in chapter 3 of [31].

With the notion of projection, we can define a collineation on the residue from some collineations on a metasymplectic space. We define this here.

Definition 2.8.6. Let θ be a collineation of a metasymplectic space Γ_i mapping a point p to an opposite point p^θ. Then θ_p is the composition of $\theta|_{\mathsf{Res}_{\Gamma_i}(p)}$ with the projection from $\mathsf{Res}_{\Gamma_i}(p^\theta)$ to $\mathsf{Res}_{\Gamma_i}(p)$, in symbols: $\theta_p := \mathsf{proj}_p^{p^\theta} \circ \theta|_{\mathsf{Res}_{\Gamma_i}(p)}$.

Remark that this is well-defined by the previous reasoning. Then we have the following connection between global and local opposition:

Lemma 2.8.7. *Let p and q be two opposite points of some metasymplectic space Γ_i. Let U and V be two elements of the same type through p and q respectively. Then U is opposite V in Γ_i if, and only if, $\mathsf{proj}_p^q(V)$ is opposite U in $\mathsf{Res}_{\Gamma_i}(p)$, that is, $\mathsf{proj}_p^q(V)$ and U are locally opposite.*

Proof. This follows directly from Theorem 3.28 of [31]. □

2.9. **A polar line grassmannian and a hexagon.** Using Lemma 2.8.4, one shows from the axioms that Γ_1 and Γ_4 are dual to each other in the sense of Remark 2.4.7. We will not do this explicitly, as we already proved all necessary ingredients. This now allows and motivates the following terminology.

Definition 2.9.1. Let ξ_1, ξ_2 be two symplecta of a metasymplectic space, then we call ξ_1 and ξ_2

(0) *equal* if $\xi_1 = \xi_2$;
(1) *collinear* if the intersection $\xi_1 \cap \xi_2$ is a plane;
(2) *symplectic* if the intersection $\xi_1 \cap \xi_2$ is a point;

(3) *special* if the intersection $\xi_1 \cap \xi_2$ is empty and there is unique symplecton ζ intersecting both in a plane;

(4) *opposite* if the intersection $\xi_1 \cap \xi_2$ is empty and there is no symplecton intersecting both in a plane.

The duality between Γ_1 and Γ_4 makes it possible to define some more geometries embedded in metasymplectic spaces. These will be used later on.

We start by embedding the line grassmannian of the extended equator geometry into Γ_1. Therefor, we have to prove some properties of mutual positions of hyperbolic lines of an extended equator geometry of Γ_4. However, we will also need the corresponding properties of Γ_1, so we initially phrase it more generally in the next lemma.

Lemma 2.9.2. *Let p, q be opposite points in a metasymplectic space Γ_i. Let L, M be two "lines" of $E(p, q)$ and let ξ, ζ be the symplecta containing L, M, respectively. Then:*

(i) *L and M are contained in a plane of $E(p, q)$ if, and only if, ξ and ζ are collinear;*

(ii) *L and M intersect in a point, but are not coplanar in the polar space $E(p, q)$ if, and only if, ξ and ζ are symplectic;*

(iii) *L and M are disjoint but not opposite in the polar space $E(p, q)$ if, and only if, ξ and ζ are special;*

(iv) *L and M are opposite in the polar space $E(p, q)$ if, and only if, ξ and ζ are opposite.*

Proof. (i) Suppose L and M lie in a "plane" of $E(p, q)$ and denote by x the intersection of both lines. Let m be a point of $M \setminus \{x\}$. Then all points of L are collinear to the line $m^\perp \cap \xi$ and consequently this line is also contained in $x^\perp \cap m^\perp \subseteq \zeta$. So there is at least a line contained in the intersection $\xi \cap \zeta$. By Definition 2.9.1, we see that this means indeed that the symplecta are collinear.

Suppose now that the symplecta ξ and ζ intersect in a plane. Then it is clear that every point of L is close to ζ and by the possible relations between points in $E(p, q)$, every point of L must be symplectic to every point of M. So L and M are contained in a "singular subspace" of $E(p, q)$, which is a "plane" by the rank of that polar space.

(iv) First suppose that ξ and ζ are opposite symplecta of Γ_i. Then each point of ξ is symplectic to a unique point of ζ. In particular, no point of L can be symplectic (or collinear in the polar space $E(p, q)$) to all points of M. Hence L and M are opposite in $E(p, q)$.

Now assume that L and M are opposite lines in $E(p, q)$. We consider the possible relations between symplecta, taking Lemma 2.8.4 into account. The symplecta ξ and ζ do not meet in a plane as otherwise no point of ξ is opposite any point of ζ. Suppose that they meet in a point s. Points $x \in \xi$ and $y \in \zeta$ are opposite if, and only if, $x \not\perp s \not\perp y$, by the possible point-symp relations. So given that each point of L is opposite some point of M, we deduce that all points of L are opposite all points of M, a contradiction. Now suppose that there is a symplecton ω intersecting ξ in a plane α and ζ in a plane β, with ξ and ζ disjoint. Since each point of L is opposite some point of M, all points of L belong to $\xi \setminus \alpha$ and likewise $M \subseteq \zeta \setminus \beta$. But then, again, there are no symplectic point pairs from $L \times M$ since the unique point of ζ symplectic to a point of $\xi \setminus \alpha$ is contained in β and we have again a contradiction. Hence ξ is opposite ζ.

(ii) Suppose L and M intersect in a point, but are not contained in a "plane". Then the symplecta ζ and ξ must at least contain this intersection point and by (i), this is exactly their intersection, which means that the symplecta are symplectic.

Suppose now ξ and ζ intersect in a point x. By (iv), there must be a point $l \in L$ which is symplectic to all points of M and a point $m \in M$ which is symplectic to all points of

L. It is clear that l is contained in or close to ζ; we will exclude the latter, which implies $l = x$. Suppose for a contradiction that $l \neq x$, then $l \perp\!\!\!\perp x$ or $l \perp x$. Suppose first that $l \perp\!\!\!\perp x$, then the line $l^{\perp} \cap \zeta$ must be collinear to x and so the intersection $\xi \cap \zeta$ contains a plane, contradicting (i). Suppose now that $l \perp x$, then every point of M must be collinear to x and consequently close to ξ, implying that every point of M must be symplectic to every point of L, again contradicting (i). Similarly it follows that $m = x$, which proves the statement.

(iii) This follows by elimination and the previous cases. □

The previous lemma can be adapted to extended equator geometries.

Corollary 2.9.3. *Let p, q be opposite points in a metasymplectic space Γ_4. Let L and M be two lines of the polar space $\widehat{E}(p,q)$ and let ξ, ζ be the corresponding symplecta containing L and M respectively. Then:*

- (i) *L and M are contained in a plane of $\widehat{E}(p,q)$ if, and only if, ξ and ζ are collinear;*
- (ii) *L and M either intersect in a point, but are not coplanar in the polar space $\widehat{E}(p,q)$, or are contained in a "solid", but don't intersect, if, and only if, ξ and ζ are symplectic;*
- (iii) *L and M are disjoint but not opposite in the polar space $\widehat{E}(p,q)$ if, and only if, ξ and ζ are special;*
- (iv) *L and M are opposite in the polar space $\widehat{E}(p,q)$ if, and only if, ξ and ζ are opposite.*

Proof. By Lemma 2.9.2 it suffices to prove that every pair of lines of $\widehat{E}(p,q)$ is embedded in a common equator geometry $E(a,b)$, except when the lines are contained in a common "solid" but not in a common "plane" and that in this particular case the corresponding symplecta are symplectic.

If L and M span a "plane" π of $\widehat{E}(p,q)$, then we can take two different maximal singular subspace through this submaximal singular subspace, that give rise to two opposite points a, b symplectic to every point of π. Suppose now that L and M intersect in a point, but are not contained in a "plane". Let then α and β be two locally opposite "planes" through L. Define now a, b as points on the respective projections of M onto α, β not on L. Subsequently, let L and M be nor contained in a solid, nor opposite in $\widehat{E}(p,q)$ and denote by l and m the respective points of L and M symplectic to all points of the other line. Consider now opposite points $l' \in L \setminus \{l\}$ and $m' \in M \setminus \{m\}$. Then in the rank 3 polar space $m'^{\perp\!\!\!\perp} \cap l'^{\perp\!\!\!\perp} = E(l', m')$ we can consider two locally opposite planes through lm giving rise to opposite points a and b symplectic to all points of L and M. Suppose finally that L and M are opposite in $\widehat{E}(p,q)$. Then one can choose two opposite points in the rank 2 polar space $L^{\perp\!\!\!\perp} \cap M^{\perp\!\!\!\perp}$.

Let now L and M be two lines contained in a common "solid" but not in a common "plane". Then it is clear that the symplecta ξ and ζ cannot be disjoint as they contain both the point of $\widehat{T}(p,q)$ corresponding to this "solid". So we may suppose for a contradiction that ξ and ζ are collinear, and denote $\pi := \xi \cap \zeta$. Then, since every pair of points of $L \cup M$ is symplectic, all points of L and M are collinear to the same line K of π. Remark now that every point y in the "solid" spanned by L and M, must lie in a symplecton through this line K, as a solid is spanned by two opposite lines and each symplecton through a point $l \in L$ and a point $m \in M$ contains clearly the line K. Similar as above, y must now be collinear to K, but then each point of the line K is collinear to the same hyperbolic solid of $\widehat{E}(p,q)$, contradicting Lemma 2.7.5. By elimination we now see that ξ and ζ are symplectic in this case. □

Now we will see that the previous lemma and corollary allow us to embed some more geometries in a metasymplectic space.

Definition 2.9.4. The polar (line) Grassmanian $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$ is the point-line geometry with point set the lines of the polar space $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$ and line set the planar line pencils of $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$ (i.e. all the lines in a certain plane π through a certain point $v \in \pi$; v is called the *vertex* of the pencil; π is called the *base plane*).

Lemma 2.9.5. *The polar line Grassmanian* $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$ *is a parapolar space of diameter* 3 *and with uniform symplectic rank* 3.

Proof. See Paragraph 17.1.1 of [24]. □

Points of $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$ at distance 3 shall be called *opposite*; they indeed correspond to opposite lines of $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$.

Proposition 2.9.6. *All points of any two arbitrary opposite symplecta of* Γ_1 *are contained in a subspace* Ω *of* Γ_1 *which, viewed as a point-line geometry, is isomorphic to* $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$ *enjoying the following property:*

(Isom) *Two points of* Ω *are collinear, symplectic, special or opposite in* Ω *if, and only if, they are collinear, symplectic, special or opposite, respectively, in* Γ_1.

Proof. Let, under the natural duality between Γ_1 and Γ_4, the two given opposite symplecta of Γ_1 correspond to the two opposite points p,q of Γ_4. Then let Ω be the set of points of Γ_1 corresponding (under the natural duality) to the symplecta of Γ_4 intersecting $\widehat{E}(p,q)$ nontrivially. Note that Lemma 2.6.18 implies that these symplecta intersect $\widehat{E}(p,q)$ in hyperbolic lines and that by definition all points of the the given opposite symplecta belong to Ω. We claim that Ω is a subspace. Indeed, let ξ and ζ be two collinear symplecta intersecting $\widehat{E}(p,q)$ nontrivially. Then by Corollary 2.9.3, the intersection contains a point of $\widehat{E}(p,q)$, and hence all symplecta containing $\xi \cap \zeta$ intersect $\widehat{E}(p,q)$ nontrivially, proving the claim. Now, identifying a symp of Γ_4 intersecting $\widehat{E}(p,q)$ in a hyperbolic line with that hyperbolic line, all assertions follow from Corollary 2.9.3. □

Let Δ be a parapolar space of diameter 3, and call points at distance 3 oppposite. A subspace Ω, structured as a geometry where pairs of points are either collinear, symplectic, special or opposite, enjoying Property (Isom) (with Γ_1 replaced with Δ) shall be referred to as an *isometric* subspace.

Definition 2.9.7. The generalised hexagon $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ is the point-line geometry with point set the flags of $\mathsf{PG}(2,\mathbb{K})$ (that is, the point-line pairs (p,L) with $p \in L$), where a typical line is the set of flags containing a fixed point or a fixed line. Two points of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ will be called *special* or *opposite* if their distance is 2 or 3, respectively. When working with $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$, we often denote the projective plane $\mathsf{PG}(2,\mathbb{K})$ by $\mathsf{A}_{2,1}(\mathbb{K})$ and the dual by $\mathsf{A}_{2,2}(\mathbb{K})$, where we tacitly identify a point of $\mathsf{A}_{2,1}(\mathbb{K})$ with the line of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ consisting of all flags containing that point, and similarly for the lines of $\mathsf{A}_{2,1}(\mathbb{K})$ and the points and lines of $\mathsf{A}_{2,2}(\mathbb{K})$.

By now proving that $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ can be embedded in the geometry $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$, we can conclude that $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ can also be embedded in $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$.

Lemma 2.9.8. *Two opposite lines of* $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$ *are contained in a unique isometric subspace isomorphic to* $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$.

Proof. Let the two given opposite lines be given as the planar line pencils of $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$ with vertices u,v and base planes π,ω, respectively. Then the planes $\alpha:=\langle u,u^\perp\cap\omega\rangle$ and $\beta:=\langle v,v^\perp\cap\pi\rangle$ of $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$ are opposite. Let $\mathscr{P}$ be the set of points of $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$, viewed as set of lines of $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$, consisting of those lines that intersect both α and β nontrivially (that is, in respective points). It is an elementary exercise in polar spaces to show that $\mathscr{P}$ is a subspace of $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$ isomorphic to $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$. Obviously, $\mathscr{P}$ contains each line through u or v in the plane π or ω, respectively.

Next we show that $\mathscr{P}$ is isometric. Since collinearity is preserved, we only have to show that being special and being opposite is preserved. So suppose that $K_1,K_2\in\mathscr{P}$ are special in $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$. Then, without loss of generality, there is a line K with $K\cap\alpha=K_1\cap\alpha$ and $K\cap\beta=K_2\cap\beta$. Clearly K_1 and K_2 are disjoint, and if they were contained in a singular 3-space, then the line $\langle K\cap\alpha,K_2\cap\alpha\rangle$ would be collinear to the line $\langle K\cap\beta,K_1\cap\beta\rangle$, contradicting opposition of α and β (indeed, a point of $\alpha\setminus K_2^\perp$ is collinear to some point of the line $K_1^\perp\cap\beta$, which would then be collinear to all points of α). Hence K_1 and K_2 are also special in $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$. Conversely, suppose $K_1,K_2\in\mathscr{P}$ are special in $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$. Then there is a unique point $x_i\in K_i$ collinear to all the points of K_j with $\{i,j\}=\{1,2\}$. If $x_1\in\beta$, then $x_1\perp q_2$ with $q_2:=K_2\cap\alpha$. Now $q_2=x_2$ as it is collinear to the two different points x_1 and $q_1:=K_1\cap\alpha$ of K_1. Then K_1 and K_2 are special in $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$. So we may assume that $x_1\notin\beta$. But then two different points of K_1 are collinear to $K_2\cap\beta$ and consequently the latter point is collinear to K_1. This leads similarly to special points in $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$. Opposition is now also preserved by elimination and the previous arguments.

Left to show is uniqueness. Let $\mathscr{P}'$ be a second isometric subspace containing the two given opposite lines. Since the subspace is isometric, it is closed under taking the centre of each special pair. This implies that the set $\mathscr{P}\cap\mathscr{P}'$ defines a subplane of $\mathsf{A}_{2,1}(\mathbb{K})$ containing a point and a line not through that point, and such that, if a point x belongs to it, then also all lines of $\mathsf{A}_{2,1}(\mathbb{K})$ through x, and similarly for the lines in that intersection. It readily follows that $\mathscr{P}=\mathscr{P}\cap\mathscr{P}'=\mathscr{P}'$.

The lemma is proved. □

Corollary 2.9.9. *Two arbitrary opposite lines K_1,K_2 of $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ are contained in a unique subspace isometric and isomorphic to $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$.*

Proof. Since two planes in Γ_4 contain a pair of opposite points, by duality two opposite lines of Γ_1 are contained in opposite symplecta. Now Proposition 2.9.6 and Lemma 2.9.8 yield existence a subspace isometric and isomorphic to $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ containing the two given opposite lines. Uniqueness then follows similarly as in the last part of the proof of Lemma 2.9.8. □

2.10. **Imaginary lines.** The next lemmas are beautiful examples of how these embeddings can be used to prove properties of metasymplectic spaces. First some terminology: two opposite lines L,M of a nondegenerate quadric define a unique set of lines intersecting all lines that intersect both L and M; it is a so-called regulus of the hyperbolic quadric spanned by L and M in the ambient projective space. We hence call this set the *regulus (defined by L and M)*. The set of lines intersecting each member of that regulus is called the *opposite regulus (defined by L and M)* and is indeed a regulus itself.

Lemma 2.10.1. *Let p,q be two opposite points in Γ_1 and denote by L and M two opposite lines having a point collinear to both p and q. Denote by $\mathscr{I}_{L,M}$ the set of points x such that $|x^\perp\cap(L\cup M)|=2$. Then $\mathscr{I}_{L,M}$ only depends on p and q.*

Proof. Let L_p^*,M_p^* and L_q^*,M_q^* be the unique lines through p and q, respectively, intersecting the respective lines L,M. Let π be an arbitrary plane through M_p^*. Let ξ_p be a symplecton containing

π. Then we can find (as before by considering the standard duality) a symplecton ξ_q containing L_q^* opposite ξ_p. Proposition 2.9.6 yields an isometric subspace Ω isomorphic to $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$ containing (all points of) ξ_p and ξ_q.

In the corresponding polar space $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$, the points p and q are represented by lines A_p and A_q. The lines L and M are represented by planar line pencils $P_{x,\alpha}$ and $P_{y,\beta}$ with (opposite) vertices x, y and (opposite) base planes α and β, respectively. Each member K of $P_{x,\alpha}$ is not opposite a unique member $N \in P_{y,\beta}$ and the unique line intersecting both K and N obviously intersects both lines $x^\perp \cap \beta$ and $y^\perp \cap \alpha$. Conversely, each line intersecting both latter lines also intersects a member of $P_{x,\alpha}$ and one of $P_{y,\beta}$. We conclude that $\mathscr{I}_{L,M}$ corresponds to the regulus $\mathscr{R}$ defined by A_p and A_q, and hence is independent of L and M. In particular, the set $\mathscr{I}_{L,M}$ coincides with $\mathscr{I}_{L,M'}$, for each line M' intersecting π, opposite L and containing a point collinear to q. Since π was arbitrary, this is true for every line M' opposite L such that $p^\perp \cap M$ and $p^\perp \cap M'$ are collinear, and $q^\perp \cap M'$ is nonempty.

The lines through p form the point set of $\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$, and locally opposite lines correspond to opposite points therein. The geometry of points opposite a given point in $\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$ is connected (as follows, using standard arguments, from the connectivity of the so-called *opposite-point geometries* of generalised quadrangles, see Remark 1.7.14 of [34]). Hence $\mathscr{I}_{L,M}$ is independent of M. Now let L_0, M_0 be two arbitrary opposite lines with the property stated in the lemma. Let L_0^* and M_0^* be the lines through p intersecting L_0 and M_0, respectively. Then there exists a line R^* through p locally opposite both L_0^* and L^*, by Proposition 3.30 of [31] (alternatively, this is an easy exercise in (dual) polar spaces). Let R be the unique line concurrent with R^* and containing a point collinear to q. Then $\mathscr{I}_{R,L} = \mathscr{I}_{R,L_0}$ by the foregoing. Similarly, $\mathscr{I}_{L_0,R} = \mathscr{I}_{L_0,M_0}$ and $\mathscr{I}_{L,R} = I_{L,M}$. It follows that $\mathscr{I}_{L_0,M_0} = \mathscr{I}_{L,M}$, which completes the proof of the lemma. $\square$

We now can introduce a rather important definition.

Definition 2.10.2. For opposite points p, q of Γ_1 we denote the set of points x such that $x^\perp$ intersects each line L with $p^\perp \cap L \neq \emptyset \neq q^\perp \cap L$ by $\mathscr{I}(p,q)$ and call it the *imaginary line (through p and q)*. We also say it is *determined by p and q*.

We record an immediate consequence of this definition and the second paragraph of the proof of Lemma 2.10.1.

Corollary 2.10.3. *Let p, q be two opposite points of Γ_1 and let ξ, ζ be the corresponding symplecta in Γ_4, using the standard duality. Let $a \in \xi$ and $b \in \zeta$ be opposite points of Γ_4. Then the image under the standard duality of $\mathscr{I}(p,q)$ is the set of symplecta corresponding to the regulus of $E(a,b)$ defined by the hyperbolic lines $\xi \cap \widehat{E}(a,b)$ and $\zeta \cap \widehat{E}(a,b)$.* $\square$

Lemma 2.10.4. *Let p, q be two opposite points in Γ_1. Then every member of $\mathscr{I}(p,q)$ is symplectic to every point of $E(p,q)$. In other words, $E(p^*,q^*) = E(p,q)$ for every pair of distinct points p^* and q^* of $\mathscr{I}(p,q)$.*

Proof. Let a be an arbitrary point of $E(p,q)$. Let M^* be a line through p in $\xi(p,a)$ and let M be the unique line intersecting this line and containing a point collinear to q. Also, let M' be the line through q intersecting M. Then we can take a line L^* through p locally opposite M^* giving similarly rise to a line L opposite M having a point collinear to q. Now a is collinear with M, as it must be collinear to a point of M^* in $\xi(a,p)$ and this can only be $M \cap M^*$ (since all the other points are opposite q) and similarly it must be collinear to $M \cap M'$. Hence there is a plane α containing a and M.

Now exactly as in the first paragraph of the proof of Lemma 2.10.1 we find an isometric subspace Ω isomorphic to $\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$ containing α and L, and hence also $\mathscr{I}(p,q)$ (by its very definition based on Lemma 2.10.1). Let p,q and a correspond to the lines K_p, K_q and K_a, respectively, of $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$. Then a being symplectic to both p and q implies by Corollary 2.9.3 that either K_a and K_p is contained in a singular 3-space, and similarly for K_a and K_q, or K_a belongs to the opposite regulus defined by K_p and K_q. In both cases K_a is obviously symplectic to each member of the regulus defined by K_p and K_q, and, as we know, this regulus corresponds to $\mathscr{I}(p,q)$. This completes the proof of the lemma. □

We now come to a beautiful geometric characterization of the imaginary lines. Recall that, for two opposite points p and q of Γ_1, the equator geometry $E(p,q)$ is isomorphic to the polar space $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ and as such admits nontrivial "hyperbolic lines", indeed between quotes as to avoid confusion with the hyperbolic lines of Γ_4 which consist of points in a symplecton, whereas now the points of a "hyperbolic line" are mutually opposite.

Proposition 2.10.5. *Let p,q be two opposite points of Γ_1 and let $a,b \in E(p,q)$ also be opposite, but for the rest arbitrary. Then $\mathscr{I}(p,q) = E(p,q)^{\perp\!\!\perp} = \{p,q\}^{\perp\!\!\perp\perp\!\!\perp}$. Also, $\mathscr{I}(p,q)$ coincides with the "hyperbolic line" of $E(a,b)$ defined by p and q.*

Proof. For ease of notation, we set $A = \{p,q\}^{\perp\!\!\perp\perp\!\!\perp}$, $B = \mathscr{I}(p,q)$ and C is the "hyperbolic line" defined by p and q in $E(a,b)$.

We first assume that we are in the inseparable case. Then the extented equator geometry $\widehat{E}(p,q)$ exists.

By the definition of a hyperbolic line (Definition 2.6.8) and the fact that $E(p,q) \subseteq \widehat{E}(p,q)$, we already conclude $A = C$. Also, by Lemma 2.10.4 we already have $B \subseteq A$. We now prove $A \subseteq B$.

Let $z \in E(p,q)^{\perp\!\!\perp}$ be a point. Then $z \in \widehat{E}(p,q)$. Pick any line N^* through p and let N be the line intersecting N^* (say, in the point p') and containing a point collinear q' collinear to q. Considering a symplecton through N^*, we find that $p' \in \widehat{T}(p,q)$. Similarly $q' \in \widehat{T}(p,q)$. hence $N \subseteq \widehat{T}(p,q)$. Let π be the "plane" of $\widehat{E}(p,q)$ all points of which are collinear to N. All points of π are clearly symplectic to both p and q, hence, since $z \in \widehat{E}(p,q)$ is symplectic to all points of $E(p,q)$, it is symplectic to all points of π and lies in a "solid" together with π. Now Proposition 2.7.6 implies that all points of that "solid" are collinear to some point of N. Hence z is collinear to some point of N. By the arbitrariness of N, we conclude that $z \in B$, which concludes the proof in this case.

Secondly, assume we are in the separable case. Since we do not have an extended equator geometry to our disposal now in Γ_1, the proof is slightly more technical. We again first show that $A = B$. By Lemma 2.10.4 we already have $B \subseteq A$. We now prove $A \subseteq B$.

So let $x \in A$ be a point and let L be a line with $p^\perp \cap L \neq \emptyset \neq q^\perp \cap L$. Let π be an arbitrary plane containing p and $p^\perp \cap L$. This plane corresponds to a "line" h in $E(p,q)$. If we denote by ξ the symplecton containing h, then $h = K^\perp \cap J^\perp$, with $K = \xi \cap p^\perp \subseteq \pi$ and $J = \xi \cap q^\perp$. The lines K and J contain the respective points $p^\perp \cap L$ and $q^\perp \cap L$, as the lines K and J consist of the points in $\langle p, K\rangle$ and $\langle q, J\rangle$, respectively, which are collinear to a point of the other plane. Suppose now first for a contradiction that $x \in \xi$. Then a point $y \in E(p,q) \setminus h$ opposite some point $z \in h$ must be close to ξ as it is symplectic to two points of ξ (x and some point of h), contradicting the opposition to $z \in \xi$. Hence, as x is symplectic to each point of h, the set $x^\perp \cap \xi$ is a line which is contained in $h^\perp$. As $\xi \cong \mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ separable, $h^\perp$ is a grid (a hyperbolic quadric in a 3-dimensional projective space), spanned by K and J. Now there are two possibilities: $x^\perp \cap \xi$ is a line intersecting L (since L is contained in that grid) or $x^\perp \cap \xi$ is a line intersecting K and J. We eliminate the latter, which proves the assertion. Suppose $x^\perp \cap K$ is a point $p' \perp p$ and $x^\perp \cap J$ is a point $q' \perp q$. Select some point $p'' \in K \setminus p'$ and let $q'' \in J$ be collinear to p'' (which differs obviously from q'). Let now $z \notin h$ be a point in the "plane" through h corresponding to $p''q''$ (so each point of that "plane" is the intersection of a symplecton through pp' and one through qq'). Then z is a point in $E(p,q) \setminus h$ collinear to $p''q''$, but by Corollary 2.5.4, x is then opposite z contradicting the fact that $x \perp\!\!\!\perp z$ by assumption. Hence $A = B$.

In $E(a,b)$, the "hyperbolic line" through p and q is by definition the set of points symplectic to all points symplectic to both p and q. The latter set belongs to $E(p,q)$ and hence $A \subseteq C$. We now show that $C \subseteq A$, which will prove the proposition.

Let $x \in C$ be arbitrary. Then x is symplectic to each point of $E(p,q) \cap E(a,b)$, and to a and b. We claim that, if x is symplectic to a point $u \in E(p,q)$ and to all points of two "lines" h_1, h_2, which are themselves symplectic to u, intersect in a unique point w and are not contained in a common "plane", then x is symplectic to all points of the "line" uw. Let ξ_i be the symplecton containing h_i, $i = 1,2$. Let $p^\perp \cap \xi(u,w) = L$ and $q^\perp \cap \xi(u,w) = M$. Also, set $L_i = p^\perp \cap \xi_i$ and $M_i = q^\perp \cap \xi_i$, $i = 1,2$. Then the assumptions imply that L_1 and L_2 intersect L in distinct points a_1, a_2. Also, as before, $x^\perp \cap \xi_i$ is contained in the grid defined by L_i and M_i, and hence intersects two distinct lines of the grid defined by L and M. This implies that $x^\perp \cap \xi$ is a line of the grid defined by L and M and hence belongs to $(uw)^\perp$. The claim follows.

Applying the previous paragraph to $u = a$ and h_1, h_2 two intersecting "lines" in $E(p,q) \cap E(a,b)$, we deduce that each point of $E(p,q)$ symplectic to a (and similarly to b) belongs to $x^{\perp\!\!\!\perp}$. Now let v be an arbitrary point of $E(p,q)$ not in $a^{\perp\!\!\!\perp} \cup b^{\perp\!\!\!\perp}$, and not in the hyperbolic line defined by a and b. Then there is a "plane" α_1 through v intersecting $E(a,b) \cap E(p,q)$ in a unique point v'. Then $\alpha_1 \cap a^{\perp\!\!\!\perp}$ and $\alpha_1 \cap b^{\perp\!\!\!\perp}$ are two distinct "lines" k_1, k_2 intersecting in v'. Pick a point $u' \in k_2 \setminus \{v'\}$, and choose a plane α_2 through vu' distinct from α_1. Then $b^{\perp\!\!\!\perp} \cap \alpha_2$ is a line through u'. By our claim above, x is symplectic to v. Hence x is symplectic to all points of $E(p,q)$, except possibly the hyperbolic line through a and b. But that now also easily follows. □

2.11. **Chambers and apartments; domesticity.** Finally we need some results that come from Tits' theory of spherical buildings, since we prove existence of the domestic collineations using that theory.

Definition 2.11.1. A *chamber of a metasymplectic space* is a set $\{p, L, \alpha, \xi\}$, with p a point, L a line, α a plane and ξ a symplecton, satisfying $p \in L \subseteq \alpha \subseteq \xi$. A *flag* (of a metasymplectic space) is a subset of a chamber.

Definition 2.11.2. A *panel of a metasymplectic space* is the set of all the elements which can be added to a flag, consisting of all the elements of a chamber except one, to form a chamber.

Definition 2.11.3. An *apartment of a metasymplectic space* is an isometrically embedded thin metasymplectic space, i.e. a metasymplectic space where every panel has only two elements.

We assume that the reader is familiar with apartments of polar spaces of rank n. These consist of the singular subspaces generated by the points of *skeleton*, that is a set of $2n$ points such that each point of that set has a unique opposite in that set.

An important (defining) property of spherical buildings, and hence of metasymplectic spaces, is that every pair of chambers is contained in an apartment, which is unique as soon as the chambers are opposite.

Next to these general properties of apartments in buildings, we will also use the following two lemmas, the first of which is specific for apartments in metasymplectic spaces.

Lemma 2.11.4. *Let p, q be two opposite points of a metasymplectic space Γ_i. If Λ' is an apartment of the equator geometry $E(p, q)$, then p, q and Λ' are contained in a unique apartment Λ of Γ_i.*

Proof. Let $x_1, x_2, x_3, y_1, y_2, y_3$, with x_i opposite y_i, $i = 1, 2, 3$, be the skeleton of Λ'. Then these points span eight "planes" in $E(p, q)$. Each such plane α corresponds to a line L_α through p and a line M_α through q. Denote by p_α the unique point of L_α special to q, and similarly by q_α the unique point of M_α special to p. Now we determine a unique apartment by the set $A := \{p, x_1, \ldots, y_3, q\}$ as follows. Let $C = \{p, L, \pi, \xi\}$ be the chamber consisting of the point p, the line $L := pp_\alpha$, the plane $\pi := \langle p, p_\alpha, p_\beta \rangle$ and the symplecton $\xi := \xi(p, x_1)$ (where $\alpha := \langle x_1, x_2, x_3 \rangle$ and $\beta := \langle x_1, x_2, y_3 \rangle$) and let $C' = \{q, L', \pi', \xi'\}$ be the chamber consisting of the point q, the line $L' := qq_{\alpha'}$, the plane $\pi' := \langle q, q_{\alpha'}, p_{\beta'} \rangle$ and the symplecton $\xi' := \xi(p, y_1)$ (where $\alpha := \langle y_1, y_2, y_3 \rangle$ and $\beta := \langle y_1, y_2, x_3 \rangle$). These chambers are clearly opposite and hence they determine a unique apartment (which must be included in an apartment spanned by A.) So it suffices to prove that A is contained in this apartment. By projecting the chambers to each other, one sees immediately that p, q, x_1, y_1 are contained in the apartment. As also the "lines" x_1x_2 and y_1y_2 corresponding to π and π', respectively, must be contained in the apartment, also the projection y_2 of x_1 onto y_1y_2 is contained in the apartment (and similarly also x_2). Projecting these "lines" on the "planes" α' and α, respectively, gives that also x_3, y_3 are contained in the apartment, which concludes the proof. □

Lemma 2.11.5. *Given some point p and some apartment Λ of a metasymplectic space Γ_i. Then there exists a point $p' \in \Lambda$ opposite p.*

Proof. This follows from the fact that every chamber outside a given apartment has at least two opposite chambers inside the apartment, see Proposition 3 in [32]. However, the interested reader can easily prove this statement only using the axioms of a metasymplectic space. □

We can now define domesticity. We start very general, but then restrict ourselves to polar spaces and metasymplectic spaces.

Definition 2.11.6. (*i*) A *domestic automorphism* of a building is an automorphism that does not map any chamber to an opposite one.

(*ii*) A collineation of a polar or metasymplectic space that does not map an object of type $*$ to an opposite is called a $*$-*domestic* collineation. This in particular applies to $* \in \{$point, line, plane, solid, symplecton$\}$.

(*iii*) A collineation of a polar or metasymplectic space is *capped* if, whenever it maps two object of types ℓ_1 and ℓ_2, respectively, to an opposite, then it maps an incident pair of objects of these respective types to an opposite.

It is shown in [16] that, whenever a building has no residue isomorphic to the projective plane with 3 points per line (that is, the so-called *Fano plane*), then any automorphism is capped. On the other hand, all domestic automorphisms of metasymplectic spaces with Fano planes are classified in [17]. Hence in the present paper we may assume that $|\mathbb{K}| > 2$ and hence that the considered collineations are capped. Then it is proved in [16] that there are three types of nontrivial domestic collineations (and no dualities) in metasymplectic spaces, and these correspond to the opposition diagrams given in Fig. 2. All possible opposition diagrams are explained in Table 2. A point-symp flag is a symplecton containing the point. We just additionally note that, for diagram $\mathsf{F}_{4;2}$, other symplecta might exist that are also mapped to an opposite, but do not contain any point mapped to an opposite, and likewise for points.

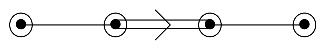
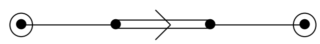
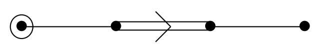
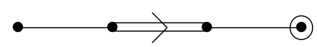
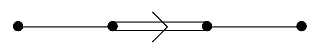
Notation	Diagram	Interpretation in Γ_1
$\mathsf{F}_{4;4}$		Some chamber is mapped to an opposite chamber. The collineation is not domestic.
$\mathsf{F}_{4;2}$		Some point-symp flag is mapped to an opposite. No line nor plane is mapped to an opposite.
$\mathsf{F}^1_{4;1}$		Some point is mapped to an opposite. No line, plane nor symp is mapped to an opposite.
$\mathsf{F}^4_{4;1}$		Some symp is mapped to an opposite. No point, line nor plane is mapped to an opposite.
$\mathsf{F}_{4;0}$		Nothing is mapped to an opposite. The collineation is the identity.

TABLE 2. Opposition diagrams of metasymplectic spaces

Remark 2.11.7. It is a general fact that, if the opposition diagram is "empty", that is, if no element is mapped onto an opposite, then the collineation is the identity. This was already proved by Leeb [11] and Abramenko & Brown [1].

We end this section with defining what we mean with central elation, long root elation, central short root elation and perpendicular central elations, so that the statement of the Main Theorem is clear.

Definition 2.11.8. (*i*) A *central elation of* Γ_i *(with centre* c*)* is a collineation that fixes the point c and stabilises all the lines that have at least one point collinear to c. The group of central elations with centre c is called the *root group with centre* c.

(*ii*) *Perpendicular* central elations of a metasymplectic space are central elations with symplectic corresponding centres.

(*iii*) A *long root elation* is a central elation in Γ_1; a *central short root elation* is a central elation in Γ_4 in Class (M).

We already note the following property of central elations (more properties are proved in Section 6):

Lemma 2.11.9. *A central elation of* Γ_i *with centre* c *fixes all points symplectic to* c*. Also,* θ *preserves each imaginary line containing* c*.*

Proof. Indeed, a point x symplectic to c is contained in at least two distinct lines that have a point collinear to c (look in $\xi(c,x)$). The second assertion follows directly from Proposition 2.10.5. □

3. SOME RESULTS IN POLAR SPACES

In this section, we prove some auxiliary results on polar spaces. As it will turn out that domestic collineations of metasymplecic spaces induce under certain circumstances domestic collineations of symplecta, equator and extended equator geometries, we also include some specific results concerning domesticity in polar spaces (and which can not be found in [19]). Also, in order to recognise or rule out certain collineations in metasymplectic spaces, we need to know something about existence and uniqueness of their counterparts in polar spaces. So this section is mainly about classes of collineations of polar spaces. However, we begin with some purely geometric properties.

Note that in this section (and the rest of the paper), we will speak about *separable polar spaces*. In the orthogonal case, these are the quadrics for which the associated polarity ρ in the ambient projective space is nondegenerate. In the Hermitian case, we will only need the polar spaces $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$, see Definition 2.3.2, with $\mathbb{A}$ not an inseparable field extension of $\mathbb{K}$.

3.1. **Two geometric lemmas.** The first lemma proves a correspondence between subspaces of an embeddable polar space and those of the underlying projective space. We only need it for rank 4, but the proof does not become simpler in this specific case, so we present the result in full generality. First some definitions.

Definition 3.1.1. A subspace S of a polar space of rank $n \geq 2$ is said to *have corank* r, $0 \leq r < n$, if every singular subspace of dimension r intersects S nontrivially, and some singular subspace of dimension $r-1$ is disjoint from S. An *ovoid* is a subspace of corank $n-1$ without lines. Equivalently, and more traditionally, it is a set of points intersecting every maximal singular subspace in a unique point.

Due to Tits' classification of polar spaces of rank at least 3, these come in two flavours: the embeddable ones and the nonembeddable ones. The latter are either related to projective 3-spaces over noncommutative skew fields, or are isomorphic to $\mathsf{C}_{3,1}(\mathbb{O},\mathbb{K})$. The former are related to so-called pseudoquadratic forms (including Hermitian and quadratic forms). For every such polar space, there exists a so-called *universal embedding*, which can be thought of as the embedding from which each other embedding is derived by projection. For instance, for orthogonal polar spaces, the universal embedding is the one realised as a quadric; for $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$, $\mathbb{A} \neq \mathbb{O}$, the universal embedding happens in $\mathsf{PG}(5,\mathbb{A})$ (see chapter 6 of [31], where this is called a *dominant embedding*).

The next lemma will be used only in the rank 3 and 4 cases, but we state and prove it for general rank.

Lemma 3.1.2. (*i*) *Let Δ be a polar space of rank r with universal embedding in the projective space Ω. Let S be a subspace of Δ such that some line of S is disjoint from some maximal singular subspace of S. Let $\langle S\rangle$ be the subspace of Ω generated by S. Then we have that $\langle S\rangle \cap \Delta = S$. Furthermore S has corank $i < r$ if, and only if, $\langle S\rangle$ has codimension i in Ω.*

(*ii*) *Let Δ be a polar space of rank r embedded in a projective space Ω. Let T be a subspace of Ω of codimension i at most $r-1$, and let S be the intersection of T with the point set of Δ. Then the corank of S in Δ is equal to the codimension of T in Ω.*

Proof. In [3] the authors prove that in Case (i) under the stated assumption, $\langle S\rangle \cap \Delta = S$. We prove the other statement and (ii) simultaneously by induction on i. Since in (ii) the codimension of T is at least the corank of S, it suffices to show indeed that $\mathrm{codim}(\langle S\rangle)$ is equal to the corank of S (with generation in Ω). Hence, we can use the notation $\langle S\rangle$ throughout and ignore T.

If $i = 0$, a geometric subspace S of corank 0 spans by the definition of an embedding the whole space, so $\langle S\rangle = \mathsf{PG}(V)$, and is consequently a subspace of codimension 0. The converse statement is trivial in this case.

Suppose now that the statement is true for every $j \leq i$ and that S is a subspace of Δ of corank $i+1$. Then by the induction hypothesis, we may assume that $\langle S\rangle$ has codimension at least $i+1$. Suppose now that the codimension of $\langle S\rangle$ is strictly bigger than $i+1$. Let x be a point of $\Delta \setminus S$. Then $\langle S, x\rangle$ is a subspace of codimension at least $i+1$, so by the induction hypothesis it is impossible that $S' := \langle S, x\rangle \cap \Delta$ has corank strictly smaller than $i+1$. Let U be an arbitrary singular subspace of Δ of dimension i and let V be a singular subspace of Δ of dimension $i+1$ containing U. Then by the assumption that S is a geometric subspace of corank $i+1$, there is a point y contained in the intersection $S \cap V$.

If $x \in U$, then clearly $U \cap S'$ is nonempty. However if $x \notin U$ and $U \subseteq x^{\perp}$, then $\langle U, x\rangle$ has dimension $i+1$ and intersects S in at least a point s. Now S' intersects $\langle U, x\rangle$ in at least the line xs and again $U \cap S'$ is nonempty. We now prove that this is also the case if $U \nsubseteq x^{\perp}$.

If x is not collinear to y, one can look at the projection of x onto V, spanning a singular subspace of dimension $i+1$ together with x, to get a point x' collinear to x contained in S. Now the line xx' is contained in S', intersecting V in y'. The line yy' is similarly contained in S', but also in V and contains consequently a point of $U \cap S'$.

If x is collinear to y, we look at the space W spanned by x and its projection on U. This space has dimension i and y corresponds to an $i+1$-space through it. So in the residue of W we can take a point opposite to the point corresponding to y. This point corresponds to an $i+1$-space W' through W for which the set of points collinear to y is exactly W. Now W' has a point $\tilde{y}$ in S by assumption on S. If $\tilde{y}$ is contained in W, then the line $x\tilde{y}$ is contained in $W \cap S'$ and intersects the hyperplane $U \cap W$ of W in a point. So U contains a point of S'. If $\tilde{y}$ is not contained in W, we can replace x by $\tilde{x} \in x\tilde{y} \setminus \{x, \tilde{y}\}$ and apply the previous paragraph as now $\tilde{x}$ is not collinear to y.

In every case an arbitrary singular subspace of Δ of dimension i intersects S', so S' has corank at most i, which is a contradiction.

Suppose now that $\langle S\rangle$ is a subspace of Ω of codimension i. Then of course every subspace of dimension $i+1$ intersects $\langle S\rangle$, and in particular every singular subspace of Δ of dimension $i+1$ intersects S. The corank of S is consequently at most $i+1$ and by the induction hypothesis it is also at least $i+1$, so the corank of S is exactly $i+1$. □

There is also a counterpart of this lemma for the inseparable case.

Lemma 3.1.3. *Let $\mathscr{O}$ be an ovoid of a polar space Δ of rank $r \geq 3$. Then there is no point $x \in \Delta$ collinear to all the points of $\mathscr{O}$.*

Proof. Suppose every point of $\mathscr{O}$ is collinear to $x \in \Delta$, then x is clearly not contained in $\mathscr{O}$. Let M be an arbitrary maximal singular subspace through x and denote by f the point of $\mathscr{O}$ in M. Let U be a hyperplane in M not through x nor f and let M' be a maximal singular subspace containing U but distinct from M. Then by our hypothesis, the point of $\mathscr{O}$ in M' must be contained in U. But then M would contain two points of $\mathscr{O}$, a contradiction. □

3.2. **Central and axial elations.**

The first type of collineations we will discuss here are the so called central (axial) elations or collineations. The basic idea is that they fix everything "close" to something.

Definition 3.2.1. A *central elation of a polar space* with *centre c* is a collineation that fixes the point c and all points collinear to c. Two central elations are called *perpendicular* if their respective centres are distinct but collinear.

Lemma 3.2.2. *Let θ be a central elation of a polar space Δ of rank $r \geq 3$ with centre c fixing a point q not collinear to c. Then θ is the identity.*

Proof. First we claim that the set of fixed points is a subspace. Indeed, let x and y be two collinear fixed points; we may assume that the line xy is not collinear to c. Then we may assume that $x \perp c$. A line L through c locally opposite cx is obviously opposite xy and so each point of L is collinear to a unique point of xy, which is fixed. The claim is proved. Now $c^\perp$ is a maximal geometric hyperplane and so the unique subspace containing c and q is the whole polar space. □

This lemma has a quite strong consequence for general collineations, which we will use regularly.

Corollary 3.2.3. *Let ξ be a polar space of rank at least 3. Let p, q be two noncollinear points. Let θ be a collineation of ξ that pointwise fixes $(p^\perp \cap q^\perp) \cup \{p, q\}$ and an additional point $x \in p^\perp \setminus q^\perp$. Then θ is the identity.*

Proof. Every plane through px is pointwise fixed as it contains a pointwise fixed line and two points not contained in this line. By the connectivity of the residue of p, we see similarly that $p^\perp$ is pointwise fixed and so θ is a central elation of ξ with centre p fixing the point q not collinear to p. Then, by Lemma 3.2.2, θ is the identity. □

Lemma 3.2.4. *Let Δ be a separable orthogonal polar space of rank $r \geq 2$. Then there are no nontrivial central elations in Δ.*

Proof. Suppose θ is a central elation in Δ. Denote by ρ the defining nondegenerate polarity. Then $c^\perp$ spans c^ρ and so c^ρ is pointwise fixed. Dually every hyperplane through c is stabilised and so every line through c is stabilised. Now an arbitray point p not collinear to c is also fixed as the line cp in the underlying projective space is stabilised and intersects the quadric only in c and p. □

Now we take a closer look at axial elations in these polar spaces.

Definition 3.2.5. An *axial elation of a polar space* with *axis L* is a collineation that stabilises the line L and all lines intersecting L. Two axial elations are called *perpendicular* if their axes either intersect but are not coplanar, or are collinear but do not intersect.

This definition has immediately an equivalent formulation given in the next corollary.

Corollary 3.2.6. *A collineation of a polar space is an axial elation with axis L if, and only if, it pointwise fixes L and all points collinear to L and maps any other point p to a collinear point q such that the line pq intersects L nontrivially.*

Lemma 3.2.7. *Let Δ be a separable orthogonal polar space of rank $r \geq 3$ and let θ be a collineation that fixes pointwise some line L and all points collinear to L. Then θ is an axial elation of Δ with axis L.*

Proof. By Corollary 3.2.6, it suffices to prove that every line intersecting L, but not coplanar with L, is stabilised under θ. Let M be such a line and denote by p the intersection of L and M. In the residue of p, L corresponds to a point l and θ fixes $l^\perp$ pointwise. By Lemma 3.2.4 the residue is now pointwise fixed. Hence the line M corresponding to any point m of the residue not collinear to l is stabilised. □

Lemma 3.2.8. *Let U_L be the group of axial elations of an orthogonal polar space Δ of rank $r \geq 2$ with axis L. Let M be a line intersecting L in p, but not coplanar with L. Then U_L acts sharply transitively on $M \setminus \{p\}$. In particular the only element of U_L fixing a point not collinear to L is the identity.*

Proof. We may suppose by coordinatisation over the field $\mathbb{K}$ that there exist a fixed n so that L is given by $x_1 = x_3 = x_i = 0$ for all $5 \leq i \leq n$, M is given by $x_1 = x_4 = x_i = 0$ for all $5 \leq i \leq n$ and Δ is given by $x_1x_2 + x_3x_4 + \ldots = 0$. Then every axial collineation acts trivially on the coordinates x_i for $5 \leq i \leq n$ and it is an elementary exercise to calculate that the action on the first coordinates is given by a matrix of the form A_k:

$$A_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ -k & 0 & 0 & 1 \end{pmatrix},$$

with $k \in \mathbb{K}$ arbitrary. It is clear that these matrices act sharply transitively on $M \setminus L$. □

Lemma 3.2.9. *Let Δ be a separable polar space isomorphic to $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$. Then there are no nontrivial axial elations in Δ.*

Proof. We prove this by contradiction, so assume that θ is a nontrivial axial elation of Δ. This implies with Corollary 3.2.6 that there exists a grid spanned by the axis L and some opposite line L'. We will now proof that such a grid doesn't exist in Δ.

First let Δ be a nonsplit polar space. As every quaternion and octoninion division algebra contains a quadratic Galois extension as a subalgebra, we may assume that $\mathbb{A} = \mathbb{L}$ with the notation taken from Table 1. So Δ is given by $\overline{x}_{-3}x_3 + \overline{x}_{-2}x_2 + \overline{x}_{-1}x_1 \in \mathbb{K}$. We order the coordinates of $\mathsf{PG}(5, \mathbb{L})$ according to increasing indices. Denote now:

$$\begin{aligned} L &:= \langle (0,1,0,0,0,0), (0,0,1,0,0,0) \rangle, \\ L' &:= \langle (0,0,0,1,0,0), (0,0,0,0,1,0) \rangle, \\ M &:= \langle (0,1,0,0,0,0), (0,0,0,1,0,0) \rangle, \\ M' &:= \langle (0,0,1,0,0,0), (0,0,0,0,1,0) \rangle. \end{aligned}$$

It is clear that L' is a line opposite L, with the point $(0,1,a,0,0,0)$ of L collinear to the point $(0,0,0,1,-\overline{a},0)$ of L' for every $a \in \mathbb{L}$. It is also clear that the lines M and M' are opposite, belong to the grid spanned by L and L' and the point $(0,1,0,b,0,0)$ of M is collinear to the point $(0,0,1,0,-\overline{b},0)$ of M' for every $b \in \mathbb{L}$. Expressing now that the lines $\langle (0,1,a,0,0,0), (0,0,0,1,-\overline{a},0) \rangle$ and $\langle (0,1,0,b,0,0), (0,0,1,0,-\overline{b},0) \rangle$ must intersect gives that $a\overline{b} = \overline{a}b$, contradicting the arbitrariness of a and b.

Now if Δ is a split polar space, then, using a standard alternating form, collinearity on Δ is given by $x_{-3}y_3 - x_3y_{-3} + x_{-2}y_2 - x_2y_{-2} + x_{-3}y_3 - x_3y_{-3} = 0$. By using the same definitions for L, L', M, M', we can apply the previous paragraph by remarking that the point $(0,1,a,0,0,0)$ of L is now collinear to the point $(0,0,0,1,-a,0)$ of L' and the point $(0,1,0,b,0,0)$ of M is now collinear to $(0,0,1,0,b,0)$. Expressing now the intersection of the corresponding lines gives that $ab = -ab$, which contradicts the arbitrariness of a and b combined with the fact that the characteristic is not 2 in the split separable case. □

3.3. **(Generalised) Baer collineations.** Some examples of domestic collineations in metasymplectic spaces are analogues of the generalised Baer collineations in polar spaces introduced in Section 6 of [28]. We repeat the definition and prove some (new) facts.

Definition 3.3.1. (*i*) A *Baer subplane* π' of a projective plane π is a proper subplane with the property that every line of $\pi \setminus \pi'$ contains exactly one point of π' and every point of $\pi \setminus \pi'$ is contained in exactly one line of π'.

(*ii*) A *Baer collineation of a projective plane* is a collineation that has as fix structure a Baer subplane.

The following examples are perhaps less familiar, so we provide a short proof.

Example 3.3.2. Let $(\mathbb{B},\mathbb{A})$ be one of $(\mathbb{K},\mathbb{L})$, $(\mathbb{L},\mathbb{H})$ or $(\mathbb{H},\mathbb{O})$. Then $\mathsf{PG}(2,\mathbb{B})$, viewed as a subplane of $\mathsf{PG}(2,\mathbb{A})$ by restricting coordinates, is a Baer subplane of $\mathsf{PG}(2,\mathbb{A})$. Moreover, there exists a Baer collineation of $\mathsf{PG}(2,\mathbb{A})$ with fix structure $\mathsf{PG}(2,\mathbb{B})$.

Indeed, this is easy and well known for $\mathbb{A}=\mathbb{L}$. So let $\mathbb{A}\in\{\mathbb{H},\mathbb{O}\}$. It suffices to show that every line of $\mathsf{PG}(2,\mathbb{A})$ has a point in $\mathsf{PG}(2,\mathbb{B})$; the dual then also holds. It is also easy to see that, after introducing affine coordinates in the standard way, this is equivalent to showing that for every $q\in\mathbb{A}$ and every $m\in\mathbb{A}\setminus\mathbb{B}$ there exist $x,y\in\mathbb{B}$ such that $y=mx+q$. Writing elements $u\in\mathbb{A}$ as pairs $(u_1,u_2)\in\mathbb{B}\times\mathbb{B}$ and using the Cayley-Dickson process mentioned in Section 2.2, we see that this is equivalent with showing that the following system of equations in the unknowns $x_1,y_1\in\mathbb{B}$, with $m_2\neq 0$, has a (unique) solution in $\mathbb{B}$:

$$\begin{cases} y_1 &= m_1x_1+q_1,\\ 0 &= x_1m_2+q_2. \end{cases}$$

This is of course obvious, as $m_2\neq 0$, and $\mathbb{B}$ is associative.

As Baer collineation we can take for instance the automorphism $\theta_c:\mathbb{A}\to\mathbb{A}:(x,y)\mapsto(x,yc)$, for all $x,y\in\mathbb{B}$ and $c\in\mathbb{B}^\times$ with $c\bar{c}=1$. Note that this is not necessarily an involution.

Lemma 3.3.3. *A Baer collineation θ of a projective plane $\pi\cong\mathsf{PG}(2,\mathbb{L})$ over a field $\mathbb{L}$ is an involution.*

Proof. Let $\pi'\cong\mathsf{PG}(2,\mathbb{K})$ be the pointwise fixed subplane of $\pi\cong\mathsf{PG}(2,\mathbb{L})$ under θ. Then we can see $\mathbb{L}$ as a field extension of $\mathbb{K}$ (extend a coordinatisation of π' to one of π). We now claim that $\mathbb{L}$ is quadratic over $\mathbb{K}$. Indeed, suppose not and let $1,e_1,e_2$ be independent elements of $\mathbb{L}$ viewed as vector space over $\mathbb{K}$. Expressing that every line of π contains a point of π' means that for every $q\in\mathbb{L}$ and every $m\in\mathbb{L}\setminus\mathbb{K}$ there exist $x,y\in\mathbb{K}$ such that $y=mx+q$. But now we see that there does not exist such x and y for $m=e_1$ and $q=e_2$, proving the claim.

Now choosing a suitable basis, we can assume that θ is given by the identity matrix and a companion nontrivial field automorphism σ fixing $\mathbb{K}$ pointwise. Then σ belongs to the Galois group of the extension $\mathbb{L}/\mathbb{K}$ of degree 2 and hence has order 2. □

These Baer collineations of projective planes can be generalised to collineations of polar spaces of rank 3.

Definition 3.3.4. A *generalised Baer collineation of a polar space of rank 3* is a collineation satisfying the following properties:

(*i*) it induces a Baer collineation in every stabilised plane;
(*ii*) it stabilises all planes through any stabilised line;
(*iii*) it stabilises at least one plane.

Lemma 3.3.5. *A generalised Baer collineation θ of a polar space Δ of rank 3 with planes over a field $\mathbb{L}$ is an involution.*

Proof. It is easy to see that there exist opposite fixed points p and q in Δ. Then every fixed point c in $p^\perp \cap q^\perp$ corresponds to a stabilised line pc through p. Now by Definition 3.3.4 every plane through pc is stabilised and θ induces a Baer collineation in it, hence by Lemma 3.3.3 θ^2 acts trivially on those planes. Consequently θ^2 acts trivially on all the lines through c in $p^\perp \cap q^\perp$ and so the pointwise fixed subquadrangle of $p^\perp \cap q^\perp$ is ideal and full in the terminology of Section 1.8 of [34] and θ^2 fixes $p^\perp \cap q^\perp$ pointwise (use Propositions 1.8.1 and 1.8.2 of [34]).

As the argument in the previous paragraph shows that θ^2 also fixes the line pc pointwise, we can now appeal to Corollary 3.2.3 to see that θ^2 fixes indeed all points of Δ. □

With arguments quite similar to those in the proof of the previous lemma, we can show that Baer collineations don't always exist. We will do this for some polar space in the next lemma. After that, we show existence in some cases.

Lemma 3.3.6. *A symplectic polar space of rank* 3*, over a field of characteristic different from* 2*, does not admit any generalised Baer collineation.*

Proof. Suppose for a contradiction that θ is a generalised Baer collineation of a symplectic polar space Δ of rank 3, over a field of characteristic different from 2. Then we claim that the fix structure of the quadrangle $p^\perp \cap q^\perp$, with p and q opposite fixed points, is an ideal subquadrangle. This is the case as every line in $p^\perp \cap q^\perp$ through a fixed point $c \in p^\perp \cap q^\perp$ corresponds to a plane through the stabilised line pc in Δ. So by Definition 3.3.4 it corresponds to a stabilised plane and these lines of $p^\perp \cap q^\perp$ are consequently also stabilised, which proves the claim. But by Proposition 5.9.4 of [34], symplectic quadrangles not over a field of characteristic 2 don't have (proper and thick) ideal subquadrangles. So $p^\perp \cap q^\perp$ is pointwise fixed and with Lemma 3.2.3 θ must be the identity, a contradiction. □

Lemma 3.3.7. *If a collineation of a polar space* $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$*, with* $\mathbb{A}$ *a separable quadratic extension of* $\mathbb{K}$ *or a quaternion division algebra over* $\mathbb{K}$*, fixes exactly a sub polar space* $\mathsf{C}_{3,1}(\mathbb{B},\mathbb{K})$*, with* $\mathsf{dim}_{\mathbb{B}}(\mathbb{A}) = 2$*; then it is a generalised Baer collineation.*

Proof. Property (*iii*) of Definition 3.3.4 is trivially satisfied and Property (*i*) holds by Example 3.3.2. So we must only prove the second property, i.e. that every plane from $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ through a line L from $\mathsf{C}_{3,1}(\mathbb{B},\mathbb{K})$ is in fact a plane of $\mathsf{C}_{3,1}(\mathbb{B},\mathbb{K})$. Recall that $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ is the hermitian polar space in $\mathsf{PG}(5,\mathbb{A})$ with point set

$$\overline{x}_{-3}x_3 + \overline{x}_{-2}x_2 + \overline{x}_{-1}x_1 \in \mathbb{K}.$$

So by choosing a coordinatisation so that $L = \langle e_{-1}, e_{-2}\rangle$, we see that every plane corresponds to a unique point collinear with the opposite line $L' = \langle e_1, e_2\rangle$ and that are exactly the points of the form $\langle e_{-3} + ke_3\rangle$, with $k \in \mathbb{K}$. As these points are independent from $\mathbb{A}$, these planes through L are exactly the same in $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ as in $\mathsf{C}_{3,1}(\mathbb{B},\mathbb{K})$, which concludes the proof. □

Letting the automorphism θ_c defined in Example 3.3.2 act on the (affine) coordinates of $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ as given in [6] produces examples of generalised Baer collinations.

3.4. **Two lemmas for inseparable polar spaces.** Certain examples of domestic collineations of separable metasymplect spaces will have no analogue in the inseparable case. The main reason is the next lemma. Also, in the inseparable case the metasymplectic spaces Γ_1 and Γ_4 both play the same role, so we will have to recognise certain examples through the dual setting. Proposition 3.4.2 will be used to recognise products of central elations in Γ_1 through products of axial elations of an extended equator geometry in Γ_4. Note that it will follow independently from Lemma 3.5.1 that a central elation does not map any point to a distinct collinear one (because a central elation is clearly line domestic).

Lemma 3.4.1. *If θ is a collineation of an inseparable polar space $\Delta \cong \mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ pointwise fixing a hyperbolic line h and its perp, then θ is the identity.*

Proof. If $\mathbb{A}=\mathbb{K}$, let Δ be the symplectic polar space in $\mathsf{PG}(5,\mathbb{A})$ corresponding to the alternating form

$$x_{-3}y_3 + x_3y_{-3} + x_{-2}y_2 + x_2y_{-2} + x_{-1}y_1 + x_1y_{-1},$$

and if $\mathbb{A}\neq\mathbb{K}$, let Δ be the polar space in $\mathsf{PG}(5,\mathbb{A})$ given by

$$x_{-3}x_3 + x_{-2}x_2 + x_{-1}x_1 \in \mathbb{K}.$$

Then we can assume that $p_{-3}=(1,0,0,0,0,0)$ and $p_3=(0,0,0,0,0,1)$ are contained in h. So the matrix corresponding to θ is diagonal and the field automorphism corresponding to θ is trivial (as this subspace contains a line). Now expressing that also the point $(1,0,0,0,0,1)$ is fixed, gives that the matrix is of the form

$$\begin{pmatrix} k & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & k \end{pmatrix}.$$

Expressing finally that the points $(0,1,0,0,0,1)$ and $(1,0,0,0,1,0)$ must stay collinear after applying θ, yields $k^2+1=0$, which is equivalent to $k=1$ in characteristic 2. □

Proposition 3.4.2. *Let Δ' be the rank 4 polar space with equation $x_{-4}x_4+x_{-3}x_3+x_{-2}x_2+x_{-1}x_1 \in \mathbb{K}$ in $\mathsf{PG}(7,\mathbb{K}')$, where $\mathbb{K}'$ is a nontrivial inseparable quadratic field extension of $\mathbb{K}$ (necessarily in characteristic 2), i.e. $\mathbb{K}'^2 \leq \mathbb{K} < \mathbb{K}'$, and let Δ be the associated ambient symplectic polar space whose point set coincides with the point set of $\mathsf{PG}(7,\mathbb{K}')$. Let θ_1 and θ_2 be two perpendicular central elations of Δ so that the product $\theta_1\theta_2 =: \theta$ is a nontrivial collineation of Δ' with the property that at least one maximal singular subspace through each fixed submaximal singular subspace is stabilised. Then exactly one of the following holds.*

- *(i) θ fixes each point collinear with its image and the centres of both θ_1 and θ_2 are points of Δ'. In this case both θ_1 and θ_2 act on Δ' and θ is not the product of two perpendicular axial elations with axes in Δ'.*
- *(ii) θ fixes each point collinear with its image and the centres of both θ_1 and θ_2 do not belong to Δ'. Then the fix structure of θ is a generalised quadrangle obtained by intersecting Δ' with the perp of the line joining the centres of θ_1 and θ_2.*
- *(iii) θ maps some point to a distinct but collinear one and the centres of both θ_1 and θ_2 belong to Δ'. In this case θ is always a product of two perpendicular axial elations with both axes belonging to Δ' and a product of two perpendicular axial elations with both axes not belonging to Δ'.*
- *(iv) θ maps some point to a distinct but collinear one and the centres of both θ_1 and θ_2 do not belong to Δ'. In this case θ is the product of two perpendicular axial elations with both axes belonging to Δ'.*

Proof. We order the coordinates of $\mathsf{PG}(7,\mathbb{K}')$ according to increasing indices. Let L be the line joining the centres e_1 and e_2 of θ_1 and θ_2, respectively. There are three possibilities.

Remark first that θ is in every case an involution, which can be easily seen by choosing the two centra as first and second base points of Δ. Then the product is a matrix as in (1) below, which is an involution in characteristic 2. Furthermore we see from this matrix that the fixed points of θ are exactly those of $L^\perp$, where $\perp$ is the defining polarity of Δ.

(1) *The line L is a line of Δ'.* Then we take $e_1 = p_{-4}$ and $e_2 = p_{-3}$. The matrix of θ looks like

$$\begin{pmatrix} 1 & 0 & 0 & 0 & k \\ 0 & 1 & 0 & \ell & 0 \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (1)$$

with $k, \ell \in \mathbb{K} \setminus \{0\}$ (as the images of the points p_3 and p_4 under θ have to belong to Δ'). One calculates that the point $(x_{-4}, x_{-3}, \ldots, x_4)$ is mapped onto a collinear but distinct point if, and only if, $(x_3, x_4) \neq (0,0)$ and $kx_4^2 = \ell x_3^2$. If $k\ell \notin (\mathbb{K}')^2$, then the assumptions of (i) are satisfied. We claim that also the conclusions are satisfied. Indeed, θ_1 (obtained from the above matrix by setting ℓ equal to 0) and θ_2 (setting k equal to 0) clearly act on Δ'. Suppose now that θ would be a product of two perpendicular axial elations. If θ was the product of two axial elations with intersecting axes, then all points collinear with this intersection point would be mapped to collinear points, hence are fixed, contradicting the fact that θ is not a central elation (indeed, no point on the line p_3p_4 is fixed as $(k, \ell) \neq (0,0)$). The set of fixed points of the product of two axial elations with nonintersecting collinear axes is precisely the solid spanned by the axes, and so this can never be a geometric subhyperplane. This shows (i).

Next suppose that $k\ell \in (\mathbb{K}')^2$. By rescaling, we may assume without loss of generality that $k = \ell$. Hence clearly some point is mapped to a distinct collinear point. We claim that we are in Case (iii). Now the matrix of θ equals the product

$$\begin{pmatrix} 0 & 1 & 0 & d & 0 \\ 1 & 0 & 0 & 0 & d \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & d' & 0 \\ 1 & 0 & 0 & 0 & d' \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

with $d + d' = k$, which induce axial elations with axes $\langle (1,1,0,0,0,0,0,0), (0,d,0,0,0,0,1,1) \rangle$ and $\langle (1,1,0,0,0,0,0,0), (0,d',0,0,0,0,1,1) \rangle$, respectively. Since $k \in \mathbb{K}$, it is clear that either both d and d' belong to $\mathbb{K}$ or both do not. This concludes Case (iii).

(2) *The line L does not belong to Δ', but has a (unique) point p in common with Δ'.* First we prove that it is impossible that p is the centre of one of our central elations. Suppose for a contradiction that p is the centre of θ_1. Then projecting e_2 onto a solid Σ of Δ' through p gives a fixed plane π through p. However there are no stabilised solids through this plane, as a central elation with centre e_2 does not map any point of $\Sigma \setminus \pi$ to a collinear one (as we noted in the beginning of this subsection), a contradiction to our assumptions.

So without loss of generality we may take $e_1 = (1, 0, \ldots, 0, a)$, with $a \in \mathbb{K} \setminus \mathbb{K}'$, $p = (0, 1, 0, \ldots, 0)$ and $e_2 = (1, 1, 0, \ldots, 0, a)$. The matrix of a central elation with centre e_1 looks like

$$\begin{pmatrix} 1 + a\ell & 0 & 0 & 0 & \ell \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a^2\ell & 0 & 0 & 0 & 1 + a\ell \end{pmatrix}.$$

Indeed, all points collinear to e_1 have coordinates of the form $(x_{-4}, x_{-3}, \ldots, x_3, ax_{-4})$ and are obviously fixed, expressing that the elation must preserve collinearity gives that the ℓ on the first row is the same as on the last row and expressing that Δ' is preserved gives that $a + \ell^{-1} \in \mathbb{K}$. Likewise, the following matrix represents an arbitrary central elation with centre e_2:

$$\begin{pmatrix} 1+a\ell' & 0 & 0 & \ell' & \ell' \\ a\ell' & 1 & 0 & \ell' & \ell' \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a^2\ell' & 0 & 0 & a\ell' & 1+a\ell' \end{pmatrix}.$$

The product has then as matrix

$$\begin{pmatrix} 1+a(\ell+\ell') & 0 & 0 & \ell' & \ell+\ell' \\ a\ell' & 1 & 0 & \ell' & \ell' \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a^2(\ell+\ell') & 0 & 0 & a\ell' & 1+a(\ell+\ell') \end{pmatrix}.$$

Now θ pointwise fixes the plane $\langle p_{-1}, p_{-2}, p_{-3} \rangle$ through p and so it has to stabilise a solid through p. Let q belong to that solid S, and choose q so that it is not collinear to e_1. Then q^θ belongs to the plane $\langle q, L \rangle$, as each projective plane through L is stabilised, since all hyperplanes through L are stabilised as their images under $\perp$, i.e. the defining polarity of Δ, are contained in $L^\perp$ and consequently fixed. The plane $\langle q, L \rangle$ intersects S in the line pq, so q is mapped to a point on that line and consequently that line is stabilised. A generic point collinear to p has coordinates $(x_{-4}, x_{-3}, *, 0, x_4)$ and is mapped onto a collinear point if, and only if,

$$(1 + a(\ell+\ell'))x_{-4}x_4 + (\ell+\ell')x_4^2 + a^2(\ell+\ell')x_{-4}^2 + (1 + a(\ell+\ell'))x_4x_{-4} = 0,$$

which is equivalent to

$$(\ell+\ell')(x_4 + ax_{-4})^2 = 0.$$

As q is not fixed (since it is not collinear to e_1 and we remarked at the begin of the proof that all the fixed points are collinear to both centra), we have that $x_4 + ax_{-4} \neq 0$, and so $\ell = \ell'$. This implies that all points collinear to p are mapped onto collinear ones, once one non fixed point is. Now the matrix of θ becomes

$$\begin{pmatrix} 1 & 0 & 0 & \ell & 0 \\ a\ell & 1 & 0 & \ell & \ell \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a\ell & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \ell c & 0 \\ \ell d & 1 & 0 & 0 & \ell c \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \ell d & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & \ell c' & 0 \\ \ell d' & 1 & 0 & 0 & \ell c' \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \ell d' & 1 \end{pmatrix},$$

as soon as

$$\begin{cases} c + c' & = 1, \\ d + d' & = a, \\ cd' + c'd & = \ell^{-1}, \end{cases} \iff \begin{cases} c' & = 1 + c, \\ d & = \ell^{-1} + ac, \\ d' & = a + \ell^{-1} + ac. \end{cases}$$

Now the above matrix in c and d is an axial elation with axis spanned by p and the point $(c, 0, \ldots, 0, d)$. Hence the axis belongs to Δ' if and only if $cd \in \mathbb{K}$. So if we want θ to be the product of two axial elations of Δ', then $cd \in \mathbb{K}$ and $c'd' \in \mathbb{K}$. Examples are given by

$$(c, d, c', d') = (1, \ell^{-1} + a, 0, \ell^{-1}) \text{ and } (c, d, c', d') = (a^{-1}\ell^{-1}, 0, 1 + a^{-1}\ell^{-1}, a).$$

This is Case (iv).

(3) *The only remaining case is when L has no points of Δ'.* In this case θ pointwise fixes the geometric subhyperplane $L^\perp \cap \Delta'$. This clearly contains (opposite) lines, but no planes as the span of such a plane and L would be a 4-dimensional singular subspace, a contradiction. If θ mapped a non fixed point $p \in \Delta'$ to a collinear one, then the line pp^θ would be stabilised (since θ is an involution). No point of that line belongs to L, hence there are fixed points collinear with a unique point of pp^θ, implying that pp^θ contains a fixed point x. Since $x \notin L$, $L^\perp \nsubseteq x^\perp$ and we find a second fixed point on pp^θ. So $pp^\theta \subseteq L^\perp$, contradicting that $p \neq p^\theta$. Hence the fix structure is exactly $L^\perp$ restricted to Δ'. Since this structure is a subspace of a polar space, not containing planes, but containing two opposite lines, this must be a (nondegenerate) generalised quadrangle. This is Case (ii)

This completes the proof of the proposition. □

3.5. **Domestic collineations in polar spaces.**

We will also have to deal with domestic collineations of some polar spaces. A lot of properties are proved in [19] and [28], and we will refer to those when needed. We also need a more detailed version of one of the results there, and a new, more specific result for separable orthogonal polar spaces. We prove these two results here.

Note that we freely use the notation for opposition diagrams as established in [16]. However, we will always shortly explain when we mention a specific opposition diagram for the first time.

Lemma 3.5.1. *The set of fixed points of any line-domestic collineation θ of any polar space is a geometric hyperplane. Also, if a point is not fixed, it is mapped onto an opposite one. Each line that is stabilised is pointwise fixed.*

Proof. If θ is trivial, then so is the assertion. If θ is nontrivial, then Theorem 5.1 of [28] asserts that the set of fixed points is a hyperplane H. Let L be a stabilised, but not pointwise fixed line. Then L contains a unique fixed point x. Since no other point of L is fixed, all fixed points are collinear to x. Take a line M intersecting L not in x and such that $M \nsubseteq x^\perp$. Then M does not contain a fixed point, a contradiction.

If now a point x were mapped onto a collinear one, then, since the line xx^θ contains a fixed point, that line would be preserved, but not pointwise fixed, contradicting the previous paragraph. □

The next proposition will be applied to extended equator geometries in Γ_4. Nevertheless we phrase it for general rank as the proof remains the same.

First some definitions.

Definition 3.5.2. (i) A *subhyperplane* of a polar space is a subspace that intersects each singular plane nontrivially, and such that some line is disjoint from it.

(ii) We say that a subspace (in particular, a hyperplane) of a polar space is *nondegenerate (of rank r)* if it defines itself a polar space of rank r (hence is not contained in $p^\perp$ for some of its points p).

(iii) A *skeleton* of a polar space of rank r is a set of $2r$ points with the property that each of these points is opposite a unique other point of the set. Equivalently, it is the set of points of an apartment.

(iv) A *generalised homology* in a polar space of rank r is a collineation that pointwise fixes a skeleton and also (pointwise fixes) at least one line determined by two collinear points of the skeleton.

Proposition 3.5.3. *Let θ be a plane-domestic and solid-domestic nontrivial collineation of a separable orthogonal polar space $\Delta = (Q, \mathscr{L})$ of rank $r \geq 4$. Then exactly one of the following holds.*

(1) *θ is point-domestic and is an axial elation;*
(2) *θ is line-domestic and the set of fixed points is a nondegenerate geometric hyperplane, necessarily of rank r or $r-1$;*
(3) *θ is neither point-domestic nor line-domestic and exactly one of the following holds.*
   (*i*) *θ is the product of two perpendicular axial elations;*
   (*ii*) *θ is a generalised homology;*
   (*iii*) *θ fixes a nondegenerate subspace of rank $r-2$ or $r-1$ which is at the same time a geometric subhyperplane.*

Proof. Since θ does not map planes and solids to opposites, the opposition diagram is one of $\mathsf{B}^1_{n;1}$ (there is a point mapped to an opposite one, no element of another type is mapped to an opposite one), $\mathsf{B}^2_{n;1}$ (there is a line mapped to an opposite one, no element of another type is mapped to an opposite one) or $\mathsf{B}^1_{n;2}$ (there is a point-line flag mapped to an opposite one, no element of another type is mapped to an opposite one), by Corollary 4 of [16].

- *The opposition diagram is $\mathsf{B}^1_{n;1}$.*
  Then θ is line-domestic and so, by Lemma 3.5.1, the fixed points form a geometric hyperplane H. Assume for a contradiction that H is not nondegenerate. Then θ is a central elation, which must be the identity by Lemma 3.2.4, a contradiction. So H is nondegenerate and this now leads to (2).
- *The opposition diagram is $\mathsf{B}^2_{n;1}$.*
  Then θ is point-domestic. It follows from Proposition 3.11 in [19] that θ is an axial elation. This is (1).
- *The opposition diagram is $\mathsf{B}^1_{n;2}$.*
  By Theorem 6.1 of [28] θ pointwise fixes a subhyperplane S of Δ. The subspace $\langle S\rangle$ viewed in the ambient projective space Ω has codimension 2 (by Lemma 3.1.2(*i*)). By the assumption of the nondegeneracy of the underlying polarity ρ, the subspace $L := \langle S\rangle^\rho$ is a line. There are now four possibilities.
  - $L \in \mathscr{L}$.
    Then θ is an axial elation by Lemma 3.2.7 and hence point-domestic, contradicting the opposition diagram.
  - $|L \cap Q| = 2$.
    Set $\{p, q\} = Q \cap L$. Hence $S = p^\perp \cap q^\perp$ is pointwise fixed. There are two possibilities.
    * *The points p and q are interchanged by θ, that is, $p^\theta = q$ and $q^\theta = p$.*
      We may assume that $p = p_1$ and $q = p_2$ for a basis $(p_1, p_2, \ldots)$ where Q has equation $X_1X_2+X_3X_4+\ldots = 0$. Since Q is preserved and $p^\perp \cap q^\perp$ is fixed pointwise, one checks that θ acts on the coordinates as follows: $(x_1, x_2, x_3, x_4, \ldots) \mapsto (ax_2, a^{-1}x_1, x_3, x_4, \ldots)$, with $a \in \mathbb{K}^\times$. It follows that the points with coordinates $(ax_1, x_1, x_3, x_4, \ldots)$ are fixed. Hence θ fixes a hyperplane pointwise and consequently θ is line-domestic, contradicting the opposition diagram.
    * *The points p and q are fixed.*
      Choosing a skeleton in $p^\perp \cap q^\perp$, we can complete it to a skeleton in Δ pointwise fixed by θ. By considering some pointwise fixed line in $p^\perp \cap q^\perp$, we see that θ is a generalised homology. This implies Case (3)(*ii*).
  - $L \cap Q = \{p\}$.
    Select $q \in L \setminus \{p\}$ and choose $x \in (Q \cap p^\perp) \setminus q^\rho$. Then the plane $\langle x, L\rangle$ intersects Q

in a pair of lines, as it is a conic containing a line ($M := px$) and a point not on that line (on qx different from x, since qx is not a tangent). Since x and q are contained in p^ρ (note that L is a tangent since it intersects Q in exactly one point), we deduce that the second line M' in the intersection of that plane with Q contains p. Choose $z \in Q \setminus p^\perp$. Then the solid $\langle z, x, L\rangle$ intersects Q in a hyperbolic quadric Q' (as it contains two intersecting lines and a point opposite to their intersection). Moreover, $Q'^\perp \subseteq L^\rho = \langle S\rangle$, so that we can choose the basis in such a way that Q has equation $X_1X_2 + X_3X_4 + X_5X_6 + \cdots = 0$, with $\{p_1, p_2, p_3, p_4\} \subseteq \langle x, z, L\rangle$, and the subspace $\langle p_5, p_6, \ldots\rangle$ pointwise fixed by θ. We can also assume $p = p_3$ and $q = (a, b, 0, 0, \ldots)$, $a, b \in \mathbb{K}^\times$. The action of θ on the coordinates x_i, $i \geq 5$, is trivial and consequently also the corresponding field automorphism is trivial. So we may concentrate on the (matrix-)action of θ on (x_1, x_2, x_3, x_4). After some elementary calculations, expressing that p and the points with coordinates $(a, -b, 0, 0, *, *, \ldots)$ belong to L^ρ, and that θ preserves the quadric Q, we see that there are two possibilities.

* **Case 1:** θ is of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^\theta = \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 0 & -b \\ b & a & 1 & -ab \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -b \\ b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

  In this case θ is the product of two axial elations with respective axes $x_2 = x_4 = x_5 = \ldots = 0$ and $x_1 = x_4 = x_5 = \ldots = 0$, by Lemma 3.2.7. It is clear that these axes intersect in the point p, and that they are not coplanar. Hence, according to Definition 3.2.5, the axial elations are perpendicular. We are in Case (3)(i).

* **Case 2:** θ is of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^\theta = \begin{pmatrix} 0 & -ab^{-1} & 0 & a \\ -ba^{-1} & 0 & 0 & b \\ b & a & 1 & -ab \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

  Now θ is clearly an involution fixing all points of a hyperplane whose coordinates satisfy $bx_1 + ax_2 = abx_4$. Hence θ is line-domestic, contradicting the opposition diagram.

– $L \cap Q = \emptyset$.

Since $L^\rho = \langle S\rangle$ has codimension 2 in Ω, the singular subspaces contained in S have maximal dimension at least $r - 3$. Suppose for a contradiction that S is degenerate, say $S \subseteq s^\perp$, for some $s \in S$. Since $L \cap Q = \emptyset$, obviously $s \notin L$ and so $\langle L, s\rangle$ is a plane. But $S \subseteq \langle L, s\rangle^\perp$, and as the latter spans a subspace of codimension 3, we obtain a contradiction. Hence S is nondegenerate.

* If S has rank r, then selecting the points of a skeleton in two pointwise fixed opposite singular subspaces of dimension $r-1$, we see that θ is a homology, and we are in Case (3)(ii).
* If S has rank $r - 1$ or $r - 2$, then we deal with Case (3)(iii).

The proposition is proved. □

4. Domesticity of certain collineations

Here we show that the relevant collineations of the Main Result are actually domestic with given opposition diagram.

4.1. **Central elations and products.**

Proposition 4.1.1. *A nontrivial central elation θ in a metasymplectic space Γ_i is a domestic collineation with opposition diagram $\mathsf{F}^i_{4;1}$.*

Proof. Let θ be a central elation in Γ_i with centre c. Then θ is symp-domestic as every symplecton contains at least one point symplectic to c, which is fixed by the definition of a central elation. So it follows from Table 2 that the opposition diagram is $\mathsf{F}^i_{4;1}$.

□

Proposition 4.1.2. *The product of two perpendicular central elations in Γ_1 is a domestic collineation with opposition diagram $\mathsf{F}_{4;2}$.*

Proof. Let θ be the product of two perpendicular central elation θ_1, θ_2 in Γ_1. Denote by c_j the centre of θ_j, $j = 1, 2$ and set $\xi := \xi(c_1, c_2)$. First we prove that θ maps a point to an opposite one. Let ζ be a symplecton though c_1 locally opposite ξ and let p be a point in ζ opposite c_2. Then p is mapped to an opposite point since it is preserved by θ_1 and mapped to an opposite by θ_2 by Lemma 6.5.1.

Now we claim the dual, i.e. that θ maps a symplecton to an opposite one. Let ζ' be a symplecton opposite ξ. Denote the projection of c_j on ζ' by x_j, $j = 1, 2$. By the previous paragraph, θ_1 maps ζ' to a symplecton ζ'' locally opposite ζ' through x_1, that is, ζ' and ζ'' are symplectic. As x_1 is not collinear to x_2 (since c_1 is not collinear to c_2), it is opposite c_2. Consequently, ζ'' is far from c_2. Then θ_2 maps ζ'' again to a symplecton ζ''' locally opposite ζ'' through the projection of c_2 onto ζ''. Now by the dual of Axiom 2.4.5(2), the symplecton ζ' is opposite the symplecton $\zeta''' = \zeta'^{\theta}$.

Finally we claim that θ maps no plane to an opposite. Let π be a plane. Note that every symplecton collinear to ξ is stabilised. Now every plane corresponds to a line in the dual and consequently, by the dual of Corollary 2.5.2, either there exists a symplecton through π collinear to ξ or there exist two (mutually) collinear symplecta ζ_1, ζ_2 with $\pi \subseteq \zeta_1$ and ζ_2 collinear to ξ. In the first case it is clear that π cannot be mapped to an opposite plane, so suppose we are in the second case. Denote by π' the intersection of ζ_1 and ζ_2. By the dual of Corollary 2.5.3, it suffices to prove that π' is not mapped to an opposite plane in ζ_2.

First note that θ_j, $j = 1, 2$, induces in ζ_2 an axial collineation, as immediately follows from Definitions 2.11.8(i) and 3.2.5 (possibly trivial, in particular when $c_j \in \zeta_2$) with axis $c_j^{\perp} \cap \zeta_2$. Hence θ induces in ζ_2 the product of two axial collineations. If their axes coincide, we see that θ acts point-domestically on ζ_2 and by the possible opposition diagrams of $\mathsf{B}_{3,1}$ in Corollary 4 of [16] it must also act plane-domestically. So suppose now that these axes intersect in a point (they are of course contained in the plane $\xi \cap \zeta_2$). Then this point is collinear to a line L of π' and this line is consequently mapped to an intersecting line. This proves that π' is not mapped to an opposite plane in ζ_2.

The above claims prove the statement, recalling the possible opposition diagrams of Table 2. □

4.2. **The fix structure is a generalised quadrangle.**

Proposition 4.2.1. *Let θ be a collineation of a metasymplectic space Γ_i such that its fix structure consists of points and symplecta only, and these form a generalised quadrangle. Assume additionally that the set of fixed points in some symplecton forms an ovoid and dually, the fixed symplecta through some point form an ovoid in the residue. We also assume that these ovoids in the symplecta or point residuals isomorphic to $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ are closed under taking the hyperbolic line through two distinct points. Then θ is domestic with opposition diagram $\mathsf{F}_{4;2}$.*

Proof. We argue in Γ_4.

Let ξ and ξ' be two fixed symplecta in Γ_4 sharing no fixed points. Then ξ and ξ' are disjoint (otherwise the intersection is fixed, and the intersection is either a plane or a point). If ξ and ξ' are not opposite, then the unique symplecton intersecting both ξ and ξ' in respective planes is fixed, and so are these planes, a contradiction. Hence ξ and ξ' are opposite. Using projections, we now see that the fixed points in each fixed symplecton form an (isomorphic) ovoid. Also the dual holds.

Now let, for a contradiction, C be a chamber of Γ_4 mapped onto an opposite chamber C^θ. Let $p \in C$ be a point. We first claim that there is a fixed point f opposite p. Indeed, remark that p cannot be contained in a fixed symplecton, as it is mapped to an opposite point by assumption. So now p is close or far from any fixed symplecton. Suppose that p is close to some fixed symplecton ξ, then ξ contains a fixed point q special to p, as an ovoid can never be collinear to a point (Lemma 3.1.3). Now another fixed symplecton through q must be far from p, as the centre $\mathfrak{c}(p,q)$ is not contained in this symplecton. So p is far from at least one fixed symplecton ζ. Again by the fact that an ovoid cannot be collinear to a point, we see that ζ contains a fixed point f opposite p and the claim is proved.

We now claim that p is symplectic to two mutually opposite fixed points. By the previous paragraph, we may assume that some fixed point f is opposite p. Consider an arbitrary fixed symplecton ξ through f, then p must be far from ξ. So there is a unique point $x \in \xi$ symplectic to p. If x is fixed for at least two choices of ξ through f, then the claim again follows (since a generalised quadrangle does not contain triangles). So we may assume that x is not fixed. Then p is special to at least two fixed points x_1, x_2 of ξ. Let L_i be the unique line containing x_i, $i = 1, 2$, and containing $p \bowtie x_i$. Then, by assumption, there is at least one fixed symplecton ξ_i containing L_i. Now p is close to ξ_i for all $i \in \{1, 2\}$ and it is clear that each ξ_i contains at least two fixed points a_i, b_i symplectic to p (note that $p^\perp \cap \xi_i$ can not have fixed points and then we can look at the fixed points in two locally opposite planes through this line). It is also obvious that the symplecta ξ_1 and ξ_2 are opposite as a generalised quadrangle does not contain triangles. Now a_1 can't be symplectic to both a_2, b_2 in the generalised quadrangle and so we find two mutually opposite fixed points symplectic to p.

Hence let x_1, x_2 be two opposite fixed points symplectic to p. Then $x_1, x_2 \in E(p, p^\theta)$, and hence $\widehat{E} := \widehat{E}(x_1, x_2) = \widehat{E}(p, p^\theta)$ is fixed by θ. Let S be a "solid" of $\widehat{E}$. We claim that S contains a fixed point. Indeed, we may assume that S does neither contain x_1, nor x_2. Let S_1 be the solid of $\widehat{E}$ generated by x_1 and the plane $\pi_1 := x_1^{\perp\!\!\!\perp} \cap S$. Let π be the plane $x_2^{\perp\!\!\!\perp} \cap S_1$ of $\widehat{E}$, but also of $E(x_1, x_2)$. Then, by definition of $E(x_1, x_2)$, there exists a line $L_1 \ni x_1$ such that π is the set of points symplectic to x_2 and contained in a symplecton through L_1. By assumption, there exists a unique symplecton $\xi_1 \supseteq L_1$ fixed under θ. Since x_2 is also fixed, the unique point $x \in \xi_1$ symplectic to x_2 is also fixed and belongs to π. Again by assumption, each point of the hyperbolic line h through x_1 and x, is fixed. By definition of $\widehat{E}(x_1, x_2)$, it is a line of the polar space, contained in S_1, and so it contains a point $y \in \pi_1 \subseteq S$. Our claim is proved.

Now let L be the line in the chamber C. As above, it defines a plane α in $E(p,p^\theta)$, and hence a solid S of $\widehat{E}$ generated by α and p. The previous paragraph yields a fixed point $x \in S$. Hence the symplecton $\xi(p,x)$ is mapped onto $\xi(p^\theta,x)$, which implies $x \in \alpha$. Consequently, the symplecton $\xi(p,x)$ contains L. This, in turn, implies that x is collinear to a point y of L. So the point $y \in L$ is close to $\xi(x,p^\theta) = \xi(x,p)^\theta$ and therefore cannot be opposite any point of it; in particular it is not opposite any point of L^θ. But then L and L^θ are not opposite, the final contradiction implying that θ is domestic.

Now we claim that the opposition diagram is $\mathsf{F}_{4;2}$. Let ξ be a fixed symplecton. As θ does not fix a geometric hyperplane in ξ, but only a geometric subhyperplane, the contraposition of Lemma 3.5.1 implies that θ is not line-domestic in ξ. Then let L be a line of ξ mapped to an opposite line of ξ. Then a point x of Γ_4 collinear to L, but not contained in ξ is mapped to an opposite one by Corollary 2.5.4. Dually there is also a symplecton mapped to an opposite one, which concludes the proof. ☐

4.3. **When an apartment is pointwise fixed.**

Proposition 4.3.1. *(i) If in Class (K), the collineation θ of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$ has fix structure an extended equator geometry and its tropics geometry, then θ has opposition diagram $\mathsf{F}^4_{4;1}$.*

(ii) If in Class (L), the collineation θ of $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{L})$ has fix structure a metasymplectic (sub)space canonically isomorphic to $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{K})$, then θ has opposition diagram $\mathsf{F}^4_{4;1}$.

(iii) If in Class (L), the collineation θ of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{L})$ has fix structure an extended equator geometry and its tropics geometry, then θ has opposition diagram $\mathsf{F}_{4;2}$.

(iv) If in Class (H), the collineation θ of $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{H})$ has fix structure a metasymplectic (sub)space canonically isomorphic to $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{L})$, with $\mathbb{L}$ a separable quadratic extension of $\mathbb{K}$ contained in $\mathbb{H}$ as a 2-dimensional subalgebra and pointwise fixed under some automorphism of $\mathbb{A}$, then θ has opposition diagram $\mathsf{F}_{4;2}$.

Proof. We first claim that in Cases (i) and (iii) every fixed symplecton in Γ_4 has as fix structure a hyperbolic line and its perp. Let ξ be a fixed symplecton intersecting the fixed extended equator geometry $\widehat{E}$, then θ clearly fixes the hyperbolic line, say $h(x,y)$, appearing as intersection $\xi \cap \widehat{E}$ (see Lemma 2.6.18) and the perp of this hyperbolic line, i.e. $x^\perp \cap y^\perp =: S$, as all these points are contained in $\widehat{T}$. Suppose now that there is some other point z also fixed. This point cannot be contained in $\widehat{E}$, again by Lemma 2.6.18, so it must be contained in $\widehat{T}$. If $z \perp S$, it would be contained in $h(x,y)$, so we may pick $s \in S$ not collinear to z. Then $\xi = \xi(z,s)$ and by Proposition 2.7.6 (ii), the hyperbolic line $\beta(z) \cap \beta(s)$ must be contained in this symplecton, again a contradiction. Now suppose that ξ is a fixed symplecton not containing a point of $\widehat{E}$. Pick arbitrarily two opposite points $p,q \in \widehat{E}$ and extend an apartment of $E(p,q)$ as in Lemma 2.11.4 to an apartment of Γ_4 containing p and q. Note that all the 24 symplecta in this apartment contain a point of $\widehat{E}$ and have consequently a fix structure as described above. So by projection, it suffices to prove that ξ is opposite some symplecton of Λ. But that is exactly the dual of Lemma 2.11.5.

We now claim that the fixed points in a fixed symplecton of Γ_1 form a hyperplane in Cases (i) and (ii) and they form a subhyperplane in Case (iii) and (iv). In Cases (ii) and (iv) this follows quite easily: Denote by ζ' the fix structure in a fixed symplecton ζ. In Case (ii) we have that $\zeta' \cong \mathsf{B}_{3,1}(\mathbb{K},\mathbb{K})$ is clearly a geometric hyperplane of $\zeta \cong \mathsf{B}_{3,1}(\mathbb{K},\mathbb{L})$ by Definition 2.3.1 and the fact that $\mathsf{dim}_{\mathbb{K}}(\mathbb{L}) - \mathsf{dim}_{\mathbb{K}}(\mathbb{K}) = 1$. In Case (iv) we have that $\zeta' \cong \mathsf{B}_{3,1}(\mathbb{K},\mathbb{L})$ is clearly a geometric subhyperplane of $\zeta \cong \mathsf{B}_{3,1}(\mathbb{K},\mathbb{H})$ by Definition 2.3.1 and the fact that $\mathsf{dim}_{\mathbb{K}}(\mathbb{H}) - \mathsf{dim}_{\mathbb{K}}(\mathbb{L}) = 2$. In Cases (i) and (iii), we look at the dual space Γ_4. A fixed symplecton ζ of Γ_1 corresponds to a fixed point z of Γ_4, which must lie in the fixed extended equator geometry or its corresponding tropics geometry. If z is contained in $\widehat{E}$, then every symplecton through z is stabilised by Lemma 2.6.18 and consequently every point in ζ is fixed. If z is contained in $\widehat{T}$, the set of symplecta containing a hyperbolic line of $\beta(z)$ define a polar space isomorphic to a Klein quadric (that is, a hyperbolic quadric in $\mathsf{PG}(5,\mathbb{K})$), taking Lemma 2.7.1 into account. Remark that all these symplecta also contain the point z, so they form a subspace of the residue of z, which is $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$. If $\mathbb{A}=\mathbb{K}$, this is a quadric in $\mathsf{PG}(6,\mathbb{K})$, and if $\mathbb{A}$ is a quadratic field extension over $\mathbb{K}$, this is a quadric in $\mathsf{PG}(7,\mathbb{K})$. So by dimensional arguments, the fixed Klein quadric is a geometric hyperplane in the first case and a geometric subhyperplane in the second case. We now claim that no other symplecton through z can be fixed. Indeed, every such symplecton must contain a fixed point symplectic to z by the first paragraph. This point cannot be contained in $\widehat{E}$ by Lemma 2.7.4. So every fixed symplecton through z is of the form $\xi(z,z')$, with $z'\in\widehat{T}$. Such a symplecton then contains $\beta(z)\cap\beta(z')$ and is consequently contained in the Klein quadric described before. This proves the claims and hence every plane of a fixed symplecton contains a fixed point.

Dually we claim that in every fixed symplecton of Γ_4 each plane contains a fixed point. For Cases (i) and (iii) this follows immediately from the first paragraph, noticing that, with the notation of that paragraph, $x^\perp\cap y^\perp$ is a subhyperplane. For Cases (ii) and (iv), the symplecta are isomorphic to $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ and the fix structure is a (canonical) sub polar space $\mathsf{C}_{3,1}(\mathbb{B},\mathbb{K})$ (with $\mathsf{dim}_{\mathbb{B}}(\mathbb{A})=2$). Then from Lemma 3.3.7 we infer that θ induces in this residue a generalised Baer collineation. Theorem 7.1 of [28] implies that every plane of this residue contains a fixed point, which proves the claim.

If the fix structure in some symplecton of Γ_i is not a hyperplane, then as in the previous proposition, we can use Corollary 2.5.4 and Lemma 3.5.1 to conclude that some point of Γ_i is mapped onto an opposite. Hence the previous paragraphs already show that, if θ is domestic, then the opposition diagram is $\mathsf{F}_{4;2}$ in Cases (iii) and (iv), and either $\mathsf{F}_{4;2}$ or $\mathsf{F}^4_{4;1}$ in Cases (i) and (ii). So it suffices to show that, in Cases (i) and (ii) no point of Γ_1 is mapped onto an opposite, and in the other cases, no point-line flag of Γ_1 is mapped onto an opposite.

To that purpose, let p be any point of Γ_1. If p is contained in a fixed symplecton, then it cannot be mapped onto an opposite, nor can any line through it be mapped onto an opposite.

We now assume that p is not contained in any fixed symplecton. We claim that p is close to some fixed symplecton ξ of Γ_1. By assumption we know that there is some pointwise fixed line and so we find a fixed point x special, symplectic or collinear to p. Suppose first that p and x are collinear. Then p is close to any fixed symplecton through x (which exists in abundance by the first part of the proof). Suppose now that p and x are at distance 2 and let L be a line through x containing a point collinear to p. By the third claim above, there is a fixed symplecton ξ containing L. Since p is collinear to some point of L, it is close to ξ and the claim is proved.

Now set $K:=\xi\cap p^\perp$. In Cases (i) and (ii), the line K has a fixed point by the second claim. Hence p^θ is at distance at most 2 from p for every point p and this shows (i) and (ii) by the possible opposition diagrams in Table 2.

Now assume we are in Case (iii) or (iv). Let P be any line through p and assume that $\{p,P\}$ is mapped onto an opposite flag. Since every plane in ξ through K contains a fixed point, by the second paragraph, we can select a fixed point y in ξ collinear to K. Set $\zeta:=\xi(p,y)$. Suppose first that P is contained in ζ. Then the projection of P^θ from p^θ onto p is not locally opposite P, as both are contained in ζ (as the projection is an isomorphism and $\xi(p^\theta,y)$ is projected onto ζ). Hence by Lemma 2.8.7 the lines P and P^θ are not opposite. Now suppose that P is not contained in ζ. Then there is a unique line P' in ζ coplanar with P and by the first part of the proof we find a fixed symplecton ξ' containing y and some point u' of P'. Note that P is collinear to u', and consequently we find a symplecton ζ' containing P and intersecting ξ' in a plane α' (indeed, set $\zeta' = \xi(p, b)$, with $a \in P \setminus \{p\}$ and $b \in (\xi' \cap a^{\perp}) \setminus \{u'\}$). Let y' be a fixed point in α' and let $w \in P$ be collinear to y' (w exists since y' and P are contained in the same polar space ζ'). Again, there exists a fixed symplecton through wy', and this implies that w and w^{θ} are contained in the same symplecton, contradicting the assumption that P^{θ} is opposite P.

Hence no point-line flag is mapped onto an opposite and so θ is domestic. As argued above, this proves all assertions. □

5. Proof of the Main Result

In this section we classify all domestic collineations of Γ_i. There are two levels of preparations: First we prove some general properties of collineations (Section 5.1), mainly to be able to recognise some specific (domestic) collineations in a geometric way. Secondly, we prove some general properties of domestic collineations (Section 5.2), mainly to restrict the displacement of points and symplecta, to derive other geometric properties of domestic collineations and to allow us to use the results on polar spaces by reducing to equator and extended equator geometries.

5.1. **Some general properties of collineations of metasymplectic spaces.**

Lemma 5.1.1. *Let θ be a collineation of the metasymplectic space Γ_i that fixes some point c and all the points collinear or symplectic to c. Then θ is a central elation of Γ_i with centre c. Also, θ induces an axial elation in every symplecton close to c.*

Proof. Using Corollary 2.5.2 and Definition 2.11.8, it suffices to prove that every line with exactly one point collinear to c and all the other points special to c is stabilised. Let M be such a line, with m the unique point collinear to c and denote by ξ a symplecton through M. Then c is close to ξ and by taking two locally ξ-opposite planes through $c^{\perp} \cap \xi$ in ξ, we get two symplectic fixed points in ξ and hence ξ is stabilised. By the arbitrariness of ξ, we see that M is stabilised.

The argument in the previous paragraph shows that every symplecton ξ close to c is stabilised. By the definition of central elation, all lines of ξ meeting $c^{\perp} \cap \xi$ are stabilised and so the last assertion follows. □

Lemma 5.1.2. *Let θ be a central elation of a metasymplectic space Γ_i with centre c. If θ fixes one more point q (special to or opposite c), then θ is the identity.*

Proof. If q is special to c, then it lies on a line L containing a point collinear to c. Let L' be opposite L and also containing a point collinear to c. Then also the point $q' \in L'$ special to q is fixed, and so is the point $\mathfrak{c}(q, q')$, which is opposite c. So we may assume that q is opposite c.

Using Lemma 2.11.4 with respect to c, q and $E(c, q)$, we find a pointwise fixed apartment containing c. Since also all points symplectic to c are fixed, we deduce from Theorem 4.1.1 of [31], see also Theorem 6.3.1, that θ is the identity. □

Lemma 5.1.3. *Let Γ_4 be a separable metasymplectic space. Then there are no nontrivial central elations in Γ_4.*

Proof. Let θ be a central elation of Γ_4 with centre c. The last assertion of Lemma 5.1.1 combined with Lemma 3.2.9 yields the trivial action of θ on each symplecton close to c. Consequently also all points special to c are fixed. Either using Lemma 5.1.2, or directly showing that also all points of Γ_4 opposite c are fixed (which is easy), we conclude that θ is the identity. □

Lemma 5.1.4. *Let p,q be opposite points of Γ_1. Let $\{x,y\}$ be a pair of opposite points in $E(p,q)$ and let ξ be a symplecton through p intersecting $E(p,q)$ in a point $z \in x^{\perp\!\!\perp} \cap y^{\perp\!\!\perp}$.*

(*i*) *Each collineation pointwise fixing $E(p,q) \cup (E(x,y) \cap \xi)$ is a central elation with centre p.*
(*ii*) *Each collineation pointwise fixing $E(p,q) \cup \xi$ is a central elation with centre p.*
(*iii*) *Each collineation pointwise fixing $\{q\} \cup E(p,q) \cup (E(x,y) \cap \xi)$ is the identity.*
(*iv*) *Each collineation pointwise fixing the union of the two "perpendicular" equator geometries $E(p,q)$ and $E(x,y)$ is the identity.*

Proof. Under the hypotheses of (i), we have to show that the said collineation θ pointwise fixes $\{p\} \cup p^\perp \cup p^{\perp\!\!\perp}$. We first claim that θ pointwise fixes ξ, thus reducing (i) to (ii). Indeed, since θ pointwise fixes $E(p,q)$, it stabilises each symplecton through p, and so it stabilises every line through p. Hence, by projecting, it stabilises every line through z, and so it fixes $p^\perp \cap z^\perp$ pointwise. Since $\{p,z\} \subseteq \{x,y\}^{\perp\!\!\perp}$, the points x and y are close to ξ and so, if we denote $L = x^\perp \cap \xi$ and $M = y^\perp \cap \xi$, we see that $E(x,y) \cap \xi = L^\perp \cap M^\perp$.

By definition the symplecta of Γ_1 are polar spaces $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ and we can look at this situation in the ambient projective space $\mathsf{PG}(\ell,\mathbb{K})$ of ξ corresponding to the universal embedding. This is a projective space of dimension $\ell \geq 6$. The subspace U_1 generated by $p^\perp \cap z^\perp$ has dimension $\ell - 2$ and is pointwise fixed; the set $L^\perp \cap M^\perp$ spans a subspace U_2 of dimension $\ell - 4$ and is also fixed pointwise. By the Grassmann identity, and since $U_1 \cup U_2$ spans $\mathsf{PG}(\ell,\mathbb{K})$ (because this span contains z, p and $z^\perp \cap p^\perp$), these spaces intersect in a subspace of dimension $\ell - 6$. Hence they share a point and so θ fixes ξ pointwise. The claim is proved.

Now, if ξ' is a symplecton through p intersecting ξ in a plane, denote $z' := \xi' \cap E(p,q)$; then similarly $p^\perp \cap z'^\perp$ is fixed pointwise and since the plane $\xi \cap \xi'$ is pointwise fixed, Corollary 3.2.3 implies that also ξ' is pointwise fixed. By connectivity of the residue at p, we conclude that every symplecton through p is pointwise fixed, which concludes the proof of the first assertion by Lemma 5.1.1. The other assertions (iii) and (iv) now follow from Corollary 5.1.2. ☐

Lemma 5.1.5. *Let p,q be opposite points of Γ_4.*

(*i*) *Each collineation pointwise fixing $E(p,q)$ and a symplecton ξ that contains p is a central elation with centre p (and hence trivial in the separable case).*
(*ii*) *Each collineation pointwise fixing $\widehat{E}(p,q)$ and a symplecton that intersects $\widehat{E}(p,q)$ nontrivially is the identity.*
(*iii*) *Each collineation pointwise fixing $\widehat{E}(p,q)$ is the identity as soon as we are in the inseparable case.*

Proof. First assume that θ is a collineation pointwise fixing $E(p,q)$ and a symplecton ξ that contains p. Copying the proof of Lemma 5.1.4(ii) we find that θ pointwise fixes each symplecton through p; hence Lemma 5.1.1 implies (i).

Now assume θ pointwise fixes $\widehat{E}(p,q)$ and some symplecton ξ that intersects $\widehat{E}(p,q)$ nontrivially. We may assume $p \in \xi$. Then (i) imlies that θ is a central eleation with centre p, and since also q opposite p is fixed, Lemma 5.1.2 shows that θ is the identity.

Suppose now that we are in the inseparable case and that θ pointwise fixes $\widehat{E}(p,q)$. To prove (iii), it suffices to show that θ pointwise fixes some symplecton with nontrivial intersection with $\widehat{E}(p,q)$. Let ξ be a symplecton containing p. Then by Lemma 2.6.18, the intersection $\xi \cap \widehat{E}(p,q) =: h$ is a hyperbolic line. Now each point of $h^\perp$ belongs to $\widehat{T}(p,q)$ and is hence fixed. Now Lemma 3.4.1 concludes the proof. ☐

The next lemma does not hold in the separable case as the constructions in Proposition 6.5.2 are counterexamples, with an induced trivial axial elation.

Lemma 5.1.6. *A collineation of an inseparable metasymplectic space Γ_1, which induces an axial elation θ in an extended equator geometry of Γ_4, is a central elation.*

Proof. Let θ have axis A, contained in the unique symplecton ξ of Γ_4. Since all hyperbolic lines of the extended equator geometry, and hence all symplecta of Γ_4, sharing a point with A are stabilised, all planes of ξ through any point of A are stabilised. This implies that $A^\perp$ is pointwise fixed. Since also A is pointwise fixed, Lemma 3.4.1 implies that ξ is pointwise fixed. Let a be the point of Γ_1 corresponding to ξ. Then we just argued that all symplecta through a are stabilised.

Now let B be an arbitrary "line" opposite A in the extended equator geometry, and let b be the point of Γ_1 corresponding to B. Then B^θ is clearly contained in the regulus defined by A and B. Corollary 2.10.3 implies that $b^\theta \in \mathscr{I}(a,b)$ and so $E(a,b) = E(a,b^\theta)$. Together with what we proved in the first paragraph this implies that $E(a,b)$ is pointwise fixed.

By the definition of axial elation, all symplecta in Γ_4 through an arbitrary point of A are stabilised; hence the corresponding symplecton in Γ_1, which contains a, is pointwise fixed. Lemma 5.1.5(i) completes the proof. □

5.2. **Some general properties of domestic collineations of metasymplectic spaces.** We now finally come to the core of tis paper: proving properties of domestic collineations that will allow us to classify these objects.

Lemma 5.2.1. *A domestic collineation θ of any metasymplectic space Γ_i does not map any point to a special one. In particular, it (dually) induces in each fixed symplecton a plane-domestic collineation.*

Proof. Suppose for a contradiction that θ maps a point x to a special point x^θ and set $p = \mathfrak{c}(x, x^\theta)$. Let L be a line through x locally opposite both xp and $xp^{\theta^{-1}}$. This line corresponds to a plane in the polar space $\mathrm{Res}_{\Gamma_i}(x)$ opposite the planes corresponding to xp and $xp^{\theta^{-1}}$. Such a plane exists in a thick polar space of rank 3 (this is an easy exercise on the theory of polar spaces, or use Proposition 3.30 in [31]). Now every point of $L \setminus \{x\}$ is special to p. By Lemma 2.5.3, every such point is opposite x^θ and similarly x is opposite every point of $L^\theta \setminus \{x^\theta\}$. We conclude, with Definition 2.8.1(2), that L is opposite L^θ, which contradicts the possible opposition diagrams for domestic collineations in Table 2.

Suppose some plane π of a fixed symplecton ξ is mapped onto a ξ-opposite one. Then each symplecton ζ distinct from ξ through π is mapped onto a special symplecton contradicting the dual of the first statement. Remark that ζ and ζ^θ are indeed disjoint as a point in their intersection would also be contained in ξ, since it would be collinear to a symplectic pair of points from π and π^θ. □

Corollary 5.2.2.

(i) *If a symplecton ξ is mapped onto an adjacent symplecton ξ^θ by a domestic collineation θ of a metasymplectic space Γ_i, then*
  (a) *the intersection $\xi \cap \xi^\theta$ of the two symplecta is fixed pointwise;*
  (b) *at least one symplecton containing $\xi \cap \xi^\theta$ is fixed.*

(ii) *If a point p is mapped onto a collinear point p^θ by a domestic collineation θ of a metasymplectic space, then*

(a) *all planes and symplecta containing the line pp^θ are stabilised by θ (in particular the line pp^θ is stabilised);*
(b) *at least one point on the line pp^θ is fixed.*

(*iii*) *If a symplecton ξ is mapped onto a symplectic symplecton ξ^θ by a domestic collineation θ of a metasymplectic space, then the intersection point $\xi \cap \xi^\theta$ of the two symplecta is fixed.*

(*iv*) *If a point p is mapped onto a symplectic point p^θ by a domestic collineation θ of a metasymplectic space, then the symplecton $\xi(p, p^\theta)$ containing these two points is stabilised.*

Proof. We will start by proving the statements in $(i)(a)$ and (iii). The statements in $(ii)(a)$ and (iv) then follow by standard duality. Afterwards we will prove $(ii)(b)$ and again, by dualising, $(i)(b)$ follows immediately.

Suppose the symplecton ξ is mapped onto the adjacent symplecton ξ^θ and set $\pi = \xi \cap \xi^\theta$. Assume for a contradiction that some line $L \subseteq \pi$ is not fixed. Then, since each line of a symplecton is contained in at least three planes of the symplecton, we can find a plane $\alpha \neq \pi$ in ξ containing L such that $\alpha^\theta \cap \pi = L^\theta \cap \pi$. Now we pick a point $q^\theta \in \alpha^\theta \setminus L^\theta$ not collinear to L. Then $L = q^\perp \cap \pi \neq (q^\theta)^\perp \cap \pi$ and, by the possible point-symp relations, q and q^θ are special, contradicting Lemma 5.2.1. Hence each line of π is stabilised and so each point of π is fixed.

Now suppose a symplecton ξ is mapped onto a symplectic symplecton ξ^θ and set $p = \xi \cap \xi^\theta$. Assume for a contradiction that $p \neq p^\theta$. Then in the polar space ξ^θ we can pick a point q^θ collinear to p^θ, but not to p. Then $q \perp p$ is close to ξ^θ, but q^θ is special to q as q^θ is not collinear to p, again contradicting Lemma 5.2.1.

Suppose finally that a point p is mapped onto a collinear point p^θ. Consider an arbitrary plane π through $L := pp^\theta$, which is stabilised by $(ii)(a)$. If every point in $\pi \setminus L$ is fixed, it is clear that also L must be pointwise fixed, contradicting the fact that $p \neq p^\theta$. So we may assume that some $q \in \pi \setminus L$ is not fixed and as π is stabilised q must be collinear to its image. Applying $(ii)(a)$ again yields the stabilised line qq^θ. So the intersection $qq^\theta \cap pp^\theta$ is a fixed point on the line pp^θ. □

Corollary 5.2.3. *Let p, q be two opposite points of Γ_4. A domestic collineation θ of Γ_4 that stabilises $\widehat{E}(p, q)$ stabilises a hyperbolic solid through every stabilised hyperbolic plane of $\widehat{E}(p, q)$.*

Proof. Let π be a stabilised hyperbolic plane of $\widehat{E}(p, q)$ and let L be the stabilised line of $\widehat{T}(p, q)$ corresponding to π by Proposition 2.7.6. Then L must contain a fixed point by Corollary 5.2.2. This means that the hyperbolic solid corresponding to this point must be stabilised. □

Corollary 5.2.4. *Let Ω be a subspace of Γ_1 isometric and isomorphic to $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ and suppose that a domestic collineation stabilises Ω. Then θ acts type-preserving on the underlying projective plane $\mathsf{PG}(2, \mathbb{K})$, and is either a Baer involution, an elation or a homology.*

Proof. As Ω is not isomorphic to the smallest projective plane, it does not admit domestic dualities by Theorem 3.5 of [16]; this implies that, if θ induced a duality in $\mathsf{PG}(2, \mathbb{K})$, then it would map some point to an opposite line. This would mean that θ would map a line to an opposite, contradicting domesticity. Now let z be a point of $\mathsf{PG}(2, \mathbb{K})$ and suppose it corresponds to the line Z of Ω. We claim that zz^θ is stabilised. Indeed, z, zz^θ and z^θ correspond to three lines Z, W, Z^θ of Ω such that Z, W and W, Z^θ are locally opposite. The intersection point $Z \cap W$ is mapped to a point of Z^θ which must coincide with $W \cap Z^\theta$ by Lemma 5.2.1. By Lemma 5.2.2$(ii)(a)$ this means that the line W in Γ_1 is stabilised. Consequently also the line zz^θ in $\mathsf{PG}(2, \mathbb{K})$ is stabilised. So we can apply Proposition 3.3 of [19] and the result follows. □

Now we prove the analogue to Lemma 5.3 in [18]. We provide a detailed proof as the special case of a building of F_4 imposes some simplifications, whereas the assumption of not being necessarily split causes some complications. It is exactly this proposition that allows us to use the earlier derived results of domestic collineations of polar spaces applied to the equator and extended equator geometries. It provides the basis of our classification.

Proposition 5.2.5. *Let θ be a domestic collineation of $\Gamma_i = \mathsf{F}_{4,\mathsf{i}}(\mathbb{K}, \mathbb{A})$, $i = 1, 4$. Suppose θ maps some point p to an opposite.*

(i) If $i = 1$ and $\mathbb{A}$ is separable, then θ stabilises $E(p, p^\theta)$.
(ii) If $i = 1$, the opposition diagram of θ is $\mathsf{F}^1_{4;1}$ and $\mathbb{A}$ is separable, then θ pointwise fixes $E(p, p^\theta)$.
(iii) If $i = 4$ and the opposition diagram of θ is $\mathsf{F}^4_{4;1}$, then θ stabilises $\widehat{E}(p, p^\theta)$.

Proof. We will denote the residue of Γ_i in p as Δ. Recall that θ_p, from Definition 2.8.6, is a collineation of the polar space Δ. From the classification in Table 2, neither lines nor planes are mapped to opposite ones by θ in Γ_i. Together with Lemma 2.8.7, it follows that θ_p is line-domestic and by Lemma 3.5.1 θ_p is the identity or pointwise fixes a hyperplane H of Δ.

We argue in Δ (which is easier since we can then think of points, lines, planes instead of symplecta, planes and lines). By Corollary 3.5.1, we get that if some plane π or some line L of Δ is stabilised by θ_p, then it is pointwise fixed.

We claim that, under the assumptions of (i), (ii) and (iii), if two planes through a pointwise fixed line L in Δ are (necessarily pointwise) fixed, then all planes through L are (necessarily pointwise) fixed. If the opposition diagram is $\mathsf{F}^{\mathsf{i}}_{4;1}$ (Cases (ii) and (iii)), we see again by Lemma 2.8.7 that no element is mapped to an opposite and by Remark 2.11.7, θ_p is then the identity. So the claim is trivially true in these cases. Consequently we may assume that the opposition diagram is $\mathsf{F}_{4;2}$ and $i = 1$. In this case, $\Delta \cong \mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$ and we now have hyperbolic lines defined by the common perp of two opposite lines (Lemma 2.6.9). Let π_1 and π_2 be the fixed planes through L, let π' be another plane through L and let L' be a line opposite L. Then the projections p_1, p_2 and p' of L' onto π_1, π_2 and π', respectively, are points not on L. It is clear that p' lies on the stabilised hyperbolic line $h(p_1, p_2) = L^\perp \cap L'^\perp$. By considering now another line L'' not through p in the plane $\langle p', L' \rangle$, we similarly find a stabilised hyperbolic line $h(q_1, q_2) = L^\perp \cap L''^\perp$. The point p' is now fixed as the unique point in the intersection of these hyperbolic lines and so the plane π' is also fixed.

Translated to Γ_i, we have shown that each line through p is contained in a plane through p fixed under θ_p (as every plane in Δ contains a fixed line of the geometric hyperplane H), and that each line through p in such a plane is fixed under θ_p as soon as at least two such lines are fixed. We now forget the notation of the previous paragraphs, in particular L and so on.

So, if we want to show that for each line L through p, the unique point of L at distance 2 from p^θ is mapped onto the unique point of L^θ at distance 2 from p, it suffices to prove that for each plane π through p fixed under θ_p, the line $\pi \cap (p^\theta)^{\bowtie}$ is mapped onto $\pi^\theta \cap p^{\bowtie}$.

Let π be a plane through p fixed under θ_p. First assume that every line L through p in π is fixed under θ_p. Let M_p be the line in π such that $M_p^\theta = \pi^\theta \cap p^{\bowtie}$. If $M_p = \pi \cap (p^\theta)^{\bowtie}$, then there is nothing to prove, so suppose M_p and $\pi \cap (p^\theta)^{\bowtie}$ intersect in a unique point z. Then, since $(pz)^{\theta_p} = pz$, we see that $z^\theta \perp z$. Now let L be a line in π through p, but not through z. Since $|\mathbb{K}| > 2$, we can select a point $q \in L \setminus (\{p\} \cup M_p \cup (p^\theta)^{\bowtie})$. Let K be a line in π through q, and set $K \cap M_p =: \{u\}$, $pu \cap (p^\theta)^{\bowtie} =: \{v\}$ and $K \cap (p^\theta)^{\bowtie} =: \{w\}$. Since $(pv)^{\theta_p} = pv$, we have $v \perp u^\theta$. Hence $(qw)^{\theta_q} = (qu)^{\theta_q} = qv$. This now yields the equivalence

$$(qw)^{\theta_q} = qw \Leftrightarrow v = w \Leftrightarrow u = z \text{ or } u \in pq.$$

Consequently the collineation θ_q fixes π and exactly two lines through q in π, which contradicts our earlier observation, replacing p with q, that all lines through q in π are fixed as soon as at least two of them are fixed under θ_q. Hence $M_p = \pi \cap (p^\theta)^{\bowtie}$. In particular the images of the points on M_p are given by projection inside the symplecton determined by M_p and M_p^θ and hence θ preserved the cross-ratio of collinear points. We say that θ is a *linear* collineation.

Next assume that exactly one line L through p in π is fixed under θ_p. Since every such fixed line is contained in a fixed plane all of whose lines through p are fixed, we know by the previous paragraph that $z := L \cap (p^\theta)^{\bowtie}$ is mapped onto $z^\theta = L^\theta \cap p^{\bowtie}$, and these two points are collinear. Let M_p again be the line of π defined by $M_p^\theta = \pi^\theta \cap p^{\bowtie}$. Let, for each $x \in M_p$, x' be the unique point of π collinear to x^θ. Then, as a product of a linear collineation and a projection, which both preserve the cross-ratio, the correspondence $x \mapsto x'$ is a projectivity from M_p to $M'_p := \pi \cap (p^\theta)^{\bowtie}$. Since $z = M_p \cap M'_p$ is fixed under this correspondence, it is a perspectivity. Let c be the centre of this perspectivity, then $c \notin M_p \cup M'_p$ is opposite c^θ, and clearly θ_c is the identity restricted to π. By the first case, this implies that $M'_c = M_c$. Now we note that $M_p = M_c$ as the line M_p^θ is indeed special to c and similarly $M'_p = M'_c$.

Finally, assume that no line through p in π is fixed under θ_p. Let M_p be as before and assume $M_p \neq \pi \cap (p^\theta)^{\bowtie}$. Set $u = M_p \cap (p^\theta)^{\bowtie}$. We can select a point x on M_p such that $(x^\theta)^\perp$ does not contain u. Set $x' = (x^\theta)^\perp \cap \pi$ and select q on the line xx' different from x, x'. Then q is opposite q^θ, π is fixed under θ_q, xx' is fixed under θ_q and pq is not fixed under θ_q. By the previous case, $M_q = M'_q := \pi \cap (q^\theta)^{\bowtie}$. Similar to the previous paragraph we get that $M_q = M_p$ and $M'_q = M'_p := \pi \cap (p^\theta)^{\bowtie}$.

Now let ξ be an arbitrary symplecton through p. Every point of ξ at distance 2 from p^θ is collinear to the unique point e_ξ of ξ symplectic to p^θ, which is also the unique point of ξ belonging to $E(p, p^\theta)$. Hence the above yields that θ maps $p^\perp \cap e_\xi^\perp$ to $(p^\theta)^\perp \cap e_{\xi^{\theta_p}}^\perp$.

Now if $i = 1$, then ξ^θ is isomorphic to $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$. If the latter is separable, then p^θ and $e_{\xi^{\theta_p}}$ are the only two points of ξ^θ collinear to all points of $(p^\theta)^\perp \cap e_{\xi^{\theta_p}}^\perp$, by Lemma 2.6.10. It follows that $e_\xi^\theta = e_{\xi^{\theta_p}} \in E(p, p^\theta)$. Hence $E(p, p^\theta)$ is preserved by θ. This yields (i). Moreover, if the opposition diagram is $\mathsf{F}_{4;1}^1$, then θ_p is the identity (as above) and we have $e_\xi^\theta = e_\xi$ and so $E(p, p^\theta)$ is fixed pointwise. This yields (ii).

Now suppose $i = 4$. Then it follows that the hyperbolic line determined by p and e_ξ is mapped onto the hyperbolic line defined by p^θ and $e_{\xi^{\theta_p}}$ which is contained in $\widehat{E}(p, p^\theta)$ by Lemma 2.6.14. So e_ξ^θ is contained in $\widehat{E}(p, p^\theta)$ for every symplecton ξ through p. Let now e_{ξ_1} and e_{ξ_2} be two opposite points in $E(p, p^\theta)$. Then they determine $\widehat{E}(p, p^\theta)$ by Lemma 2.6.17. As their images $e_{\xi_1}^\theta$ and $e_{\xi_2}^\theta$ are still opposite points in $\widehat{E}(p, p^\theta)$, they also determine $\widehat{E}(p, p^\theta)$ and so $\widehat{E}(p, p^\theta)$ is stabilised. This yields (iii) and the proposition is completely proved. $\square$

Now we are finally prepared to classify the possible domestic collineations. We will do so by making a distinction between the inseparable and separable case.

5.3. **Domestic collineations in inseparable metasymplectic spaces.** Here, $\mathbb{A} = \mathbb{K}'$ is an inseparable field extension of $\mathbb{K}$ in characteristic 2. This means that the following inclusions of fields hold: $(\mathbb{K}')^2 \leq \mathbb{K} \leq \mathbb{K}'$. An important property of these metasymplectic spaces is that $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{K}') \cong \mathsf{F}_{4,4}(\mathbb{K}'^2,\mathbb{K})$. This can be easily proven, when we look at the definitions for $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ and $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ (i.e. Definitions 2.3.1 and 2.3.2, respectively) and use the isomorphisms:

$$\begin{array}{llll} \phi: & \mathsf{B}_{3,1}(\mathbb{K},\mathbb{K}') & \to & \mathsf{C}_{3,1}(\mathbb{K},\mathbb{K}'^2): \\ & (x_{-3},x_{-2},x_{-1},x_0,x_1,x_2,x_3) & \mapsto & (x_{-3},x_{-2},x_{-1},x_1,x_2,x_3) \end{array}$$

and

$$\begin{array}{llll} \psi: & \mathsf{B}_{3,1}(\mathbb{K}'^2,\mathbb{K}) & \to & \mathsf{C}_{3,1}(\mathbb{K}',\mathbb{K}): \\ & (x_{-3}^2,x_{-2}^2,x_{-1}^2,x_0,x_1^2,x_2^2,x_3^2) & \mapsto & (x_{-3},x_{-2},x_{-1},x_1,x_2,x_3). \end{array}$$

This isomorphism between metasymplectic spaces allows us for example to speak about the extended equator geometry and tropics geometry in a metasymplectic space Γ_1, as these are the geometries isomorphic to the extended equator geometry and tropics geometry in the isomorphic metasymplectic space Γ_4. Therefor in this section, we will speak about Γ instead of Γ_1 or Γ_4; this Γ is at the same time a Γ_1 and a Γ_4, but for different pairs of fields. A good example of the power of this isomorphism is the following lemma.

Lemma 5.3.1. *Let p,q be two opposite points of the inseparable metasymplectic space Γ. Then $\mathscr{I}(p,q) = E(p,q)^{\perp\perp}$ equals the "hyperbolic line" in the polar space $\widehat{E}(p,q)$ through the opposite points p and q.*

Proof. This follows straight from Proposition 2.10.5 noting that the "hyperbolic line" through p and q in $\widehat{E}(p,q)$ coincides with the "hyperbolic line" through these points in $E(a,b)$, for each pair of opposite points $\{a,b\} \subseteq E(p,q)$. □

Theorem 5.3.2. *Let θ be a domestic collineation of an inseparable metasymplectic space $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{K}')$. Then one of the following holds.*

(i) θ is a central elation and the opposition diagram is $\mathsf{F}^1_{4;1}$ or $\mathsf{F}^4_{4;1}$;
(ii) θ is the product of two perpendicular central elations, and then the opposition diagram is $\mathsf{F}_{4;2}$. There are three types of such products: those that are only products of perpendicular central elations in $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{K}')$, those that are only products of perpendicular central elations in $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K}')$ and those that are products of perpendicular central elations in both;
(iii) θ is an involution with fix structure consisting of points and symplecta forming a Moufang quadrangle of mixed type and the opposition diagram is $\mathsf{F}_{4;2}$. Here the fixed points in a fixed symplecton ξ form an ovoid, which consists of the set of points of the perp of a line L of the unique symplectic polar space in which ξ is fully embedded, but L does not contain points of ξ; however, L is a singular line with respect to the symplectic form. Also the dual holds. This third case does not occur when $\mathbb{K}'$ equals $\mathbb{K}$ (i.e. in the split case).

Proof. Let θ be a domestic collineation of Γ. Without loss of generality (possibly by going to the dual), we may assume that θ is not point-domestic. Let p be a point mapped onto an opposite. By Proposition 5.2.5, $\widehat{E}(p,p^\theta)$ is preserved by θ. Remark that by similar isomorphisms as above $\widehat{E}(p,p^\theta) \cong \mathsf{B}_{4,1}(\mathbb{K},\mathbb{K}') \cong \mathsf{C}_{4,1}(\mathbb{K},\mathbb{K}'^2)$, and the latter polar space is embedded in a symplectic polar space $\Delta \cong \mathsf{C}_{4,1}(\mathbb{K},\mathbb{K})$ defined by the standard alternating form in $\mathsf{PG}(7,\mathbb{K})$:

$$x_{-4}y_4 + x_4y_{-4} + x_{-3}y_3 + x_3y_{-3} + x_{-2}y_2 + x_2y_{-2} + x_{-1}y_1 + x_1y_{-1},$$

and we denote ρ for the associated polarity.

We claim now that θ induces a plane-domestic collineation in $\widehat{E}(p,p^\theta)$. Suppose for a contradiction that π is a hyperbolic plane in $\widehat{E}(p,p^\theta)$ mapped to an opposite plane π^θ. As there are no symplectic polarities in a plane (cf. [27]), there must be a point $q\in\pi$ mapped to an opposite point q^θ (otherwise $x\mapsto(x^\theta)^{\perp\!\!\!\perp}\cap\pi$ would be a symplectic polarity of π). Now the "line" $\pi\cap(q^\theta)^{\perp\!\!\!\perp}$ corresponds to a plane α through q by Proposition 2.6.11 and must be mapped to an opposite "line" in π^θ, corresponding to a plane β through q^θ. Now again by Proposition 2.6.11, α must be opposite β as the corresponding lines in $E(q,q^\theta)$ are opposite (as they are contained in π and π^θ respectively). Consequently α is mapped to an opposite plane, contradicting domesticity of θ (cf. Table 2). So the claim is proved. Since at least one point is mapped onto an opposite, Corollary 4 of [16] implies that no "solid" is mapped onto an opposite. Then by Theorem 6.1 of [28], θ pointwise fixes a (sub)hyperplane H of $\widehat{E}(p,p^\theta)$, and hence it pointwise fixes a (sub)hyperplane $\overline{H}=\langle H\rangle$ (generation in $\mathsf{PG}(7,\mathbb{K})$) of Δ. By Lemma 3.1.2(ii), $\overline{H}=x^\rho$ (x a point of Δ) or $\overline{H}=L^\rho$ (L a projective line).

Suppose first that $\overline{H}=x^\rho$, for some point x of Δ. We claim that the opposition diagram of θ must be $\mathsf{F}_{4;1}$. Indeed, we already assumed that a point is mapped to an opposite one, so the only other possibility is that the opposition diagram would be $\mathsf{F}_{4;2}$. If that is the case, we can assume that the p we chose at the beginning is part of a point-symp flag $\{p,\xi\}$ mapped to an opposite one and so the "line" in $\widehat{E}(p,p^\theta)$ corresponding to ξ would be mapped to an opposite one (Corollary 2.9.3), contradicting the fact that there is a pointwise fixed hyperplane in $\widehat{E}(p,p^\theta)$. So the claim is proved.

We now claim that $x\in\widehat{E}(p,p^\theta)$. Suppose for a contradiction that $x\notin\widehat{E}(p,p^\theta)$. As a geometric hyperplane of a polar space of rank 4 contains "planes", we get by Lemma 5.2.3 that θ must have a stabilised "solid" S in $\widehat{E}(p,p^\theta)$. By Lemma 3.5.1 this "solid" S is contained in H, contradicting $x\notin H$.

This means that θ induces a central elation with centre x in $\widehat{E}(p,p^\theta)$. Lemma 5.3.1 and Lemma 6.5.1 imply that there is a central elation θ' with centre x mapping p to p^θ. Clearly, θ' induces a central elation in $\widehat{E}(p,p^\theta)$ mapping p to p^θ. Lemma 3.2.2 implies that θ and θ' coincide over $\widehat{E}(p,p^\theta)$ and Lemma 5.1.5 then implies $\theta=\theta'$. The opposition diagram follows from Proposition 4.1.1 and hence we are in Case (i) of the theorem.

Suppose now that $\overline{H}=L^\rho$. As this is the last possible case, we may assume that this is the case for every extended equator geometry determined by an opposite pair $\{q,q^\theta\}$ and that there is no pointwise fixed geometric hyperplane in the corresponding Δ. There are now two possible cases for the line L: L can be singular or nonsingular with regard to the underlying polarity ρ of Δ. We prove that also the singularity of L is the same for every extended equator geometry related to an opposite pair $\{q,q^\theta\}$, by proving that θ is an involution if, and only if, L is singular.

<u>Suppose first that L is nonsingular</u>, i.e. $L\nsubseteq\overline{H}$. Suppose for a contradiction that θ were an involution. Then θ would also induce an involution on the plane α of the underlying projective space of Δ spanned by L and a point $h\in H$. If now every point in $\alpha\setminus L$ is fixed, then L is clearly also pointwise fixed and if a point $a\in\alpha\setminus L$ is not fixed then the intersection of the stabilised lines aa^θ and L is fixed, so in every case L contains a fixed point b. Now θ induces on every line bh, with $h\in\overline{H}$, a linear involution fixing two points. Hence θ pointwise fixes bh and hence also $\langle b,\overline{H}\rangle$, implying that θ induces in Δ a central elation with centre b. So by renaming $\overline{H}$ as this span, we would be in the case $\overline{H}=x^\rho$, contradicting our assumptions.

<u>Suppose now that L is singular</u>, i.e. $L\subseteq\overline{H}$. Then dually all hyperplanes through L in the underlying projective space of Δ are stabilised and consequently also all planes through L are stabilised. Suppose now that $y\notin L$ is not fixed and set $\alpha:=\langle y,L\rangle$. As θ pointwise fixes the line L of this

plane, it induces a perspectivity in α. Suppose first that this is a homology and also the point $q \in \alpha \setminus \{L\}$ is fixed. Let β be a symplectic plane through L, then β is pointwise fixed. Projecting q onto β yields a (pointwise fixed) line M containing a fixed point $q' \in \beta \setminus \{L\}$ collinear to q. Then the line qq' is stabilised. Now every point q'' on this line is fixed, as it is the intersection of a stabilised line with a stabilised plane $\langle q'', L\rangle$. Consequently the plane $\langle q, M\rangle$ contains at least two pointwise fixed lines and is pointwise fixed. Hence also the plane α contains two pointwise fixed lines, a contradiction. So θ induces an elation in α and by the arbitrariness of y we now have that θ is an involution.

We now choose an appropriate skeleton as basis, i.e. two points on L (corresponding to x_{-2}, x_{-1}), four in H (corresponding to $x_{\pm 3}, x_{\pm 4}$) and the two others on neither H nor L. Then we see this choice can be made so that collinearity is given by the standard alternating bilinear form $x_{-4}y_4 + x_{-3}x_3 + x_{-2}y_2 + x_{-1}y_1 + x_1y_{-1} + x_2y_{-2} + x_3y_{-3} + x_4y_{-4}$. By the choice of the coordinates, θ acts trivially on the coordinates x_i with $i = \pm 3, \pm 4$ and the associated field automorphism is trivial. Now one can easily calculate that the action of θ on the subspace $\langle e_{-2}, e_{-1}, e_1, e_2\rangle$ of the projective space underlying Δ is given by the following matrix:

$$A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $a, b, c \in \mathbb{K}'^2$. Suppose first that $b = c = 0$. Then we see that θ acts on Δ, and hence also on $\widehat{E}(p, p^\theta)$, as an axial elation with axis L. This contradicts the fact that the point p is mapped to an opposite point. So we may suppose without loss of generality that $c \neq 0$. Then we get that

$$A = \begin{pmatrix} 1 & 0 & a & \frac{a^2}{c} \\ 0 & 1 & c & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & b + \frac{a^2}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now we see that θ is clearly the product of two central elations with respective centres $e := \langle ae_{-2} + ce_{-1}\rangle$ and $e' := \langle e_{-2}\rangle$. Note that these centres are collinear in Δ, in particular they are contained in L. Now we can apply Proposition 3.4.2 on $\widehat{E}(p, p^\theta)$.

The Cases (i), (iii) and (iv) of that proposition yield immediately Case (ii) of this theorem, taking Proposition 4.1.2 and Lemma 5.1.6 into account.

So we may assume that the fix structure of θ in $\widehat{E}(p, p^\theta)$ is a generalised quadrangle obtained by intersecting L^ρ with $\widehat{E}(p, p^\theta)$ and there is no point of $\widehat{E}(p, p^\theta)$ mapped to a "collinear" one.

We claim that θ does not fix any line. Indeed, suppose for a contradiction that the line R of Γ is fixed by θ. Suppose first that some point r of R is not fixed by θ. Then select some point $q \perp r$ such that $r^\theta \in q^{\bowtie}$. Then $q^\theta \in r^{\bowtie}$ (as R is fixed and q is special to all points of R except r) and $\{q, q^\theta\}$ is an opposite pair, by Lemma 2.5.3. Clearly the line qr is fixed by θ_q, implying that the corresponding plane in $E(q, q^\theta)$ is fixed, a contradiction with the described fix structure of $\widehat{E}(p, p^\theta)$ in the previous paragraph and the fact that we may make the same assumptions on $\widehat{E}(q, q^\theta)$ as on $\widehat{E}(p, p^\theta)$ as remarked above. Hence R is pointwise fixed.

Select two "collinear" (in $\widehat{E}(p, p^\theta)$) fixed points u, v and denote $\xi := \xi(u, v)$. Suppose first that R is contained in ξ. Since at least one point of R is collinear to u, we may even assume that $u \in R$. Select $u' \in \widehat{E}(p, p^\theta)$ opposite u and also fixed by θ. Then the plane of $E(u, u') \subseteq \widehat{E}(p, p^\theta)$ corresponding to R is stabilised, a contradiction. Hence R does not belong to ξ. If some point

$r \in R$ is collinear to a unique line R' of ξ, which is also fixed we get a contradiction by replacing R with R'. If, lastly no point of R is close to ξ, then by the possible point-line relations, we have that every point of ξ symplectic to a point of R is symplectic to a unique point of R. Now the projections of two points of R onto ξ yield a line R' in ξ which is stabilised by θ, again a contradiction to the previous cases. The claim is proved.

So no line is fixed by θ. Suppose now that a plane π is stabilised by θ. Then as no line is fixed, there is a point mapped to a distinct, necessarily collinear point, but then the line determined by these points is stabilised by Corollary 5.2.2 $(ii)(a)$, a contradiction. It now easily follows that the fixed points and fixed symplecta form the point set and line set of a generalised quadrangle. We now claim that the fixed points in a fixed symplecton form an ovoid $\mathscr{O}$ of that symplecton. Indeed, by Lemma 7.4 of [28], it suffices to show that θ restricted to ξ is domestic, which follows from Lemma 5.2.1. The claim is proved and clearly also the dual holds.

We claim that we are now in Case (iii) of the theorem. Let a, b be two "collinear" points of the fixed quadrangle in $\widehat{E}(p, p^\theta)$ described above, then clearly the symplecton $\xi(a, b)$ is fixed and we claim that the fixed points in this symplecton are as described in the statement. Indeed, the dual holds by considering two opposite fixed points and noting that the fixed structure in the residue at one of them is isomorphic to the fix structure in the equator geometry defined by them. Since this is the last possibility for an involution, we may assume that also the dual holds. This case does clearly not occur when $\mathbb{K} = \mathbb{K}'$, as the unique symplectic polar space wherein ξ is fully embedded is then ξ itself and does not contain singular lines disjoint from ξ. The opposition diagram follows from Proposition 4.2.1. The quadrangle is Moufang of mixed type by the main result in [14].

Suppose finally again that L is nonsingular. It suffices now to prove that this leads to a contradiction. We first claim that no point of $\widehat{E}(p, p^\theta)$ is mapped to a "collinear" point. Suppose for a contradiction that some non-fixed point t is mapped onto a collinear point $t^\theta \neq t$. Denote $x_L := t^\rho \cap L$ and $x_{\overline{H}} := tx_L \cap \overline{H}$. The projective plane $\pi = \langle L, x_{\overline{H}} \rangle$ is preserved by θ, and so $\langle t, t^\theta \rangle$ is contained in π. As all lines through $x_{\overline{H}}$ in π are singular w.r.t. the polarity ρ and there are no other singular lines in π, since π contains the nonsingular line L, we see that $\langle t, t^\theta \rangle = \langle x_L, x_{\overline{H}} \rangle$. Hence θ stabilises $\langle t, t^\theta \rangle$ and so fixes the point x_L. So the matrix of θ restricted to $\widehat{E}(p, p^\theta)$, and with respect to an appropriate basis, i.e. a skeleton consisting of the point x_L (first base point), another point on L (second base point) and the rest in H (the other base points), is a block matrix of the following form:

$$\begin{pmatrix} a & b & & & \\ 0 & c & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}. \tag{2}$$

The companion field automorphism is trivial since H is fixed pointwise and contains full lines. We may assume that the polar space $\widehat{E}(p, p^\theta)$ is described by the standard equation (as given above, namely $x_{-4}x_4 + x_{-3}x_3 + x_{-2}x_2 + x_{-1}x_1 \in \mathbb{K}'^2$) in this basis. So expressing that the matrix represents a collineation of that polar space $\widehat{E}(p, p^\theta)$, gives that $c = a^{-1}$ and $ab \in \mathbb{K}'^2$. Note that $a \neq 1$ as otherwise we are in the case that a geometric hyperplane is pointwise fixed. Hence we see that θ fixes the additional point $y_L = (1, ab^{-1} + a^{-1}b^{-1}, 0, \cdots, 0)$ in $\widehat{E}(p, p^\theta)$ on L.

Consequently θ pointwise fixes $\{x_M, y_M\} \cup E(x_M, y_M) \subseteq \widehat{E}(p, p^\theta)$. This implies that θ fixes each line through x_L and each line through y_L. Pick two locally opposite lines M, N through x_L and denote $x_M := y_L^{\bowtie} \cap M$, $x_N := y_L^{\bowtie} \cap N$, $y_M := \mathfrak{c}(y_L, x_M)$, $y_N := \mathfrak{c}(y_L, x_N)$. Corollary 2.9.9 implies that these six points are contained in a subspace Ω isometric and isomorphic to $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$. Since

θ stabilises the ordinary hexagon given by the six points $x_L, x_M, y_M, y_L, y_N, x_N$, Corollary 5.2.4 implies that θ induces either the identity, a Baer involution, or a homology in the corresponding projective plane π. The second option is impossible since θ pointwise fixes hyperbolic lines in $\widehat{E}(p, p^\theta)$, implying it cannot act as a semilinear collineation on the "hyperbolic line" defined by x_L and y_L.

If the lines $x_M y_M$ and $x_N y_N$ are pointwise fixed, then $\mathscr{I}(x_L, y_L)$ is pointwise fixed and by Lemma 5.3.1 this means that the line L is pointwise fixed in the underlying projective space of Δ. Then by Lemma 3.4.1 $\widehat{E}(p, p^\theta)$ is pointwise fixed, which contradicts the fact that $p \neq p^\theta$. So we may assume that $x_M y_M$ is not pointwise fixed and consequently θ induces a nontrivial homology in π. Then we see that exactly one of the lines M and N is pointwise fixed. Now we get a contradiction as follows. Take a line K through x_L locally opposite both M and N (this is possible by Proposition 3.30 of [31]). Applying now the previous arguments to the sets $\{M, K\}$ and $\{N, K\}$, we see that this line must be at the same time pointwise fixed and not pointwise fixed, which is of course impossible. So in all cases we get a contradiction, hence θ does not map any non-fixed point of $\widehat{E}(p, p^\theta)$ to a collinear point.

We now claim that θ does not stabilise any "plane" of $\widehat{E}(p, p^\theta)$. Suppose for a contradiction that $\widehat{E}(p, p^\theta)$ contains a stabilised hyperbolic plane π. By Corollary 5.2.3, there exists a stabilised hyperbolic solid S through π and as no point is mapped to a collinear one in $\widehat{E}(p, p^\theta)$, we deduce that S is pointwise fixed. The geometric subhyperplane H cannot contain S, as this would span a 4-space in Δ with a point of L. So $\langle S, H\rangle$ spans at least a hyperplane $\overline{H}'$ in Δ. But then Δ is fixed or we have a point $x \in \overline{H}' \cap L$, for which $x^\rho \supseteq \langle x, \overline{H}\rangle = \overline{H}'$, which contradicts our assumptions on the current case.

Now we can apply the last four paragraphs of the case that L is singular (except the last sentence about the quadrangle being Moufang of mixed type). This leads however to a contradiction as by Theorem 6.3 of [23], θ must be an involution in this case. □

5.4. **Domestic collineations in separable metasymplectic spaces.** Now we will classify the domestic collineations in the separable metasymplectic spaces. As noted before, we will make here a distinction between the different nontrivial opposition diagrams.

Theorem 5.4.1. *If a domestic collineation θ of a separable building $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$ has opposition diagram $\mathsf{F}^1_{4;1}$, then θ is a central elation in $\Gamma_1 \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$.*

Proof. Considering the corresponding metasymplectic space $\Gamma_1 \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$, and a point p mapped onto an opposite, Proposition 5.2.5 implies that θ pointwise fixes $E(p, p^\theta)$. This implies by Proposition 2.10.5 that the imaginary line $\mathscr{I}(p, p^\theta)$ is stabilised. Now for every path $p \perp x \perp y \perp p^\theta$, the line $L := xy$ is stabilised and hence contains a fixed point f by Corollary 5.2.2$(ii)(b)$. The unique point c of $\mathscr{I}(p, p^\theta)$ collinear to f is consequently also fixed. Now select two such paths $p \perp x_i \perp y_i \perp p^\theta$ with corresponding lines $L_i = x_i y_i$, $i = 1, 2$, such that L_1 is opposite L_2 (it suffices to choose px_1 locally opposite px_2 to achieve that).

Corollary 2.9.9 implies that L_1 and L_2 are contained in a unique common subspace Ω isometric and isomorphic to $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ which, by Definition 2.10.2, contains $\mathscr{I}(p, p^\theta)$. Let M_i be the unique line of Ω containing c and intersecting L_i in a point, say z_i, $i = 1, 2$. Since L_i is the intersection of the symplecta defined by the "lines" in the "plane" of $E(p, p^\theta)$ consisting of the points corresponding to the symplecta containing px_i, $i = 1, 2$, the line L_i is stabilized by θ. Hence θ induces a collineation in Ω fixing the points c, z_1, z_2. Using Corollary 5.2.4 we see that, if θ fixes no more points on the lines M_1 and M_2, then it induces a homology in the underlying projective plane and has to pointwise fix the lines L_1 and L_2. This, however, contradicts the fact that $p \neq p^\theta$. Hence at least one point on $(M_1 \cup M_2) \setminus \{c, z_1, z_2\}$, say of M_1, is fixed. But now Corollary 3.2.3 yields a pointwise fixed symplecton ξ through M_1. Subsequently Lemma 5.1.4(ii) implies that θ is a central elation with centre c. □

Theorem 5.4.2. *If a domestic collineation θ of a separable metasymplectic space $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ has opposition diagram $\mathsf{F}^4_{4;1}$, then one of the following holds:*

- (i) *$\mathbb{A}$ is a quadratic extension of $\mathbb{K}$ and θ is an involution pointwise fixing a metasymplectic space canonically isomorphic to $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$;*
- (ii) *the building is split, i.e. $\mathbb{A} = \mathbb{K}$, and θ is an involution with fix structure an extended equator geometry and its tropics geometry in $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$.*

Proof. Considering the corresponding metasymplectic space $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$, and a point p mapped onto an opposite, Proposition 5.2.5 implies that θ stabilises $\widehat{E}(p, p^\theta)$. Corollary 2.9.3 (iv) now implies that θ induces a line-domestic collineation in $\widehat{E}(p, p^\theta)$. Hence by Lemma 3.5.1, θ pointwise fixes a geometric hyperplane H of $\widehat{E}(p, p^\theta)$. If H was singular, i.e. $H = u^\perp$ with $u \in \widehat{E}(p, p^\theta)$, θ would be a central elation with centre u, which is impossible by Lemma 3.2.4 as $\widehat{E}(p, p^\theta)$ is a separable orthogonal polar space. So H is nonsingular, in particular a polar subspace of rank at least 3 containing two (pointwise fixed) opposite hyperbolic lines. Lemma 2.9.3 (iv) implies that the corresponding stabilised symplecta ζ, ξ of Γ_4 are also opposite. Let a, b be the points of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ corresponding to ζ, ξ respectively. Now θ induces in ζ a point-domestic collineation as $\zeta \cong E(a, b)$ and opposition in $E(a, b)$ corresponds to opposition in Γ_1. With Corollary 4 of [16], we can now apply some propositions of [19].

In the **nonsplit case**, we apply Proposition 3.11 of that article and see that θ induces either an axial elation, or a generalised Baer collineation in ζ. Since the polar space ζ is isomorphic to $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$, it does not admit axial elations by Lemma 3.2.9. It follows that θ induces a generalised Baer collineation in ζ.

We now claim that each fixed point a^* in Γ_1 (corresponding to a stabilised symplecton ζ^* in Γ_4) admits an opposite fixed point b^*, so we can apply the previous paragraph to these points and the corresponding symplecta. We first show the claim for fixed points collinear to a. Let x be such a point and set $L := ax$. In $E(a, b)$, the line L corresponds to a stabilised "plane" α. Select, using the previous paragraph, a "stabilised" plane β in $E(a, b)$ opposite α. Then β corresponds to a fixed line $M \ni b$. Since α and β are opposite, also the lines L and M are opposite (this follows from Lemma 2.8.7). The unique point b' of M special to a is then opposite x (see Lemma 2.5.3) and is fixed. Hence the claim follows for $x \perp a$. Now let z be an arbitrary fixed point. If z is opposite a, there is nothing to prove. If z is special to a, then the point $a \bowtie z$ is also fixed, and the foregoing implies first that $\mathfrak{c}(a, z)$ admits an opposite fixed point, and then also $z \perp \mathfrak{c}(a, z)$ admits an opposite fixed point. If $z \perp\!\!\!\perp a$, then the symplecton $\xi(a, z)$ is fixed, and hence corresponds to a fixed point $f \in E(a, b)$. Selecting a fixed plane of $E(a, b)$ through f (which is possible by the previous paragraph), we obtain a fixed line R through a in $\xi(a, z)$. Now R contains a fixed point collinear to z, and the claim follows again from the previous paragraph. Finally, if $z \perp a$, then we already showed the claim.

Let L now be any stabilised line in Γ_1 (such a line exists as in the residue of a fixed point a^* we find a stabilised plane corresponding in Γ_1 to a stabilised line through a^*). We claim that L is then pointwise fixed. By Corollary 5.2.2 $(ii)(b)$, L contains at least one fixed point c. By the previous paragraph c has an opposite fixed point c' and θ induces on $E(c, c')$ a generalised Baer collineation. So there exists a symplecton through L that is not fixed and then by Corollary 5.2.2 $(i)(a)$ we have that L is pointwise fixed and the claim is proved.

Since the fixed points and lines in α, the "plane" corresponding to L in $E(c,c')$, form a Baer subplane, the fixed planes and fixed symplecta through L also form a Baer subplane of the residue of L. So there exist a stabilised plane π and a stabilised symplecton ζ through L and consequently one finds a stabilised chamber $C := (x, L, \pi, \zeta)$, with $x \in L$ arbitrary. We also have a fixed point x' opposite x and as the residue of x' contains a fixed Baer polar space, we can find a fixed line L' through x' opposite L. Similarly we find a fixed plane π' and a fixed symplecton ζ' so that the fixed chamber $C' = (x', L', \pi', \zeta')$ is opposite C. Now these two chambers span a fixed apartment in Γ_1.

Now we apply some arguments of [13]. Let G denote the group generated by the automorphism θ. The fact that an apartment is stabilised elementwise means that the group is type preserving and fixes two opposite chambers. Thus, in the terminology of [13], all fixed chambers are G-chambers. Since there are two opposite G-chambers, every G-panel contains at least two G-chambers by 22.34(ii) in [13]. Now, since all points on all lines of the fixed apartment are fixed, one can apply 22.14(iii) in [13] to conclude that the set of fixed chambers forms a subgeometry of type $\mathsf{F}_{4,1}$ over $\mathbb{K}$. Hence the fix structure is a metasymplectic space $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{B})$, where $\mathbb{B}$ is a quaternion subalgebra of $\mathbb{A}$ if $\mathbb{A}$ is octonion (since Baer subplanes of octonion planes are quaternion planes), $\mathbb{B}$ is a quadratic extension of $\mathbb{K}$ if $\mathbb{A}$ is quaternion (since Baer subplanes of quaternion planes are planes over a quadratic extension), and $\mathbb{B} = \mathbb{K}$ if $\mathbb{A}$ is a separable quadratic extension of $\mathbb{K}$ (since Baer subplanes of a plane over a quadratic extension $\mathbb{L}$ of $\mathbb{K}$ are isomorphic to $\mathsf{PG}(2, \mathbb{K})$ (note that the fixed subfield of $\mathbb{L}$ must be an algebra over $\mathbb{K}$ and hence must coincide with $\mathbb{K}$)).

Now we show that $\mathbb{A}$ is a quadratic extension of $\mathbb{K}$. Let ξ be a stabilised symplecton of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$. Then the fix structure of θ in ξ is a subquadric $\zeta \subseteq \xi$ of Witt-index 3 the anisotropic kernel of which corresponds to the norm of $\mathbb{B}$. If $\mathbb{A}$ is octonion or quaternion, then the codimension of $\langle \zeta \rangle$ in the ambient projective space of ξ is 4 or 2, respectively. Hence ζ is not a geometric hyperplane (see Lemma 3.1.2) and so, by Lemma 3.5.1, θ does not act line-domestically in ξ, . Let L be a line of ξ mapped to a ξ-opposite line L^θ. Then Corollary 2.5.4 yields points mapped to opposites in Γ_1, a contradiction.

So now we may assume that we are in the **split case** and we can apply Theorem 3.13 of [19]. So ζ is the symplectic polar space corresponding to the alternating form $x_{-3}y_3 - x_3y_{-3} + x_{-2}y_2 - x_2y_{-2} + x_{-1}y_1 - x_1y_{-1}$, and θ acts on $\zeta \cong E(a,b)$ by the following matrix:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is now clear that we have a fixed apartment in this symplecton ζ given by the points $p_{-3} = (1,0,\dots,0), p_{-2} = (0,1,0,\dots,0), \dots, p_3 = (0,\dots,0,1)$. As above we can use the isomorphism between ζ in Γ_4 and $E(a,b)$ in Γ_1 to get a similar fixed apartment Λ' in $E(a,b)$. By Lemma 2.11.4 this apartment together with the fixed points a and b gives rise to an elementwise fixed apartment Λ of Γ_1. We now claim that every line in this apartment Λ is pointwise fixed (and consequently also every plane is pointwise fixed as an apartment of a projective plane exists of three lines). Let L for instance be the line in $\xi(a, p_1)$ collinear to both a and p_1 corresponding to the "line" p_1p_2 in $E(a,b)$. This line is pointwise fixed as every point on the line corresponds to a "plane" through

p_1p_2 and these planes are stabilised as θ acts point-domestic on $E(a,b)$, as noticed in the first paragraph. Similarly as in the last paragraph of the proof of Theorem 5.4.3, one now shows that θ fixes exactly an extended equator geometry and its tropics geometry of Γ_4.

We finally show that θ is in both cases an **involution**. As the hyperplane H from the first paragraph is nonsingular, $H = v^\rho \cap \widehat{E}(p,p^\theta)$, with v a point in the underlying projective space of $\widehat{E}(p,p^\theta)$ and ρ the polarity in that space defining the polar space $\widehat{E}(p,p^\theta)$. As $\widehat{E}(p,p^\theta)$ is a quadric, every line through v intersects $\widehat{E}(p,p^\theta)$ in at most two points which can be fixed or swapped. In every case θ^2 acts trivial on $\widehat{E}(p,p^\theta)$. In both cases θ^2 also fixes the symplecton ζ of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ (remark that in the nonsplit case, θ acts involutive on ζ by Lemma 3.3.5). It follows then by Lemma 5.1.5 that θ^2 is the identity. □

Theorem 5.4.3. *Let θ be a domestic collineation of a separable metasymplectic space $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ with opposition diagram $\mathsf{F}_{4;2}$. Then one of the following holds:*

(i) θ is the product of two perpendicular central elations in $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$;
(ii) the fix structure of θ consists of points and symplecta forming a generalised quadrangle. Here the fixed points in a fixed symplecton form an ovoid, which arises as the intersection with a subspace, in the unique projective embedding. Dually the fixed symplecta through a fixed point form an ovoid in the residue of that point and in the corresponding $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ these ovoids in symplecta are closed under taking hyperbolic lines. This second case does neither occur when $\mathbb{A}$ is an octonion division algebra, nor in the split case;
(iii) $\mathbb{A}$ is a separable quadratic extension of $\mathbb{K}$ and θ is a generalised homology with fix structure an extended equator geometry and its tropics geometry in $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$;
(iv) $\mathbb{A}$ is a quaternion division algebra over $\mathbb{K}$ and θ is a generalised homology pointwise fixing a metasymplectic space canonically isomorphic to $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{L})$, where $\mathbb{L}$ is a subalgebra of $\mathbb{A}$ of dimension 2 fixed under some automorphism of $\mathbb{A}$ (hence $\mathbb{L}$ is a field).

Proof. Consider the corresponding metasymplectic space $\Gamma_1 := \mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$, and a point-symp pair (p,ν) mapped onto an opposite. Proposition 5.2.5 implies that θ stabilises $E(p,p^\theta)$ and the opposition diagram implies that θ_p (see Definition 2.8.6) is line-domestic by Lemma 2.8.7, so θ induces a nontrivial line-domestic collineation in $E(p,p^\theta)$. It follows from Lemma 3.5.1 that the fix structure of θ in $E(p,p^\theta)$ is a geometric hyperplane H. There are two possibilities.

Suppose first that H is singular, that is, H is the perp of a point $x \in E(p,p^\theta)$, so $H = x^{\perp\!\!\!\perp} \cap E(p,p^\theta)$. Let ξ be any symplecton through x intersecting $E(p,p^\theta)$ in a line O of $E(p,p^\theta)$. As in the proof of Lemma 2.6.7, $O = L^\perp \cap (L^\theta)^\perp$, with $L = p^\perp \cap \xi$. Since every line M in the plane $\langle p, L\rangle$ through p corresponds to a plane α of $E(p,p^\theta)$ through x, and each such plane is (even pointwise) fixed by θ, we conclude that $M^\theta = \mathsf{proj}^p_{p^\theta}(M)$. From there we deduce that $z \perp z^\theta$, for each point $z \in L$. By Corollary 5.2.2(ii), at least one point $u \in zz^\theta$ is fixed, giving rise to a fixed point q in the imaginary line $\mathscr{I}(p,p^\theta)$.

Now select a point $y \in E(p,p^\theta)$ opposite x (for instance $y = \nu \cap E(p,p^\theta)$). Since $y \notin H$, y^θ is opposite y by Lemma 3.5.1 and we can consider $E(y,y^\theta)$. Since $p \in E(y,y^\theta)$, the induced collineation of $E(y,y^\theta)$ is nontrivial, but line-domestic (as in the first paragraph). The corresponding fixed geometric hyperplane H' contains $y^{\perp\!\!\!\perp} \cap (y^\theta)^{\perp\!\!\!\perp} \cap E(p,p^\theta) = y^{\perp\!\!\!\perp} \cap x^{\perp\!\!\!\perp} \cap E(p,p^\theta)$ and $q \perp\!\!\!\perp E(p,p^\theta)$. Hence, since H' is a subspace, H' is again singular and collinear to q as every line L through q in $E(y,y^\theta)$ contains two fixed points, namely q and the projection of p onto it (i.e. $L \cap E(p,p^\theta)$); use also Lemma 7.5.1 of [24] that says that the complement of a hyperplane of a polar space of rank at least two is always connected and consequently there are no hyperplanes properly contained in proper hyperplanes.

Now θ composed with the product of the inverse of two suitable central elations of Γ_i with centres x and q pointwise fixes $E(p,p^\theta)\cup E(y,y^\theta)$, the union of two perpendicular equator geometries. Hence this composition is the identity by Lemma 5.1.4 and θ is the product of two perpendicular central elations of Γ_1. This is (i).

We now claim that we are in the previous case, as soon as we find an imaginary line $\mathscr{C}$ stabilised by θ but not pointwise fixed, and a point $c\in\mathscr{C}$ fixed by θ. We may then assume $p\in\mathscr{C}$, but we cannot longer assume that some symplecton through p is mapped onto an opposite, so θ_p may as well be trivial.

Suppose for a contradiction that H (as defined above) is nonsingular and proper and let O be a pointwise fixed "line" in $E(p,p^\theta)$. That line corresponds to a plane π of Γ_1 through c, containing a unique line L all points of which are collinear to all points of O (so L is the intersection of the symplecton containing O and $c^\perp$). Clearly $L^\theta=L$. Corollary 5.2.2(ii) implies that L contains some fixed point f. Then the line cf is stabilised, and so is the corresponding "plane" of $E(p,p^\theta)$ through O. But this now contradicts Lemma 3.5.1 and the fact that H is nonsingular and as such does not contain planes (since H is obtained from the intersection of a hyperplane of $\mathsf{PG}(5,\mathbb{A})$ with the embedded hermitian polar space by Lemma 3.1.2(i), it does not contain two opposite planes; note $\mathbb{A}$ is not octonion since by [5], see also [21], a thick non-embeddable polar space does not contain nonsingular hyperplanes).

So we may assume that H coincides with $E(p,p^\theta)$. Select two locally opposite lines L_1,L_2 through c (remark that these are stabilised as every symplecton through c is stabilised) and complete them to a unique isometric subspace isomorphic to $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ containing p and p^θ. Using Corollary 5.2.4, we conclude similarly as in the proof of Theorem 5.4.1 that there exists at least one fixed point f on L_1 or L_2 different from c and opposite p.

Hence there is a symplecton ξ through c stabilised by θ, and containing a fixed point f collinear to c, but not to $e:=\xi\cap p^{\perp\!\!\!\perp}$. As each symplecton through c is stabilised, each line through c is stabilised and consequently $e^\perp\cap c^\perp$ is fixed pointwise. So we can apply Corollary 3.2.3 to conclude that ξ is pointwise fixed. Lemma 5.1.4(ii) then implies that θ is a central elation. However, θ then has opposition diagram $\mathsf{F}^1_{4;1}$ by Proposition 4.1.1, a contradiction.

So we may from now on assume that H is (always) nonsingular and that each stabilised imaginary line is either pointwise fixed, or contains no fixed points. In particular $\mathbb{A}\neq\mathbb{O}$ from now on. Select two pointwise fixed "lines" A and B of $E(p,p^\theta)$ which are opposite; their symplecta ξ and ζ are also opposite by Lemma 2.9.2(iv) and they represent opposite points a and b, respectively, of the dual $\Gamma_4:=\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$. The corresponding extended equator geometry $\widehat{E}(a,b)$ is stabilised by θ and we claim that the latter induces a plane-domestic collineation in $\widehat{E}(a,b)$. Indeed, suppose for a contradiction that π is a plane mapped onto an opposite plane π^θ. By Proposition 2.7.6(1) these planes correspond to lines L and L^θ in $\widehat{T}(a,b)$. Again Proposition 2.7.6 implies easily that L and L^θ are opposite, a contradiction.

Now we claim that θ maps some point of $\widehat{E}(a,b)$ to an opposite. Indeed, recall that A and B are two opposite pointwise fixed "lines" of $E(p,p^\theta)$. Let $r\in A^{\perp\!\!\!\perp}\cap B^{\perp\!\!\!\perp}$ be a point. Then, since the "plane" spanned by r and A is not pointwise fixed (as H is nonsingular and consequently does not contain planes), Lemma 3.5.1 says that r is opposite r^θ. Then $\xi(p,r)$ and $\xi(p,r^\theta)$ are locally opposite in p. Since $\xi(p,r^\theta)$ is the projection of $\xi(p^\theta,r^\theta)$ onto $\mathsf{Res}_{\Gamma_1}(p)$, Lemma 2.8.7 implies that $\xi(p^\theta,r^\theta)$ is (globally) opposite $\xi(p,r)$ in Γ_1. Hence we may redefine ν as $\xi(p,r)$. Recall that ξ is the symplecton corresponding to A. Suppose for a contradiction that ν and ξ are disjoint. Pick $a\in A$. Then $a\perp L\subseteq\nu$ and $r\perp M\subseteq\xi$. Since $r\perp\!\!\!\perp a$, we also have $r\perp L$ and $a\perp M$. Then the planes $\langle r,L\rangle$ and $\langle a,M\rangle$ are contained in the symplecton $\xi(a,r)$. It follows that ν and ξ are special. But interchanging the roles of r and p, we obtain a different plane in ν (namely, one through p) which lies together with a plane of ξ in a common symplecton, a contradiction. Remark that ν and ξ also don't intersect in a plane, as then p, r and every point of A must be collinear to the same line of this plane, which implies that $r \in A$. Hence ν and ξ intersect in a point d. Similarly ζ and ν intersect in some point e. Hence there is a point n of $\widehat{E}(a,b)$ corresponding to ν, and since ν^θ is opposite ν, the point n is mapped onto an opposite. The claim is proved.

So the opposition diagram of θ on $\widehat{E}(a,b)$ has the first node encircled, and not the third. It follows from the list of feasible opposition diagrams in [16] that the fourth node is not encircled, that is, θ acts both plane- and solid-domestically on $\widehat{E}(a,b)$. So we can apply Proposition 3.5.3. We refer to "Case X of Proposition 3.5.3" briefly by "Case X". We claim that θ either induces in $\widehat{E}(a,b)$ a generalised homology or pointwise fixes a nondegenerate polar subspace of rank 2. We rule out the other cases.

Case (1). We already showed above that there is a point of $\widehat{E}(a,b)$ mapped onto an opposite, hence θ is not point-domestic on $\widehat{E}(a,b)$.

Case $(3)(i)$. Here θ induces the product of two axial elations with respective axes A and A'. Then A and A' intersect in a point, while not contained in a hyperbolic solid, or they are contained in a hyperbolic solid and don't intersect, by Lemma 2.9.3.

First suppose A and A' intersect in a point, but are not contained in a hyperbolic solid. Let B be a hyperbolic line intersecting A and opposite A'. Then B is stabilised by the first elation with axis A, by Definition 3.2.5. But B is mapped to a "line" B' still intersecting A by the second elation with axis A'. Obviously, A, B and B' are contained in a regulus and by Corollary 2.10.3 the corresponding points in Γ_1 are contained in a common imaginary line, which is stabilised, not pointwise fixed, but contains a fixed point, a contradiction to our assumptions.

Now suppose A and A' are contained in a common hyperbolic solid, but don't intersect. Let B be a hyperbolic line in a common solid with A, but opposite A'. Then B is stabilised by the first elation with axis A, by Definition 3.2.5. But B is mapped to a "line" B' opposite B, and again A', B and B' are contained in a regulus. This leads to the same contradiction as in the previous paragraph.

Cases (2) and $(3)(iii)$ with rank 3. Suppose that θ pointwise fixes a nondegenerate polar subspace S of $\widehat{E}(a,b)$ of rank 3. Let π and π' be opposite pointwise fixed "planes" of $\widehat{E}(a,b)$. By Corollary 5.2.3 there is a "solid" Σ containing π stabilised by θ. Let f be the projection of π' onto Σ. Then f is a fixed point not contained in S. Note that θ induces a homology in Σ with axis π and centre f. This homology is nontrivial as otherwise the set of fixed points of θ in $\widehat{E}(a,b)$ would either be a degenerate polar subspace, or have rank 4, contradicting our assumption. Let $x \in S \setminus \Sigma$ be a point. As the rank of S is 3, the point x is not collinear to π. So the projection of x onto Σ must contain f, as this projection is stabilised, and only "planes" through f in Σ are stabilised by θ (except for π). So x is collinear to f. Since θ maps all points of $\Sigma \setminus (\pi \cup \{f\})$ to collinear ones, Corollary 3.5.1 implies that θ is not line-domestic on $\widehat{E}(a,b)$. Since $f \in S^\perp$, the proof of Proposition 3.5.3 reveals that we are either in Case $(3)(i)$ or $(3)(ii)$ of that proposition (indeed, the line L of that proof contains the point f of the current proof). But we already ruled out Case $(3)(i)$ above, so θ again induces a generalised homology in $\widehat{E}(a,b)$. The claim is proved.

Case (2) with rank 4. Then we have a generalised homology as θ clearly fixes an apartment and a line in the nonsingular fixed hyperplane of rank 4.

First suppose that θ pointwise fixes a nondegenerate polar subspace S of rank 2 in $\widehat{E}(a,b)$. Then the proof of Proposition 3.5.3 reveals that $S^\perp$ (considered in the ambient projective space of $\widehat{E}(a,b)$) is a line disjoint from $\widehat{E}(a,b)$, so that θ does not fix any point of $\widehat{E}(a,b)\setminus S$. Indeed, every other fixed point s would give rise to a stabilised hyperplane in the ambient projective space, which must intersect the line $S^\perp$ in a fixed point. So we would have two fixed points and a pointwise fixed subhyperplane in our hyperplane, but then the hyperplane must be pointwise fixed, contradicting our assumption.

We claim now that θ does not fix any line of Γ_4. Indeed, assume for a contradiction that θ fixes the line L. By Corollary 5.2.2$(ii)(b)$, θ fixes a point x on L. Consider any fixed point y in $\widehat{E}(a,b)$. If $x=y$, then some line through y is fixed (namely, L). If $x\perp y$, then again some line through y is fixed (namely, xy). If $x \bowtie y$, then again some line through y is fixed (namely, the line joining $\mathfrak{c}(x,y)$ with y). If x is opposite y, then θ fixes the projection of L onto y and so again some line through y is fixed. If x is symplectic to two opposite points of S, then it belongs to $\widehat{E}(a,b)$ and so θ again fixes some line through some point of S. So we may assume that L contains some point $s\in S$. Let $s'\in S$ be opposite s. Let t be in $E(s,s')$ fixed, en let $\xi=\xi(s,t)$ be the corresponding symplecton. By possibly projecting L into ξ, we may assume that $L\subseteq \xi$. But L corresponds to a "plane" α in $E(s,s')$, which is stabilised by θ. Since every "plane" in $E(s,s')$ contains a unique point of S, we see that every "line" $M\not\ni t$ of α is stabilised (consider any "plane" of $E(s,s')$ through M; it contains some point of S and hence M is stabilised). Hence α is pointwise fixed, contradicting our assumptions on S. Our claim is proved.

We can now repeat the argument that we also used in the proof of Theorem 5.3.2: Suppose that a plane π is stabilised by θ. Then as no line is fixed, there is a point of π mapped to a distinct, necessarily collinear point, but then the line determined by these points is stabilised by Corollary 5.2.2 $(ii)(a)$, a contradiction.

This shows that θ only fixes points and symplecta. Hence the fix structure is a generalised quadrangle as every fixed point not incident to a fixed symplecton is far from that symplecton (otherwise, there would be a fixed line) and so there is a unique symplecton through that point intersecting that symplecton. So the basic property of generalised quadrangles is satisfied. Since the fix structure in $\widehat{E}(a,b)$ is a generalised quadrangle, the complete fix structure will also contain opposite points and opposite lines, hence it is a generalised quadrangle.

Now we claim that θ fixes an ovoid in each fixed symplecton and dually. Indeed, θ can't fix two collinear points (giving rise to a fixed line), but there must be a fixed point in every plane (by Lemma 5.2.1 combined with Theorem 7.2 of [28] using that there are no fixed planes). The claim follows. So it remains to show that in Γ_1 these ovoids arise as intersections of subspaces in their natural embeddings in projective space, and in Γ_4 these ovoids are closed under taking hyperbolic lines. It suffices to show this in one symplecton of each duality type, and then by projection, this is true in every fixed symplecton.

Note that S (defined earlier) is the intersection of $\widehat{E}(a,b)$ with a subspace (in its natural embedding) by Lemma 3.1.2(i). It follows that the same is true for $E(s,s')$, as this is a part of $\widehat{E}(a,b)$. Since $E(s,s')$ is canonically isomorphic to the symplecton of Γ_1 corresponding to the point s of Γ_4, we obtain the assertion for symplecta of Γ_1.

Now we show that in Γ_4 the said ovoids are closed under taking hyperbolic lines. Hence consider two fixed points x,x' in some symplecton ξ of Γ_4. It is easy to find a fixed point y symplectic to x' and opposite x. Then $\widehat{E}(x,y)$ is stabilised and θ induces a plane-domestic and solid-domestic collineation in it. Then the set of fixed points of θ in $\widehat{E}(x,y)$ is a subspace, except if θ induces a generalised homology (but we treat that case below), see earlier. Hence the hyperbolic line

$h(x,x')$ is pointwise fixed and we obtain (ii). Remark that $\mathbb{A} \neq \mathbb{K}$ in this case, i.e. we are in the nonsplit case as otherwise the hyperbolic line $h(x,x')$ corresponds to a line of the projective space underlying the symplectic polar space ξ. Then a plane disjoint to this line gives rise to a fixed point not contained in this line, but collinear to a point of this line, contradicting that the set of fixed points form an ovoid.

Hence from now on we may assume that θ induces a (possibly trivial) generalised homology in $\widehat{E}(a,b)$. Suppose θ is as such; in particular it fixes the points $p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4$ of a skeleton, with p_i opposite q_i, $i=1,2,3,4$. Consider the symplecton $\xi(p_1,p_2)$ and $E(p_1,q_1)$. Then the fixed hyperbolic plane $\langle p_2,p_3,p_4\rangle$ in $E(p_1,q_1)$ corresponds to some line L in $\xi(p_1,p_2)$ through p_1, which is also fixed by θ. So is the point $L \cap q_1^{\bowtie}$. Doing this for each plane $\langle p_2,p_3,q_4\rangle$, $\langle p_2,q_3,p_4\rangle$ and $\langle p_2,q_3,q_4\rangle$, we obtain a fixed skeleton (and consequently also apartment) in $\xi(p_1,p_2)$. In particular we find a fixed chamber C in $\xi(p_1,p_2)$. We find a fixed chamber D opposite C by taking a locally opposite flag to the {point,line,plane}-flag of C in the fixed apartment of $\xi(p_1,p_2)$ and projecting this on $\xi(q_1,q_2)$. It is clear that this chamber must be fixed and the opposition follows by the analogue of Lemma 2.8.7 for symplecta. Hence we find a fixed apartment of Γ_4.

We now consider this situation in Γ_1. Let x_1 and y_1 be two opposite points of a fixed apartment Λ of Γ_1. Then θ fixes an apartment $\Lambda' = \Lambda \cap E(x_1,y_1)$ of $E(x_1,y_1)$. Let Λ' consist of the points x_2,x_3,x_4,y_2,y_3,y_4 with x_i opposite y_i, $i=2,3,4$. Consider the symplecton $\xi(x_1,x_2)$. There is a line L in $\xi(x_1,x_2)$ contained in $x_1^\perp \cap x_2^\perp$ such that the plane $\langle x_1,L\rangle$ corresponds to the "line" x_2x_3 of $E(x_1,y_1)$. Each line through x_1 intersecting L in some point x corresponds to a "plane" of $E(x_1,y_1)$ through x_2x_3. Note that already at least two such planes are fixed, namely $x_2x_3x_4$ and $x_2x_3y_4$. If any such plane α were not fixed, then the point $z := \alpha \cap (y_2y_3)^{\perp\!\!\!\perp}$ would be mapped onto an opposite point z', as if $z \perp\!\!\!\perp z'$ the line zz' would span a "solid" with x_2x_3 in $E(x_1,y_1)$. But the imaginary line $\mathscr{I}(z,z')$ corresponds to the hyperbolic line through z and z' in $E(x_1,y_1)$ by Lemma 2.10.5, and the latter contains $x_2x_3x_4 \cap (y_2y_3)^{\perp\!\!\!\perp} = \{x_4\}$, which is fixed, and $x_2x_3y_4 \cap (y_2y_3)^{\perp\!\!\!\perp} = \{y_4\}$, which is also fixed. Hence $\mathscr{I}(z,z')$ is stabilised. This contradicts our assumption that a stabilised imaginary line is either pointwise fixed, or no point of it at all is fixed. This shows that L is pointwise fixed. Likewise, every line of Λ is pointwise fixed, and hence every plane of Λ is pointwise fixed.

Now let α be the "plane" spanned by x_2,x_3,x_4 and let L_α be the line of Γ_1 through x_1 corresponding to α. Consider an arbitrary point $u \in \alpha$ and assume that u is not fixed. Let ξ_u be the symplecton $\xi(x_1,u)$. Since $u \in \alpha$ and α is fixed, the image ξ_u^θ intersects ξ_u in a plane by Lemma 2.9.2. Then Corollary 5.2.2(i) implies that the "line" uu^θ is stabilised and that it contains at least one fixed point f (since at least one symplecton through $\xi_u \cap \xi_u^\theta$ is fixed). According to Proposition 3.3 of [19] (recalling that α contains three noncollinear fixed points), either θ induces in α a homology, or its fix structure in α is a Baer subplane. Of course, if no such u exists in α, then θ induces the identity in α. Hence there are three possibilities to consider.

(a) Suppose θ induces the identity in α. Then the set of fixed points of θ in $E(x_1,y_1)$ is a polar subspace of rank 3. This follows from the fact that in Λ' there is a (stabilised) plane opposite α and this is consequently also pointwise fixed. So the set of fixed points contains two disjoint planes. The other axioms of a polar space are inherited from the polar space $E(x_1,y_1)$ (keeping in mind that a stabilised line is pointwise fixed by its projection on a pointwise fixed plane). This fixed polar space necessarily has to coincide with $E(x_1,y_1)$ since geometric subspaces of that polar space conform with subspaces of the ambient projective space of dimension 5 (which is generated by two opposite planes of the polar space). Now every line of $\xi(x_1,x_2)$ through x_1 is fixed and at least one plane through it belongs to Λ and is hence pointwise fixed. Since also x_2 is fixed, and hence $x_1^\perp \cap x_2^\perp$ is pointwise fixed, Corollary 3.2.3 asserts that θ acts trivially on $\xi(x_1,x_2)$. Lemma 5.1.4(ii) implies that θ is a central elation. Since θ fixes y_1, which is opposite x_1, this elation is trivial by Lemma 5.1.2. Hence this case leads to the identity, which contradicts clearly the opposition diagram.

(b) Secondly we may assume that θ induces a Baer collineation in α. As the residue of x_1 as a symplecton of Γ_4 is isomorphic to $E(x_1,y_1)$ in Γ_1, we see that θ induces a Baer collineation in a plane of the symplecton x_1. Now we see that θ induces a generalised Baer collineation in this symplecton as this is the only non-linear domestic collineation of a hermitian rank 3 polar space by Theorem 7.2 of [28]. Remark that this is impossible in the split case by Lemma 3.3.6.

Then we claim that θ induces a generalised Baer collineation in every stabilised symplecton of Γ_4. This follows from the following connectivity argument. If two adjacent fixed symplecta intersect in a fixed plane and θ induces a generalised Baer collineation in one of them, then θ induces also a generalised Baer collineation in the other. Similarly, if two fixed symplecta share a unique point x, and θ induces a generalised Baer collineation in one of them, then there is a fixed line through x in that symplecton, and hence some fixed plane sharing a line through x with each of the symplecta. Since the plane is not pointwise fixed (the lines are not), the contraposition of Corollary 5.2.2$(i)(b)$ implies that all symplecta through that plane are fixed and so the previous argument implies that θ induces a generalised Baer collineation in both symplecta. If the two symplecta are special, then we apply the first argument with these symplecta and the unique symplecton adjacent to both. Finally, if the two symplecta are opposite, then the fix structures are isomorphic by projection. The claim follows.

Now we can apply the arguments in the third and fourth paragraph of the nonsplit case of the proof of Theorem 5.4.2 and conclude that θ pointwise fixes a subspace isomorphic to $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{B})$, for $\mathbb{B}$ a subalgebra half the dimension over $\mathbb{K}$ of $\mathbb{A}$. If $\mathbb{A}$ is a separable quadratic extension of $\mathbb{K}$, then we are dealing with the opposition diagram $\mathsf{F}^4_{4;1}$ by Proposition 4.3.1. Since $\mathbb{A}$ is not octonion either, it is quaternion and we find (iv).

(c) Finally we may assume that θ induces a central collineation in α. Since the point x_4 is also fixed, it must be a homology, and the centre is one of x_2,x_3,x_4. Without loss of generality, we may assume that x_2 is the centre. Then no point of the "line" x_2x_3 other than x_2 and x_3 themselves, is fixed by θ. In the symplecton $\xi(x_2,x_3)$ this means that θ fixes x_2 and x_3, it also pointwise fixes the lines $L_x := \xi(x_2,x_3)\cap x_1^\perp$ and $L_y := \xi(x_2,x_3)\cap y_1^\perp$, and pointwise the planes $\langle L_x,x_2\rangle$, $\langle L_x,x_3\rangle$, $\langle L_y,x_2\rangle$ and $\langle L_y,x_3\rangle$ (since these belong to the apartment Λ). Hence θ induces a nontrivial linear collineation in $\xi(x_2,x_3)$ and its ambient projective space $\mathsf{PG}(n,\mathbb{K})$, with $n\in\{6,7,9,13\}$. By Lemma 5.2.1, θ induces a plane-domestic collineation in $\xi(x_2,x_3)$ and, as θ induces a linear collineation in α and has a stabilised plane in Λ', Theorem 7.2 of [28] implies that ξ contains a pointwise fixed hyperplane or subhyperplane (consisting of all fixed points contained in a pointwise fixed line). This is the intersection of the quadric with a hyperplane or subhyperplane H of $\mathsf{PG}(n,\mathbb{K})$, respectively, by Lemma 3.1.2(i). Then H contains the span of the four planes mentioned above, which has dimension 5, and must intersect the space $L_x^\perp\cap L_y^\perp$, which has dimension $n-4$, in a subspace W of dimension at least $n-6$. We now show that the dimension of W is 1, by looking at the intersection with the quadric Q. First we prove that the intersection $W\cap Q$ spans W. Let $p\in W$ be an arbitrary point. If p is contained in Q, then p is obviously contained in the span of Q. If p is now neither contained in Q nor in the tangent space to Q at x_2, then the line px_2 intersects Q in two points and so p is contained in $\langle W\cap Q\rangle$. If p is finally not contained in Q, but contained in the tangent space at x_2 to Q, then every point of the line px_3 distinct from p is contained in the span by the previous arguments and so also $p\in\langle W\cap Q\rangle$. By assumption $W\cap Q$ only contains the points x_2 and x_3, as every other point would be contained in $\xi\cap E(x_2,x_3)$, which is exactly the "line" x_2x_3 by Lemma 2.6.7. Hence the dimension of W is 1, which implies that $n\in\{6,7\}$.

As it is clear that θ can't imply one of the previous two collineations (considered in (a) and (b)) in the planes spanned by the triangles

$$\{x_2, x_3, y_4\}, \{x_2, y_3, x_4\}, \{x_2, y_3, y_4\}, \{y_2, x_3, x_4\}, \{y_2, x_3, y_4\}, \{y_2, y_3, x_4\}, \{y_2, y_3, y_4\},$$

we may assume that it induces a homology and consequently fixes exactly one line pointwise in these triangles. It is now an elementary exercise to conclude that we may assume that the lines x_3x_4, x_3y_4, y_3x_4 and y_3y_4 are pointwise fixed (use also the fact that, if a line is pointwise fixed, then so is every opposite line that is stabilised). Now we consider the symplecton $\xi_x = \xi(x_3, x_4)$ and denote $L_x := \xi(x_3, x_4) \cap x_1^\perp$ and $L_y := \xi(x_3, x_4) \cap y_1^\perp$. Again θ pointwise fixes the planes $\langle L_x, x_3\rangle$, $\langle L_x, x_4\rangle$, $\langle L_y, x_3\rangle$ and $\langle L_y, x_4\rangle$, and these generate a (pointwise fixed) 5-space U in the ambient projective space $\mathsf{PG}(n, \mathbb{K})$. Furthermore, θ now also pointwise fixes $W := \langle L_x^\perp \cap L_y^\perp\rangle$ and this has dimension $n-4$. Clearly $U \cap W = x_3x_4$ and so U and W generate $\mathsf{PG}(n, \mathbb{K})$. Since both U and W are pointwise fixed and they are not disjoint, θ pointwise fixes $\xi(x_3, x_4)$. Likewise θ pointwise fixes the symplecta $\xi(y_3, y_4)$, $\xi(x_3, y_4)$ and $\xi(y_3, x_4)$. This implies that in the extended equator geometry $\widehat{E}(s,t)$ defined by the points of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ corresponding to these symplecta, θ pointwise fixes two perpendicular equator geometries. Hence in the corresponding quadric $\widehat{E}(s,t)$, one pointwise fixes $(s^\perp \cap t^\perp) \cup (u^\perp \cap v^\perp)$, for points s, t, u, v with s not collinear to t, u not collinear to v, and both s, t collinear to both u, v. This implies that θ acts trivially on $\widehat{E}(s,t)$ using Corollary 3.2.3 taking a pointwise fixed line through s in $(u^\perp \cap v^\perp)$, and hence θ also acts trivial on the corresponding tropics geometry. Remark that this means that θ does not fix any other point in $\Gamma_4 \setminus (\widehat{E}(s,t) \cup \widehat{T}(s,t))$. This follows from the fact that $\widehat{E}(s,t) \cup \widehat{T}(s,t)$ is a geometric hyperplane, by Proposition 3.10 of [8] and the fact that a geometric hyperplane does not properly contain another geometric hyperplane in this case by Proposition 2.5 of [9]. This leads to (iii), taking Proposition 4.3.1 into account.

The theorem is proved. □

Remark 5.4.4. Ovoids of $\mathsf{C}_{3,1}(\mathbb{K}, \mathbb{L})$ closed under hyperbolic lines have the property that they arise as the intersection with a subspace in their unique projective embedding in $\mathsf{PG}(5, \mathbb{L})$. This can be shown with some elementary calculations. However, this ovoids of $\mathsf{C}_{3,1}(\mathbb{K}, \mathbb{H})$ closed under hyperbolic lines do no longer necessarily have that property. In fact, our examples in Section 6.2.4 are examples of this phenomenon.

6. Constructions

6.1. **Outline of the methodology.** In this section we prove that each type of domestic collineation mentioned in the Main Result really exists. Despite the fact that it might look obvious for most cases, a rigorous proof is required as, for instance, Lemma 5.1.5(iii) witnesses. Indeed, it is not entirely clear that for certain fields or in certain characteristics, nontrivial collineations exist that have the prescribed fix structure.

However, the most intriguing cases are of course those of (Dom14)(iii) of the Main Result. Concerning Class (M), the job has already been done in [23], where it is proved in Proposition 5.1 and Theorem 6.3, that the fix structure of a collineation θ of an inseparable building $\mathsf{F}_4(\mathbb{K}, \mathbb{K}')$ consists of vertices of types 1 and 4 only, such that , for $\{k, \ell\} = \{1, 4\}$, the fixed vertices of type k incident with a fixed vertex of type ℓ form an ovoid in the corresponding residual polar space of rank 3, if, and only if, θ is conjugate to a certain explicitly defined involution denoted θ_0 in Section 5 of [23]. The corresponding fixed quadrangle is Moufang of mixed type. The full fix group is also determined and is surprisingly large.

So, for the Case (Dom14)(*iii*) we are left with Classes (L) and (H). Concerning the other cases, the ones definitely requiring a proof are (Dom4), Class (L), and (Dom14)(*ii*) (Classes (L) and (H)). Class (K) in Case (Dom4) has been treated in [18], and the existence of central elations in Γ_1, so-called long root elations can be attributed to folklore, see also Chapter 2 and 3 of Timmesfeld's book [29]. For completeness's sake we include a construction here.

All our existence proofs rely on Tits' extension theorem 4.16 of [31]. We translate it to our setting in Theorem 6.3.1. The method is then described in detail in Section 6.3.1. In short, it suffices to find *two collineations* g, g' acting respectively on the residues of a point p and a symplecton $\xi \ni p$, agreeing on the intersection of these two residues and two apartments Λ and Λ' containing this point and symplecton such that the union of g and g' is compatible with an isomorphisms $\Lambda \to \Lambda'$. Then we can conclude by Tits' extension theorem that the union of these three maps $(g, g', \Lambda \mapsto \Lambda')$ extends uniquely to a collineation of the metasymplectic space. The only thing to check then is that this collineation is indeed of the wanted form.

We carry out this scheme in detail for the most involved and most interesting cases, namely (Dom14)(*iii*), Classes (L) and (H). It will then be clear how this works and we can treat the other cases more quickly, only concentrating on the essentials.

Case (Dom14)(*iii*) will occupy the first three subsections of this section. In the first subsection we will construct a collineation in the residue of a point and a collineation in the residue of a symplecton, which act well together, i.e. the point is contained in the symplecton and the actions of the collineations coincide on the intersection of these residues. In the second section we will then prove that such collineation extends to a domestic collineation fixing a generalised quadrangle as in Proposition 4.2.1 and (*ii*) of Theorem 5.4.3. In the third subsection we will then identify the type of these fixed quadrangles. Note that we not only get a collineation that fixes exactly the said Moufang quadrangle, but a whole group of collineations. Nevertheless we do not determine the full fix group, as this seems to require more detailed calculations which we did not perform (yet).

In the next three subsections, we assume $\mathbb{A} \in \{\mathbb{L}, \mathbb{H}\}$, with $\mathbb{L}$ a separable quadratic extension of $\mathbb{K}$ and $\mathbb{H}$ a quaternion division algebra over $\mathbb{K}$. We denote $e = \dim_{\mathbb{K}} \mathbb{A} \in \{2, 4\}$. We will work in some fixed metasymplectic space $\Gamma_1 = \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$, where $\mathbb{A}$ will be obvious from the context, and $C := \{p, L, \pi, \xi\}$ will be a fixed chamber with p a point, L a line, π a plane and ξ a symplecton in Γ_1.

6.2. **Residual collineations.**

As written above, we will construct some collineations in some residues in this section. As in the statement of Proposition 4.2.1 these collineations will fix an ovoid. Also, as required by Theorem 5.4.3, the ovoid in the symplecton arises as the intersection with a subspace in the ambient projective space, considering the symplecton as a quadric in some projective space. First we will determine a nontrivial group of collineations of the symplecton pointwise fixing this ovoid. Afterwards, we will link the different residues in such a way that there exist nontrivial collineations acting in the same way on the intersection of both residues.

6.2.1. *Ovoids and collineations of (the residue of) ξ in $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$.*

By definition, (the residue of) ξ is the polar space $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$, i.e. a quadric in $\mathsf{PG}(5 + e, \mathbb{K})$ with equation

$$x_{-3}x_3 + x_{-2}x_2 + x_{-1}x_1 = z_0\overline{z}_0 - bz'_o\overline{z}'_0, \tag{3}$$

where we view the underlying vector space as isomorphic to $\mathbb{K}^3 \oplus \mathbb{L}^{\frac{e}{2}} \oplus \mathbb{K}^3$ and $b = 0$ if $e = 2$ (this makes it possible to threat the cases $e = 2$ and $e = 4$ at the same time here). Also, $z \mapsto \overline{z}$ is the Galois involution of the separable quadratic extension $\mathbb{L}/\mathbb{K}$. That extension is given by the irreducible quadratic polynomial $x^2 - x + d$. We denote one root of this polynomial in $\mathbb{L}$ as i, and then the other one is $\bar{\imath} = 1 - i$. The corresponding norm is the map $\mathsf{N} : \mathbb{L} \to \mathbb{K} : x \mapsto \mathsf{N}(x) = x\overline{x}$ and $b \notin \mathsf{N}(\mathbb{L})$.

From Theorem 5.4.3(ii), we know that the fixed ovoid in θ must arise as the intersection with a subspace in $\mathsf{PG}(5+e, \mathbb{K})$. Let now D be the subspace of codimension 2 of $\mathsf{PG}(5+e, \mathbb{K})$ with equations

$$\left\{ \begin{array}{rcl} x_{-3} & = & a(x_3 + x_2), \\ x_{-2} & = & adx_2, \end{array} \right.$$

where $\mathsf{N}(z_1) - a\mathsf{N}(z_2) - b\mathsf{N}(z_3) = 0$ if, and only if, $z_1 = z_2 = z_3 = 0$, for all $z_1, z_2, z_3 \in \mathbb{L}$. The intersection of D with ξ is the ovoid $\mathscr{O}_\xi$ with equation

$$\left\{ \begin{array}{rcl} x_{-1}x_1 & = & z_0\overline{z}_0 - bz'_o\overline{z}'_0 - a(x_3^2 + x_3x_2 + dx_2^2), \\ x_{-3} & = & a(x_3 + x_2), \\ x_{-2} & = & adx_2. \end{array} \right. \tag{4}$$

This is clearly an ovoid because it is a geometric subhyperplane as it is the intersection with a subhyperplane of the underlying projective space, and it does not contain lines, since it can be seen as a polar space of rank 1 due to the first equality in Eq. (4) and the choice of a.

We write $z_0 =: x_6 + ix_7$ and $z'_0 =: x_4 + ix_5$. We denote the point with all coordinates zero except $x_j = 1$ with p_j, $j \in \{-3, -2, -1, 1, 2, 3, 4, 5, 6, 7\}$. We order the coordinates according to the following ordering of the indices: $-2, 2, -3, 3, -1, 1, 4, 5, 6, 7$.

Let φ be a collineation of ξ pointwise fixing $\mathscr{O}_\xi$ with matrix M (with respect to the basis in which we write the equations of course) and field automorphism τ.

The intersection of $\mathscr{O}_\xi$ with the subspace $\langle p_{-1}, p_1, p_4, p_5, p_6, p_7 \rangle$ has equations $x_{-3} = x_{-2} = x_2 = x_3 = 0$ together with $x_{-1}x_1 = z_0\overline{z}_0 - bz'_0\overline{z}_0$, which is a quadric spanning this subspace. Since it has to be pointwise fixed by φ, we see that the corresponding submatrix is the identity (and the companion field automorphism τ is trivial). Also, since $\langle p_{-1}, p_1, p_4, p_5, p_6, p_7 \rangle^\perp = \langle p_{-3}, p_{-2}, p_2, p_3 \rangle$, the matrix M is of the form

$$\begin{pmatrix} M' & 0 \\ 0 & I_{2+e} \end{pmatrix},$$

where I_{2+e} is the $(2+e) \times (2+e)$ identity matrix and M' is a 4×4 matrix.

Now we consider the subspace U spanned by $p_{-1}, p_1, p_{-2}, p_2, p_{-3}, p_3$, and it is convenient to rewrite the coordinates in this order. The points with coordinates $(1, -a, 0, 0, a, 1)$ and $(1 - ad, ad, 1, a, 0)$ are fixed under φ, which results in M' being of the form

$$\begin{pmatrix} 1+H & -adH - aD & D & -aD \\ G & 1 - adG - aC & C & -aC \\ F & -adF - aB & 1+B & -aB \\ E & -adE - aA & A & 1 - aA \end{pmatrix},$$

with $A, B, C, D, E, F, G, H \in \mathbb{K}$. Since this fixes the generic point $(adx_2, x_2, a(x_3 + x_2), x_3)$ of $\langle p_{-2}, p_2, p_{-3}, p_3 \rangle$, the matrix M as given above pointwise fixes $\mathscr{O}_\xi$, and it is a generic matrix doing so. Now we express that the matrix M preserves ξ. This results in the identity

$$\begin{aligned} & x_{-2}x_2 + x_{-3}x_3 = \\ & ((1+H)x_{-2} - (adH + aD)x_2 + Dx_{-3} - aDx_3)(Gx_{-2} + (1 - adG - aC)x_2 + Cx_{-3} - aCx_3) \\ & + (Fx_{-2} - (adF + aB)x_2 + (1+B)x_{-3} - aBx_3)(Ex_{-2} - (adE + aA)x_2 + Ax_{-3} + (1 - aA)x_3), \end{aligned}$$

which is equivalent with the following system of conditions on the parameters:

$$0 = G(1+H) + EF, \tag{5}$$
$$0 = (adH + aD)(1 - adG - aC) - (adF + aB)(adE + aA), \tag{6}$$
$$0 = CD + A(1+B), \tag{7}$$
$$0 = a^2CD - (1-aA)aB, \tag{8}$$
$$0 = C(1+H) + DG + AF + E(1+B), \tag{9}$$
$$0 = aC(1+H) + aDG - F(1-aA) + aBE, \tag{10}$$
$$0 = D(1 - adG - aC) - C(adH + aD) - A(adF + aB) - (1+B)(adE + aA), \tag{11}$$
$$0 = aC(adH + aD) - aD(1 - adG - aC) - (adF + aB)(1 - aA) + aB(adE + aA), \tag{12}$$
$$1 = (1+H)(1 - adG - aC) - G(adH + aD) - F(adE + aA) - E(adF + aB), \tag{13}$$
$$1 = -aCD - aCD - aAB + (1-aA)(1+B). \tag{14}$$

Combining (7) and (14), we obtain $B = -aA$. Then (7), (8) and (14) reduce to $CD + A - aA^2 = 0$. Combining (9) and (10), we obtain $F = -aE$. Further, if we divide (6) by a and add ad^2 times (5), a times (7) and ad times (9) to it, then we obtain

$$d(H + adG + aE) + (D + aA + adC) = 0.$$

Also, if we add (11) to $2a$ times (7) and ad times (9), then $D + aA + adC = 0$. Hence we can set

$$D = -aA - adC,$$
$$H = -aE - adG.$$

Then (5) becomes $G = a(E^2 + EG + dG^2)$, (7) becomes $A = a(A^2 + AC + dC^2)$ and (9) becomes

$$C + E = a(CE + AG + 2(AE + dCG)).$$

One can check that no other conditions can be derived from the above identities. Hence the above system of conditions is equivalent to

$$\left\{\begin{array}{rcl} B &=& -aA, \\ F &=& -aE, \\ D &=& -aA - adC, \\ H &=& -aE - adG, \\ A &=& a(A^2 + AC + dC^2), \\ G &=& a(E^2 + EG + dG^2), \\ C + E &=& a(CE + AG + 2(AE + dCG)). \end{array}\right. \tag{15}$$

So we get the matrix

$$\begin{pmatrix} 1 - aE - adG & a^2dE + a^2d^2G + a^2A + a^2dC & -aA - adC & a^2A + a^2dC \\ G & 1 - adG - aC & C & -aC \\ -aE & a^2dE + a^2A & 1 - aA & a^2A \\ E & -adE - aA & A & 1 - aA \end{pmatrix}, \tag{16}$$

which we only have to complete with an $e \times e$ identity part on the z_0 and z_0' coordinates to have the full action on the residual quadrangle Q_ξ in $\mathsf{PG}(3+e, \mathbb{K})$. Note that Q_ξ is the common perp of the points p_{-1} and p_1.

6.2.2. *Ovoids and collineations of the residue of p in $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{L})$.* Now we consider $\mathsf{Res}_{\Gamma_1}(p)$. As this is isomorphic to a symplecton ξ_p in Γ_4, we will denote this residue by ξ_p. This is a Hermitian polar space of rank 3, $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$. Although it should in principle be possible to treat both cases ($e=2,4$) at the same time, this would only make things less transparent and the computations needlessly complicated. So we first consider the case $e=2$, which will be done in this subsection. In this case by the closedness under hyperbolic lines (see Theorem 5.4.3(ii)), the ovoid also arises as the intersection with a subspace in the unique projective embedding (see Remark 5.4.4).

The equation of ξ_p, which is a Hermitian polar space of rank 3 in $\mathsf{PG}(5,\mathbb{L})$, where $\mathbb{L}$ is as in the previous subsection, is now

$$\overline{y}_{-3}y_3+\overline{y}_{-2}y_2+\overline{y}_{-1}y_1=\overline{y}_3y_{-3}+\overline{y}_2y_{-2}+\overline{y}_1y_{-1}.$$

Set $t=1-2i$, then $\overline{t}=-t$ and $t\overline{t}=-t^2=4d-1$. We intersect ξ_p with the subspace with equations

$$\begin{cases} y_{-2} &=& (1-i)y_2,\\ y_{-3} &=& -(1-i)ay_3,\end{cases}$$

with $a\in\mathbb{L}$ exactly as in the previous subsection. Then $\mathscr{O}_p$ has equations

$$\begin{cases} \overline{y}_{-1}y_1-\overline{y}_1y_{-1} &=& t\overline{y}_2y_2-at\overline{y}_3y_3,\\ y_{-2} &=& (1-i)y_2,\\ y_{-3} &=& -(1-i)ay_3.\end{cases}$$

Consider the order $(y_{-1},y_1,y_{-2},y_2,y_{-3},y_3)$ of the coordinates and let φ be a collineation of ξ_p pointwise fixing $\mathscr{O}_p$. Then the points $(1,0,0,0,0,0)$ and $(0,1,0,0,0,0)$ are fixed, and so are their perps, resulting in a 2×2 identity submatrix and trivial borders. Note that also the field automorphism is trivial since the points $(1,x,0,0,0,0)$ with $x\in\mathbb{K}$ and $(1,i,1-i,1,0,0)$ are fixed. So we concentrate on the part of the matrix involving the last four coordinates. Expressing that a generic point with coordinates $((1-i)y_2,y_2,-(1-i)ay_3,y_3)$ is fixed, we obtain the matrix

$$\begin{pmatrix} 1+B & -(1-i)B & F & (1-i)aF\\ C & 1-(1-i)C & G & (1-i)aG\\ A & -(1-i)A & 1+E & (1-i)aE\\ D & -(1-i)D & H & 1+(1-i)aH\end{pmatrix},$$

where $A,B,C,D,E,F,G,H\in\mathbb{L}$. Setting

$$\begin{cases} Y_{-2} &=& (1+B)y_{-2}-(1-i)By_2+Fy_{-3}+(1-i)aFy_3,\\ Y_2 &=& Cy_{-2}+(1-(1-i)C)y_2+Gy_{-3}+(1-i)aGy_3,\\ Y_{-3} &=& Ay_{-2}-(1-i)Ay_2+(1+E)y_{-3}+(1-i)aEy_3,\\ Y_3 &=& Dy_{-2}-(1-i)Dy_2+Hy_{-3}+(1+(1-i)aH)y_3,\end{cases}$$

we have the identity

$$\overline{y}_{-2}y_2-\overline{y}_2y_{-2}+\overline{y}_{-3}y_3-\overline{y}_3y_{-3} \quad=\quad \overline{Y}_{-2}Y_2-\overline{Y}_2Y_{-2}+\overline{Y}_{-3}Y_3-\overline{Y}_3Y_{-3}, \tag{17}$$

by expressing that ξ_p is preserved. Equating the coefficients of $\overline{y}_{-2}y_{-2}$, $\overline{y}_2y_2$ and $\overline{y}_{-2}y_2$, we obtain the relations

$$\begin{cases} B &=& iC,\\ \overline{C}-C &=& t\overline{C}C+\overline{A}D-A\overline{D}.\end{cases}$$

Equating the coefficients of $\overline{y}_{-3}y_{-3}$, $\overline{y}_3y_3$ and $\overline{y}_{-3}y_3$, we obtain the relations

$$\begin{cases} E &=& -iaH,\\ \overline{H}-H &=& -ta\overline{H}H+\overline{F}G-F\overline{G}.\end{cases}$$

Finally, equating the coefficients of $\overline{y}_{\pm 2}y_{\pm 3}$, we obtain the relations

$$\begin{cases} A &= -iaD, \\ F &= iG, \\ \overline{D} - G &= \overline{A}H + \overline{B}G - \overline{C}F - \overline{D}E. \end{cases}$$

It can now be checked that Identity (17) is equivalent to the following system of conditions:

$$\begin{cases} A &= -iaD, \\ B &= iC, \\ E &= -iaH, \\ F &= iG, \\ \overline{C} - C &= t(\overline{C}C - a\overline{D}D), \\ \overline{H} - H &= t(\overline{G}G - a\overline{H}H), \\ \overline{D} - G &= t(\overline{C}G - a\overline{D}H). \end{cases} \tag{18}$$

This yields the matrix:

$$\begin{pmatrix} 1 + iC & -dC & iG & adG \\ C & 1 - \overline{\imath}C & G & \overline{\imath}aG \\ -iaD & adD & 1 - iaH & -a^2dH \\ D & -\overline{\imath}D & H & 1 + \overline{\imath}aH \end{pmatrix}. \tag{19}$$

This matrix acts on the residual quadrangle Q_p with equation $\overline{y}_{-2}y_2 - \overline{y}_2y_{-2} + \overline{y}_{-3}y_3 - \overline{y}_3y_{-3} = 0$, that we get again as the common perp of p_{-1} and p_1.

6.2.3. *Identification of the residual collineations in* $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{L})$. We now need to find a duality between the quadrangles Q_ξ and Q_p of the previous two subsections, in such a way that there is at least one nontrivial collineation φ of ξ fixing the corresponding ovoid pointwise, and one collineation φ_p of ξ_p fixing the corresponding ovoid pointwise, and such that the action of φ on Q_ξ agrees with the action of φ_p on Q_p through the duality.

We start from the quadrangle Q_p, given by the equation $\overline{y}_{-2}y_2 - \overline{y}_2y_{-2} + \overline{y}_{-3}y_3 - \overline{y}_3y_{-3} = 0$ in $\mathsf{PG}(3, \mathbb{L})$ and use the Plücker transformation. One can calculate that the corresponding Plücker coordinates satisfy the equation

$$\overline{p}_{-2,2}p_{-2,2} = p_{-2,-3}p_{3,2} + p_{-2,3}p_{2,-3}. \tag{20}$$

We check this for a generic line, the exceptional cases can be done similarly. For the first point we assume that $y_{-2} \neq 0$, then it is of the form $(1, \overline{y}_3y_{-3} + r, y_{-3}, y_3)$, with $y_{-3}, y_3 \in \mathbb{L}$ and $r \in \mathbb{K}$. For a generic point collinear with it, we may assume that $y_{-2} = 0$, and we also assume that $y_{-3} \neq 0$, then it has coordinates $(0, \overline{y}_3 - s\overline{y}_{-3}, 1, s)$, with $s \in \mathbb{K}$. The Plücker coordinates of the corresponding line are now

$$(p_{-2,2}, p_{-3,3}, p_{-2,-3}, p_{3,2}, p_{-2,3}, p_{2,-3}) = \\ (\overline{y}_3 - s\overline{y}_{-3}, sy_{-3} - y_3, 1, y_3\overline{y}_3 - s(\overline{y}_3y_{-3} + sy_3\overline{y}_{-3}) - sr, s, r + s\overline{y}_{-3}y_{-3}),$$

which satisfy indeed Eq. (20). Furthermore, note that $p_{-2,-3}, p_{3,2}, p_{-2,3}, p_{2,-3} \in \mathbb{K}$, while $p_{-2,2} = -\overline{p_{-3,3}} \in \mathbb{L}$. So Eq. (20) corresponds indeed with the equation of Q_ξ as the common perp of p_{-1} and p_1 in (3), with $b = 0$ (as $e = 2$).

Now we calculate the corresponding matrix, applying the Plücker transformation to the matrix in (19).

$$\begin{pmatrix} 1-tC & 0 & G & adG & \bar{\imath}aG & -iG \\ 0 & 1+taH & -D & -adD & -iaD & \bar{\imath}D \\ adD & -adG & 1+iC-iaH+i^2a(DG-CH) & a^2d^2(DG-CH) & -a^2dH+ia^2d(DG-CH) & -dC+iad(CH-DG) \\ D & -G & DG-CH & 1-\bar{\imath}C+\bar{\imath}aH+\bar{\imath}^2a(DG-CH) & -C+\bar{\imath}a(DG-CH) & -H+\bar{\imath}(CH-DG) \\ -\bar{\imath}D & iG & H+i(CH-DG) & dC+\bar{\imath}ad(CH-DG) & 1+iC+\bar{\imath}aH+ad(CH-DG) & d(DG-CH) \\ iaD & -\bar{\imath}aG & C+ia(DG-CH) & a^2dH+\bar{\imath}a^2d(DG-CH) & a^2d(DG-CH) & 1-\bar{\imath}C-iaH+ad(CH-DG) \end{pmatrix}$$

Since z_0 in (3) corresponds to $p_{-2,2}$ in (20), and in the matrix extending the one in (16), the corresponding base vector was fixed, we set $G = D = 0$ and $C = -aH$. We then obtain

$$\begin{pmatrix} 1+taH & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+taH & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-2iaH+i^2a^2H^2 & a^3d^2H^2 & -a^2dH+ia^3dH^2 & adH-ia^2dH^2 \\ 0 & 0 & aH^2 & 1+2\bar{\imath}aH+\bar{\imath}^2a^2H^2 & aH+\bar{\imath}a^2H^2 & -H-a\bar{\imath}H^2 \\ 0 & 0 & H-iaH^2 & -adH-\bar{\imath}a^2dH^2 & 1-iaH+\bar{\imath}aH-a^2dH^2 & adH^2 \\ 0 & 0 & -aH+ia^2H^2 & a^2dH+\bar{\imath}a^3dH^2 & a^3dH^2 & 1+\bar{\imath}aH-iaH-a^2dH^2 \end{pmatrix},$$

with (18) reducing to only one extra restriction $\overline{H} - H = -at\overline{H}H$, which we can also write as $\overline{H}(1 + atH) = H$. This implies $H^2(1 + atH)^{-1} = \overline{H}H$ and $(H - iaH^2)(1 + taH)^{-1} = \overline{H} - ia\overline{H}H$. Since the extra condition yields $\overline{H} - ia\overline{H}H = H - a(t + i)\overline{H}H = H - \bar{\imath}a\overline{H}H$, the quantity $\zeta_1 := \overline{H} - ia\overline{H}H$ belongs to $\mathbb{K}$. Likewise $\zeta_2 := \overline{H} + \bar{\imath}a\overline{H}H$ belongs to $\mathbb{K}$. We then see that the above matrix is proportional to the blockmatrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-a\zeta_1-a^2d\overline{H}H & a^3d^2\overline{H}H & -a^2d\zeta_1 & ad\zeta_1 \\ 0 & 0 & a\overline{H}H & 1+a\zeta_2-a^2d\overline{H}H & a\zeta_2 & -\zeta_2 \\ 0 & 0 & \zeta_1 & -ad\zeta_2 & 1-a^2d\overline{H}H & ad\overline{H}H \\ 0 & 0 & -a\zeta_1 & a^2d\zeta_2 & a^3d\overline{H}H & 1-a^2d\overline{H}H \end{pmatrix},$$

which is a real matrix (meaning, all entries belong to $\mathbb{K}$). Now we have to match the nontrivial 4×4 block with the earlier obtained matrix in (19). However, we apply first an isomorphism by switching the coordinates $p_{-2,3}$ and $p_{2,-3}$. This way we obtain

$$\begin{pmatrix} 1-a\zeta_1-a^2d\overline{H}H & a^3d^2\overline{H}H & ad\zeta_1 & -a^2d\zeta_1 \\ a\overline{H}H & 1+a\zeta_2-a^2d\overline{H}H & -\zeta_2 & a\zeta_2 \\ -a\zeta_1 & a^2d\zeta_2 & 1-a^2d\overline{H}H & a^3d\overline{H}H \\ \zeta_1 & -ad\zeta_2 & ad\overline{H}H & 1-a^2d\overline{H}H \end{pmatrix} =: M^*.$$

This matrix corresponds to (19), if you set there:

$$\begin{cases} A &= ad\overline{H}H, \\ C &= -\zeta_2, \\ E &= \zeta_1, \\ G &= a\overline{H}H. \end{cases}$$

Indeed, this is obvious or easy for most entries; we do the explicit calculation for the a priori least obvious one, namely

$$a^2dE + a^2d^2G + a^2A + a^2dC = a^2d((\zeta_1-\zeta_2) + (ad\overline{H}H + a\overline{H}H)) =$$
$$a^2d((-i-\overline{\imath})a\overline{H}H + a(d+1)\overline{H}H) = a^3d^2\overline{H}H.$$

One also checks similarly the additional conditions in (15). To conclude that we now have indeed obtained our goal, we need to verify that there really exists an $H \neq 0$, so that $1 + atH \neq 0$ and $\overline{H} - H = -at\overline{H}H$. This is for example satisfied by $H = -(ad)^{-1}i$.

Remark 6.2.1. By the above collineations, no point is mapped to a collinear one (including itself) in the quadrangle Q_ξ. This can be seen as follows. A general point x of Q_ξ is given by the coordinates $(x_{-2}, x_2, x_{-3}, x_3, z_0)$ with $x_i \in \mathbb{K}$ and $z_0 \in \mathbb{L}$ and satisfies $x_{-3}x_3 + x_{-2}x_2 = z_0\overline{z_0}$. Now this point is mapped to the point $(y_{-2}, y_2, y_{-3}, y_3, z_0)$ with

$$(y_{-2}, y_2, y_{-3}, y_3)' = M^*(x_{-2}, x_2, x_{-3}, x_3),$$

which is a point collinear to x if, and only if (after some elementary calculations),

$$\begin{aligned} & x_{-2}y_2 - x_2y_{-2} + x_{-3}y_3 - x_3y_{-3} = 0 \\ \Leftrightarrow\quad & \mathsf{N}((x_{-2} - adx_2) + (x_{-3} - ax_3 - ax_2)i) = 0 \\ \Leftrightarrow\quad & x_{-2} - adx_2 = 0 \quad \wedge \quad x_{-3} - ax_3 - ax_2 = 0, \end{aligned}$$

with N the norm in $\mathbb{L}$. So, recalling Eq. (4), the point must be fixed. But then this yields

$$x_{-3}x_3 + x_{-2}x_2 = z_0\overline{z_0} \quad \Leftrightarrow \quad a\mathsf{N}(x_3 + x_2i) = \mathsf{N}(z_0),$$

which contradicts the choice of a. We say that the collineation is *anisotropic*.

6.2.4. *Ovoids and collineations of the residue of* p *in* $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{H})$. Now the closedness under hyperbolic lines does no longer imply that the fixed ovoid in the residue of p in $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{H})$ arises as the intersection with a subspace of the underlying projective space. We know that ξ_p is a Hermitian polar space of rank 3 in $\mathsf{PG}(5,\mathbb{H})$, where $\mathbb{H}$ is a quaternion division algebra over $\mathbb{K}$ containing the subfield $\mathbb{L}$ of the previous sections. Let $\mathscr{O}_p$ be the Hermitian surface in $\mathsf{PG}(5,\mathbb{L})$ with equation

$$\overline{y}_{-1}y_1 - \overline{y}_1y_{-1} = \overline{y}_0y_0 - a\overline{y}_0'y_0' - b\overline{y}_0''y_0'' + ab\overline{y}_0'''y_0'''. \tag{21}$$

We now have to prove that this is indeed an ovoid in a polar space $\xi_p \cong \mathsf{C}_{3,1}(\mathbb{H},\mathbb{K})$. It is immediately clear that this does not contain lines. So it suffices to prove that it is a subhyperplane in a polar space $\mathsf{C}_{3,1}(\mathbb{H},\mathbb{K})$.

First we prove that $\mathscr{O}_p$ is contained in such a polar space, by looking at the equation over $\mathbb{H} := \mathsf{CD}(\mathbb{L}, b)$ instead of $\mathbb{L}$. The choice for b to be the primitive element of the Cayley-Dickson process is not coincidental. It is motivated by the fact that the quadrangle over $\mathbb{H}$, which is the point residual in ξ_p, is dual to the orthogonal quadrangle appearing as point residual in ξ, which is a necessary condition. Recall the definition of the multiplication and standard involution in $\mathbb{H}$:

$$(x, y) \cdot (u, v) = (xu + b\overline{y}v, \overline{x}v + yu),$$
$$\overline{(x, y)} = (\overline{x}, -y),$$

taking into account that $\mathbb{L}$ is commutative. To prove that Eq. (21) is indeed a hermitian polar space over $\mathbb{H}$, we rewrite the equation as a pseudo-quadratic form, since this is needed and unavoidable in characteristic 2. This becomes

$$\overline{y}_{-1}y_1 - \overline{y}_0 i y_0 + a\overline{y}'_0 i y'_0 + b\overline{y}''_0 i y''_0 - ab\overline{y}'''_0 i y'''_0 \in \mathbb{K}. \tag{22}$$

This pseudo-quadratic form is indeed equivalent to (21) over $\mathbb{L}$ by expressing that the elements of $\mathbb{K}$ are exactly those of $\mathbb{L}$ that are equal to their conjugate (under the standard involution).

We now prove that the residue of p_1 (or the common perp of p_1 and p_{-1}) is a generalised quadrangle by coordinatising it as in Chapter 3 of [34]. This residue, which we denote (quite suggestively) as Q_p, is given by the following pseudoquadratic form

$$f(z_{-2}, z_2, z_{-3}, z_3) := \overline{z}_{-2} i z_{-2} - b\overline{z}_2 i z_2 - a\overline{z}_{-3} i z_{-3} + ab\overline{z}_3 i z_3 \in \mathbb{K},$$

and the collinearity is given by the Hermitian form:

$$\overline{x}_{-2} i y_{-2} - b\overline{x}_2 i y_2 - a\overline{x}_{-3} i y_{-3} + ab\overline{x}_3 i y_3 - \overline{x}_{-2}\overline{\imath} y_{-2} + b\overline{x}_2\overline{\imath} y_2 + a\overline{x}_{-3}\overline{\imath} y_{-3} - ab\overline{x}_3\overline{\imath} y_3.$$

We order the coordinates as $(z_{-2}, z_2, z_{-3}, z_3)$. We make the following assignments of coordinates, with the convention that $\gamma := (0, i), u, u', v \in \mathbb{H}$ and $\ell, \ell', \lambda \in \mathbb{L}$:

Coordinates in $\mathsf{PG}(4, \mathbb{H})$	Coordinates in Q_p
$(b, \gamma, 0, 0)$	(∞)
$(bu, \gamma u, b, \gamma)$	(u)
$(bu', \gamma u', bi, \gamma\overline{\imath})$	$(0, u')$

Now we define the points in the common perp of $(0, 0)$ and (0). It is easy to see that the point $(0, 0, 0)$ with coordinates as in the following table is part of Q_p and collinear to both. Remark also that this point is not collinear to (∞) in this polar space. However in the underlying projective space, these points are collinear and as the images of $(0, 0)$ and (0) under the defining polarity are two distinct planes, all points collinear to both must be contained in the line of this projective space through $(0, 0, 0)$ and (∞). Expressing that these points must also be contained in Q_p gives us that these points can be labeled by $(0, \ell, 0)$ with $\ell \in \mathbb{K}$, corresponding to the coordinates below. The reason for the factor a^{-1} is to obtain later the same incidence relation as for the quadrangle Q_ξ.

Coordinates in $\mathsf{PG}(4, \mathbb{H})$	Coordinates in Q_p
$(bi, \gamma\overline{\imath}, 0, 0)$	$(0, 0, 0)$
$(bi + a^{-1}\ell b, \gamma\overline{\imath} + a^{-1}\ell\gamma, 0, 0)$	$(0, \ell, 0)$

We can now calculate the coordinates of the points $(u, \ell, 0)$ as the unique point on the line $\langle(0), (0, \ell, 0)\rangle$ collinear to (u), and also of (u, ℓ, u') as the unique point on the line $\langle(u, \ell, 0), (u)\rangle$ collinear to $(0, u')$, in the standard way and we obtain:

Coordinates in $\mathsf{PG}(4,\mathbb{H})$	Coordinates in Q_p
$(bi+a^{-1}\ell b, \gamma\bar{\imath}+a^{-1}\ell\gamma, a^{-1}bi\overline{u}, a^{-1}\gamma\bar{\imath}\,\overline{u})$	$(u,\ell,0)$
$(abi+\ell b-bu\overline{u}', a\gamma\bar{\imath}+\ell\gamma-\gamma u\overline{u}', bi\overline{u}-b\overline{u}', \gamma\bar{\imath}\,\overline{u}-\gamma\overline{u}')$	(u,ℓ,u')

We now calculate the coordinates of the point $(\ell',0)$ with $\ell'\in\mathbb{K}$ as a point collinear to (∞) and $(0,0,0)$, similar to those of $(0,\ell,0)$. With the standard way to calculate the coordinates of (ℓ',u') as the unique point on $\langle(\infty),(\ell',0)\rangle$ collinear to $(0,0,u')$ we then get:

Coordinates in $\mathsf{PG}(4,\mathbb{H})$	Coordinates in Q_p
$(0,0,bi-b\ell',\gamma\bar{\imath}-\ell'\gamma)$	$(\ell',0)$
$(bu',\gamma u',bi-\ell' b,\gamma\bar{\imath}-\ell'\gamma)$	(ℓ',u')

Now we define the lines, one can check that these are indeed lines of Q_p:

$$\begin{aligned}
[\infty] &:= \langle(\infty),(0)\rangle,\\
[\ell] &:= \langle(\infty),(\ell',0)\rangle,\\
[\ell,v] &:= \langle(v),(v,\ell,0)\rangle,\\
[\ell,v,\ell'] &:= \langle(\ell,v),(0,\ell',v)\rangle.
\end{aligned}$$

This coordinatisation proves that Q_p is indeed a generalised quadrangle and consequently the space defined by the pseudoquadratic form (22) over $\mathbb{H}$ is a polar space of rank 3. Since it lives in 5-dimensional space $\mathsf{PG}(5,\mathbb{H})$, Eq. (22) implies that it is isomorphic to the polar space $\mathsf{C}_{3,1}(\mathbb{H},\mathbb{K})$ as in Definition 2.3.2, so we can denote it by ξ_p.

Now we prove that $\mathscr{O}_p$ is a subhyperplane. Let π be an arbitrary plane in ξ_p. If all points of π are collinear to p_1, then π must contain $p_1\in\mathscr{O}_p$. So we may suppose that $x\in\pi$ is not collinear to p_1, then the coordinates of x are of the form:

$$x=(1,k+f(z_{-2},z_2,z_{-3},z_3),z_{-2},z_2,z_{-3},z_3),$$

with $z_{\pm2},z_{\pm3}\in\mathbb{H}$ and $k\in\mathbb{K}$. Denote by M the projection of p_{-1} on π and by L the projection of p_1 on $\langle p_{-1},M\rangle$. Then L is a line of the quadrangle Q_p. We suppose that L is of the general form $[\ell,v,\ell']$ (the other cases are similar). Then L is spanned by the points with coordinates (ℓ,v) and $(0,\ell',v)$ in Q_p. Projecting these points onto M yields two points y and z with the following coordinates in $\mathsf{PG}(5,\mathbb{H})$:

$$\begin{aligned}
y&=(0,\alpha,bv,\gamma v,bi-\ell b,\gamma\bar{\imath}-\ell\gamma),\\
z&=(0,\beta,abi+\ell' b,a\gamma\bar{\imath}+\ell'\gamma,-b\overline{v},-\gamma\overline{v}),
\end{aligned}$$

where α and β are completely determined by expressing the collinearity to x. By the definition of $\mathscr{O}_p$ it suffices now to prove that there exists a point with coordinates in $\mathbb{L}$ in this plane. We prove this by taking a linear combination of the coordinates of x,y,z with the property that the first coordinate is 1 and the last four coordinates are contained in $\mathbb{L}$. Then the second one will also be contained in $\mathbb{L}$ since the plane is contained in ξ_p. So with the map $\mathsf{Im}:\mathbb{H}\to\mathbb{L}:(v_1,v_2)\mapsto v_2$ we have to show that the system of equations corresponding to

$$\mathsf{Im}((bv,\gamma v,bi-b\ell,\gamma\bar{\imath}-\ell\gamma)\cdot u+(abi+\ell' b,a\gamma\bar{\imath}+\ell'\gamma,-b\overline{v},-\gamma\overline{v})\cdot w)=\mathsf{Im}((z_{-2},z_2,z_{-3},z_3))$$

has a solution in $u,w\in\mathbb{H}$ for every $v,z_{-2},z_2,z_{-3},z_3\in\mathbb{H}$ and every $\ell,\ell'\in\mathbb{K}$. Writing w as (w_1,w_2) and u as (u_1,u_2), this is a linear system of four equations in four variables, so it has a solution if, and only if, the corresponding determinant is not zero. One now calculates that this corresponding determinant, when writing $v = (v_1, v_2)$, is equal to

$$\begin{vmatrix} bv_2 & b\overline{v_1} & 0 & b(a\bar{\imath}+\ell') \\ v_1 & b\overline{v_2} & a\bar{\imath}+\ell' & 0 \\ 0 & b(\bar{\imath}-\ell) & bv_2 & -bv_1 \\ \bar{\imath}-\ell & 0 & -\overline{v_1} & b\overline{v_2} \end{vmatrix} = b^2(v\overline{v} + (\bar{\imath}-\ell)(a\bar{\imath}+\ell'))^2.$$

One gets then easily that the above expression is zero if, and only if, $\mathsf{N}(v) = a\mathsf{N}(\ell - i)$, which implies that $a = \mathsf{N}(h)$ for some $h \in \mathbb{H}$. If we write $h = (h_1, h_2)$, we see that $a = h_1\overline{h_1} - bh_2\overline{h_2}$, which is impossible by the choice of a in the first subsection. This concludes the proof of the fact that $\mathscr{O}_p$ is a subhyperplane, and hence an ovoid, of ξ_p.

Now we determine the collineations of ξ_p fixing $\mathscr{O}_p$ pointwise. As p_{-1} and p_1 and their perp are fixed, the corresponding matrix must be of the form

$$\begin{pmatrix} hI_2 & 0 \\ 0 & M \end{pmatrix},$$

where I_2 is the 2×2 identity matrix and $h \in \mathbb{H}$. By possibly conjugating the associated automorphism τ of $\mathbb{H}$ with h, we may assume that $h = 1$. Also every point of the form $(1, x, 0, 0, 0, 0)$ with $x \in \mathbb{L}$ must be fixed, and consequently τ fixes $\mathbb{L}$. Since also the points $(1, i, 1, 0, 0, 0)$, $(1, -ai, 0, 1, 0, 0)$, $(1, -bi, 0, 0, 1, 0)$ and $(1, abi, 0, 0, 0, 1)$ are fixed, we now see that also M must be the identity matrix. Now τ is completely determined by the image (A, B) of $(0, 1)$ and expressing that τ is a morphism yields $A = kt$ with $k \in \mathbb{K}$ (and still $t = 1 - 2i$) and $A^2 + bB\overline{B} = b$. These collineations clearly fix all points of the ovoid and preserve the polar space.

However since one nontrivial collineation will suffice in the following, we will only consider a special type of such collineations, i.e. those with $A = 0$ and $B = u^{-1}\overline{u}$ with $u \in \mathbb{L}$. It is easy to see that these correspond to collineations with associated matrix uI_6 and trivial associated automorphism τ. Since we only need the nontrivial collineations and they are determined up to a factor of $\mathbb{K}$, we can write u as $i + \lambda$ with $\lambda \in \mathbb{K}$. In the following subsection, we will denote this collineation by θ_λ.

6.2.5. *Identification of the residual collineations in* $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{H})$. In $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{L})$, we could use the Plücker transformation to define the duality between Q_p and Q_ξ, however this is impossible in the present case. So we will use coordinatisations of these quadrangles as in Chapter 3 of [34].

For Q_p this coordinatisation was done in the previous paragraph and by the theory of coordinatisation as in Chapter 3 of [34], all incidences follow immediately from the coordinates, except for a point (u, λ, u') and a line $[\ell, v, \ell']$. One calculates that these are incident if, and only if,

$$\begin{cases} u' &= v + \ell u, \\ \ell' &= \lambda - u\overline{v} - v\overline{u} - \ell u\overline{u}. \end{cases} \tag{23}$$

Now we coordinatise the quadrangle Q_ξ given by the equation

$$x_{-3}x_3 + x_{-2}x_2 = z_0\overline{z}_0 - bz'_o\overline{z}'_0,$$

where we previously set $z_0 = x_6 + ix_7$ and $z'_0 = x_4 + ix_5$. Hence the equation becomes

$$x_{-3}x_3 + x_{-2}x_2 = x_6^2 + x_6x_7 + dx_7^2 - b(x_4^2 + x_4x_5 + dx_5^2).$$

We order the coordinates as $(x_4, x_6, x_{-3}, x_{-2}, x_2, x_3, x_7, x_5)$. We make the subsequent assignments, after elementary calculations similar to those of the previous section. Set $u := (x_4, x_5, x_6, x_7)$, $u' := (x'_4, x'_5, x'_6, x'_7)$, $\mathsf{N}(u) := x_4^2 + x_4x_5 + dx_5^2 - b(x_6^2 + x_6y_7 + dx_7^2)$ and $\mathsf{N}(u, u') := \mathsf{N}(u+u') - \mathsf{N}(u) - \mathsf{N}(u')$.

Coordinates in $\mathsf{PG}(7,\mathbb{K})$	Coordinates in Q_ξ
$(0,0,1,0,0,0,0,0)$	(∞)
$(0,0,\ell,1,0,0,0,0)$	(ℓ)
$(0,0,\ell',0,1,0,0,0)$	$(0,\ell')$
$(0,0,0,0,0,1,0,0)$	$(0,0,0)$
$(x_4,x_6,\mathsf{N}(u),0,0,1,x_7,x_5)$	$(0,u,0)$
$(x_4,x_6,\mathsf{N}(u),0,-\ell,1,x_7,x_5)$	$(\ell,u,0)$
$(x_4,x_6,\mathsf{N}(u)-\ell\ell',-\ell',-\ell,1,x_7,x_5)$	(ℓ,u,ℓ')
$(x_4',x_6',0,\mathsf{N}(u'),1,0,x_7',x_5')$	$(u',0)$
$(x_4',x_6',\ell',\mathsf{N}(u),1,0,x_7',x_5')$	(u',ℓ')

The lines here are similarly defined as in the previous coordinatisation and also now we only have to verify the incidence of a point (ℓ,v,ℓ') and a line $[u,\lambda,u'] := \langle (u,\lambda),(0,u',\lambda-\mathsf{N}(u,u'))\rangle$ (note that this is really again the line through (u,λ) intersecting the line $\langle (0),(0,u',0)\rangle$). This incidence is indeed again equivalent with

$$\begin{cases} u' &= v+\ell u, \\ \ell' &= \lambda-\mathsf{N}(u,v)-\ell\mathsf{N}(u). \end{cases}$$

This is exactly (23) and so the coordinatisation of both quadrangles is indeed dual.

We now want to verify whether there is a nontrivial collineation of Q_ξ from the previous subsection that induces through this duality a collineation of Q_p from the first subsection. By the above coordinatisation it suffices to know the images from $(\ell,0,\ell')$ and $(u,0)$ in Q_ξ. So we determine the action of the collineations θ_λ in Subsection 6.2.4, i.e. scalar matrices M_λ with elements of the form $i+\lambda$ on the diagonal, on the lines $[\ell,0,\ell']$ and $[u,0]$ of Q_p.

We start with the line $[\ell,0,\ell']$. This line is spanned by the points $(\ell,0)$ and $(0,\ell',0)$ in Q_p. We denote the transpose of a matrix by a prime. Now we calculate the image under θ_λ:

$$\begin{aligned}
\theta_\lambda((\ell,0)) &= M_\lambda\cdot(0,0,bi-b\ell,\gamma\bar{\imath}-\ell\gamma)' \\
&= (0,0,(1-\ell+\lambda)bi-(d-\ell\lambda)b,(1-\ell+\lambda)\gamma\bar{\imath}-(d+\ell\lambda)\gamma)' \\
&= \left(\frac{d+\ell\lambda}{1-\ell+\lambda},0\right); \\
\theta_\lambda((0,\ell',0)) &= M_\lambda\cdot(abi+\ell' b,a\gamma\bar{\imath}+\ell'\gamma,0,0)' \\
&= ((1+a^{-1}\ell'+\lambda)abi+(\lambda\ell'-ad)b,(1+a^{-1}\ell'+\lambda)a\gamma\bar{\imath}+(\lambda\ell'-ad)\gamma)' \\
&= \left(0,\frac{(\lambda\ell'-ad)a}{a+\ell'+a\lambda},0\right); \\
\theta_\lambda([\ell,0,\ell']) &= \left[\frac{d+\ell\lambda}{1-\ell+\lambda},0,\frac{(\lambda\ell'-ad)a}{a+\ell'+a\lambda}\right].
\end{aligned}$$

So under the duality this θ_λ, which we denote by θ^*_λ, acts on a point $(\ell,0,\ell')$ of Q_ξ as follows:

$$\begin{aligned}
\theta^*_\lambda((\ell,0,\ell')) &= M^*_\lambda\cdot(0,0,-\ell\ell',-\ell',-\ell,1,0,0)' = \left(\frac{d+\ell\lambda}{1-\ell+\lambda},0,\frac{(\lambda\ell'-ad)a}{a+\ell'+a\lambda}\right) \\
&= \left(0,0,-\frac{d+\ell\lambda}{1-\ell+\lambda}\cdot\frac{(\lambda\ell'-ad)a}{a+\ell'+a\lambda},-\frac{(\lambda\ell'-ad)a}{a+\ell'+a\lambda},-\frac{d+\ell\lambda}{1-\ell+\lambda},1,0,0\right)'.
\end{aligned}$$

Now one sees that the (4×4)-submatrix of M_λ^* determined by the coordinates $x_{\pm 2}, x_{\pm 3}$ is of the form

$$\begin{pmatrix} a\lambda^2 & ad\lambda & -a^2 d\lambda & a^2 d^2 \\ -a\lambda & a\lambda(1+\lambda) & a^2 d & a^2 d(1+\lambda) \\ \lambda & d & a\lambda(1+\lambda) & -ad(1+\lambda) \\ 1 & -(1+\lambda) & a(1+\lambda) & a(1+\lambda)^2 \end{pmatrix}. \tag{24}$$

Now we look at the action of θ_λ on the line $[u, 0]$. This line is spanned by the points (u) and $(u, 0, 0)$ in Q_p. Remark that the equalities between vectors are equalities in homogeneous coordinates, so must be interpreted afterwards as up to a scalar.

$$\begin{aligned}
\theta_\lambda((u)) &= M_\lambda \cdot (bu, \gamma u, b, \gamma)' \\
&= (b(i+\lambda)u, \gamma(\bar{\imath}+\lambda)u, b(i+\lambda), \gamma(\bar{\imath}+\lambda))' \\
&= \Big(abi + a\lambda(1 - au^{-1}\overline{u^{-1}})b - b(a\overline{u^{-1}})\overline{(-a\lambda\overline{u^{-1}})}, \\
&\qquad a\gamma\bar{\imath} + a\lambda(1 - au^{-1}\overline{u^{-1}})\gamma - \gamma(a\overline{u^{-1}})\overline{(-a\lambda\overline{u^{-1}})}, \\
&\qquad bi\overline{(a\overline{u^{-1}})} - b\overline{(-a\lambda\overline{u^{-1}})},\ \gamma\bar{\imath}\overline{(a\overline{u^{-1}})} - \gamma\overline{(-a\lambda\overline{u^{-1}})}\Big)' \\
&= \Big(a\overline{u^{-1}}, a\lambda(1 - au^{-1}\overline{u^{-1}}), -a\lambda\overline{u^{-1}}\Big); \\
\theta_\lambda((u,0,0)) &= M_\lambda \cdot (abi, a\gamma\bar{\imath}, bi\bar{u}, \gamma\bar{\imath}\,\bar{u})' \\
&= (ab(i+\lambda)i, a\gamma(\bar{\imath}+\lambda)\bar{\imath}, b(i+\lambda)i\bar{u}, \gamma\bar{\imath}(\bar{\imath}+\lambda)\bar{u})' \\
&= \Big(abi + \frac{d(u\bar{u}-a)}{1+\lambda}b - bu\frac{d\bar{u}}{1+\lambda},\ a\gamma\bar{\imath} + \frac{d(u\bar{u}-a)}{1+\lambda}\gamma - \gamma u\frac{d\bar{u}}{1+\lambda} \\
&\qquad bi\bar{u} - b\frac{d\bar{u}}{1+\lambda},\ \gamma\bar{\imath}\,\bar{u} - \gamma\frac{d\bar{u}}{1+\lambda}\Big)' \\
&= \Big(u, \frac{d(u\bar{u}-a)}{1+\lambda}, \frac{du}{1+\lambda}\Big).
\end{aligned}$$

We now determine on which line $[k, v, k']$ these two images lie. Using Eq. (23), one obtains

$$\begin{aligned}
k &= \frac{(\lambda^2+\lambda+d)u\bar{u}}{(1+\lambda)(u\bar{u}-a)} - \lambda, \\
v &= \frac{-a(\lambda^2+\lambda+d)u}{(1+\lambda)(u\bar{u}-a)}, \\
k' &= \frac{a(ad+\lambda(1+\lambda)u\bar{u})}{(1+\lambda)(u\bar{u}-a)}.
\end{aligned}$$

So we can look at the dual action of θ_λ^* in Q_ξ on the point $(u, 0)$

$$\begin{aligned}
\theta_\lambda^*((u,0)) &= M_\lambda^* \cdot (x_4, x_6, 0, \mathsf{N}(u), 1, 0, x_7, x_5)' \\
&= \left(\frac{(\lambda^2+\lambda+d)u\bar{u}}{(1+\lambda)(u\bar{u}-a)} - \lambda, \frac{-a(\lambda^2+\lambda+d)u}{(1+\lambda)(u\bar{u}-a)}, \frac{a(ad+\lambda(1+\lambda)u\bar{u})}{(1+\lambda)(u\bar{u}-a)}\right) \\
&= \Big(x_4, x_6, \frac{ad\lambda u\bar{u} - a^2 d\lambda}{a(\lambda^2+\lambda+d)}, \frac{a\lambda(1+\lambda)u\bar{u} + ad}{a(\lambda^2+\lambda+d)}, \\
&\qquad \frac{du\bar{u} + a\lambda(1+\lambda)}{a(\lambda^2+\lambda+d)}, \frac{-(1+\lambda)u\bar{u} + a(1+\lambda)}{a(\lambda^2+\lambda+d)}, x_7, x_5\Big).
\end{aligned}$$

This shows that we can extend the submatrix from (24) to the matrix M^*_λ by setting the other diagonal elements equal to $a(\lambda^2+\lambda+d)$ and filling the empty places then with zeros. We now show that this is indeed a matrix as at the end of the first subsection, i.e. as in Eq. (16). This is done by first applying an isomorphism to ξ, corresponding to cyclically permuting the coordinates (x_2, x_{-3}, x_3). Then one sees that Eq. (24) divided by $a(\lambda^2+\lambda+d)$ corresponds to Eq. (16) by setting

$$\begin{cases} A &= \frac{d}{a(\lambda^2+\lambda+d)}, \\ C &= \frac{-(1+\lambda)}{a(\lambda^2+\lambda+d)}, \\ E &= \frac{\lambda}{a(\lambda^2+\lambda+d)}, \\ G &= \frac{1}{a(\lambda^2+\lambda+d)}. \end{cases}$$

Also the extra conditions in Eq. (15) are satisfied by these choices.

Remark 6.2.2. Completely similar to Remark 6.2.1, one verifies that also these collineations are anisotropic, that is, they do not map any point of Q_ξ to a collinear one (nor itself).

6.3. **Extension to domestic collineations.** Now we prove that the collineations and duality defined in the previous section, give indeed rise to a domestic collineation of type (*ii*) in Theorem 5.4.3. We will use Tits' extension theorem in the first paragraph to extend the collineations in that way. In the second paragraph we will then prove that the obtained collineation is indeed a domestic collineation fixing a quadrangle. In the next subsection we will then finally identify these quadrangles.

6.3.1. *Tits' extension theorem.* First we translate Theorem 4.16 of [31] to our situation. Therefor, let $C = \{p, L, \pi, \xi\}$ be a chamber of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ (the chamber chosen at the beginning of this chapter) and let Λ be an apartment containing C. Let ξ_p be the symplecton of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ corresponding to p, let p_ξ be the point of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ corresponding to ξ and let α_L be the plane of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ corresponding to L. Let Q be the generalised quadrangle with point set the lines in ξ through p and line set the planes in ξ through p and let $\widetilde{Q}$ be its dual. Let C' be a second chamber, contained in a second apartment Λ' and denote everything for C' the same as for C, but furnished with a prime.

- (*i*) Denote by $E_1(C)$ the union of the set of all points of L, the set of all lines of π through p, the set of all planes through L in ξ and the set of all symplecta containing π.
- (*ii*) Denote by $E_2(C)$ the union of the set of points of π, the set of lines of π, the set of points of Q, the set of lines of Q, the set of points of α_L and the set of lines of α_L (all viewed as elements of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$).
- (*iii*) Denote by $E_3(C)$ the union of the set of points, lines and planes of ξ and the set of points, lines and planes of ξ_p (all viewed as elements of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$).

Notice that, with the above conventions, we have $E_1(C) \subseteq E_2(C) \subseteq E_3(C)$.

Theorem 6.3.1 (Tits [31, 4.16] applied to F_4)**.** *Let θ be a type-preserving and incidence-preserving bijection from $E_2(C)$ and the set of points, lines, planes and symplecta of Λ onto the union of $E_2(C')$ and the set of points, lines, planes and symplecta of Λ'. Then θ uniquely extends to a collineation of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$.*

The uniqueness part of the previous theorem follows from Theorem 4.1.1 of [31]. We also need the specification of that theorem to polar spaces of rank 3.

Theorem 6.3.2 (Tits [31, 4.1.1] applied to B_3 or C_3)**.** *Let Δ be a polar space of rank 3 and let $A = \{p_1, p_2, p_3, p_{-1}, p_{-2}, p_{-3}\}$ be a skeleton of Δ, with p_i not collinear to p_j if, and only if, $i = -j$. Let θ_1 and θ_2 be collineations of Δ which agree on A, on the set of points of the line p_1p_2, on the set of lines of the plane $p_1p_2p_3$ through the point p_1 and on the set of planes through the line p_1p_2. Then $\theta_1 = \theta_2$.*

Now we will use these theorems to extend our collineations from the previous section. The idea is that we have, by the previous section, specific collineations acting on the residues of p and of ξ. By the identifications in the previous section and Theorem 6.3.1, it suffices now to find some type-preserving and incidence-preserving bijection from the apartment Λ to another apartment compatible with the two collineations from the residues of p and ξ, respectively, which coincide under the identification, to extend these collineations to a collineation of the metasymplectic space. In the rest of this subsection, we construct such a bijection.

Let G be the group of collineations g of $\xi_p \cong \mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$ pointwise fixing the ovoid $\mathscr{O}_p \ni p_\xi$ and such that there is a collineation h of $\xi \cong \mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$ pointwise fixing the ovoid $\mathscr{O}_\xi \ni p$ and an identification of $\mathrm{Res}_\xi(p)$ and the dual of $\mathrm{Res}_{\xi_p}(p_\xi)$ on which h and g coincide. For each such $g \in G$, we may extend the domain of definition of g with that of the corresponding h and denote by g the common extension. Then G is a group of type preserving and incidence-preserving permutations of $E_3(C)$.

Now let $q_\xi \in \mathscr{O}_\xi \setminus \{p\}$ and $q_p \in \mathscr{O}_p \setminus \{p_\xi\}$ be arbitrary. We restrict each element g of G to $E_2(C)$ and denote this restricted bijection by g^*. Since $E_1(C) \subseteq E_2(C) \subseteq E_3(C)$, it follows from Theorem 6.3.2 that g is determined by g^* and the assumptions $g(q_\xi) = q_\xi$ and $g(q_p) = q_p$. This is because we can choose the skeleton A of Theorem 6.3.2, say with respect to ξ, containing p and q_ξ and the point p_1 of that skeleton equal to p.

We select a skeleton $S = \{p, q_\xi, r_1, r_2, r_{-1}, r_{-2}\}$ in ξ such that $r_1 \in L$ and $r_2 \in \pi$ (and we use the convention that r_1 is not collinear to r_{-1} and r_2 not collinear to r_{-2}). Then $\{q_\xi, q_\xi r_{-1}, q_\xi r_{-1} r_{-2}\}$ is a chamber of ξ opposite $\{p, L, \pi\}$ in ξ.

Since $q_p \in \mathscr{O}_p$ and $\mathscr{O}_p$ is a set of points of ξ_p, which is isomorphic to the residue at p, we can associate q_ξ to a symplecton ζ of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$. Denote by M the line of ζ all points of which are collinear to r_1, and by α the plane of ζ spanned by M and the points of ζ collinear to r_2. Then $\{q_\xi, q_\xi r_{-1}, q_\xi r_{-1} r_{-2}, \xi\}$ and $\{p, M, \alpha, \zeta\}$ are two chambers of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$, and so we can consider an apartment Λ (without confusion with the previously used Λ) containing both chambers, since there exists always an apartment through two chambers (by the very definition of a building in [31]). Since Λ contains ξ and M, α, it also contains L and π as the "projections" of M and α on ξ. Hence it contains S, after some more projections. Let ξ' be the unique symplecton of Λ opposite ξ and let $D = \{p', L', \pi', \xi'\}$ and $D^* = \{p', L^*, \pi^*, \xi'\}$ be the projection of $\{q_\xi, q_\xi r_{-1}, q_\xi r_{-1} r_{-2}, \xi\}$ and $\{q_\xi, q_\xi r^g_{-1}, q_\xi r^g_{-1} r^g_{-2}, \xi\}$, respectively, onto ξ'. By the dual of Lemma 2.8.7, the chambers C and D are opposite in $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$, and so are the chambers C^g and D^*, as g induces a collineation of ξ. Hence C^g and D^* define a unique apartment Λ' of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$. There is a unique isomorphism $g' : \Lambda \to \Lambda'$ mapping C to C^g and hence D to D^*, as this morphism is completely determined by the image of these two opposite chambers.

We now claim that g and g' agree on the intersection of their domains. Note that the intersection $\xi \cap \Lambda$ is the apartment in ξ spanned by the opposite chambers C and $\{q_\xi, q_\xi r_{-1}, q_\xi r_{-1} r_{-2}, \xi\}$, since an apartment can never intersect a residue in more than an apartment of the residue itself. Then it is clear that g and g' agree on the intersection of their domains in ξ, as the projection of D^* onto ξ is $\{q_\xi, q_\xi r^g_{-1}, q_\xi r^g_{-1} r^g_{-2}, \xi\}$, which equals $\{q_\xi, q_\xi r_{-1}, q_\xi r_{-1} r_{-2}, \xi\}^g$ and $C^g = C^{g'}$. Now we consider $\mathrm{Res}_{\Gamma_1}(p)$. First we note that ζ belongs to Λ' as it is the projection of ξ' onto p, since it is the only symplecton through p of Λ locally opposite ξ. Since D is mapped to D^* under g' and also p is fixed by g', we now see that also ξ' and consequently ζ are fixed under g'. Since the intersection $\mathrm{Res}_{\Gamma_1}(p) \cap \Lambda$ is a "dual" apartment determined by ζ and pr_i, $i = -2, -1, 1, 2$, and g coincides with g' on these elements, we find that g and g' coincide on $\mathrm{Res}_{\Gamma_1}(p) \cap \Lambda$.

Hence we can extend g^* to Λ using g'. Now we claim that this extension preserves incidence. Let $A \subseteq B$ be two incident elements of $\Lambda \cup \xi \cup \mathrm{Res}_{\Gamma_1}(p)$. If both elements are contained in Λ or in $\xi \cup \mathrm{Res}_{\Gamma_1}(p)$, then the claim is true, as g and g' preserve incidence. So we may suppose without loss of generality that $A \in (\xi \cup \mathrm{Res}_{\Gamma_1}(p)) \setminus \Lambda$ and $B \in \Lambda \setminus (\xi \cup \mathrm{Res}_{\Gamma_1}(p))$. This means (again without loss of generality) that $A \subseteq \xi$ and $B \nsubseteq \xi$. Denote now $C := B \cap \xi$, then C is contained in $(\xi \cup \mathrm{Res}_{\Gamma_1}(p)) \cap \Lambda$ and as now the incidence between A and C is preserved under g and the incidence of C and B is preserved under g', the incidence of A and B is preserved under g^*.

Now g^* satisfies the conditions of Theorem 6.3.2 and extends consequently to a unique collineation θ of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$. So we only have to check that θ equals g on $E_3(C)$. This follows by the second paragraph of this reasoning and the fact that $g^*(q_\xi) = q_\xi$ and $g^*(q_p) = q_p$.

6.3.2. *Domestic collineation fixing a quadrangle.* Now we will verify that the obtained collineation θ is indeed a domestic collineation with opposition diagram $\mathsf{F}_{4;2}$ with fix structure points and symplecta forming a generalised quadrangle.

Proposition 6.3.3. *The collineation θ does not fix any line of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$. Dually, it does not fix any plane either.*

Proof. We first prove that g does not fix any line of ξ. Suppose for a contradiction that $L \in \xi$ is stabilised. If L is not coplanar with p, then the unique line through p intersecting L gives rise to a fixed point in Q_ξ, contradicting Remarks 6.2.1 and 6.2.2. So L must be collinear to p, but then the plane spanned by L and p gives rise to a stabilised line in Q_ξ again contradicting Remarks 6.2.1 and 6.2.2.

Furthermore we claim that all the lines through p in ζ are mapped to noncoplanar lines by g. Suppose again for a contradiction that some line is not mapped to a coplanar one. By projecting onto ξ we may suppose that the line is contained in ξ, but then it gives again rise to a point of Q_ξ mapped to a collinear one, contradicting Remarks 6.2.1 and 6.2.2.

As we have a self-dual setting, it suffices to show that no line is fixed. Let, for a contradiction, K be a fixed line. Then K is not contained in ξ (since g does not fix any line in ξ). Also, K does not have a unique point in common with ξ as otherwise the line K' of ξ all points of which are collinear to K is also fixed by g, again a contradiction. If every point of K is far from ξ, then the set of points of ξ symplectic to a point of K is a line K' of ξ fixed by θ and hence by g, a contradiction. If a unique point u of K is close to ξ, then u is fixed and so is the line $u^\perp \cap \xi$, again a contradiction.

Hence the only remaining possibility is that each point of K is close to ξ. Let $u_1, u_2 \in K$ be distinct. Then at least one point $v_1 \in u_1^\perp \cap \xi$ is collinear to and distinct from some point $v_2 \in u_2^\perp \cap \xi$. Then there is a symplecton ξ_{12} containing $u_1 \perp v_1 \perp v_2 \perp u_2 \perp u_1$, and ξ_{12} is clearly adjacent to ξ, hence shares a plane β with it. It follows that there is a unique point v of ξ (in β) collinear to all points of K. Naturally, v is fixed and hence belongs to $\mathscr{O}_\xi$. However, $v \neq p$ as this would contradict the action of g on $\mathrm{Res}_{\Gamma_1}(p)$ and clearly v is not collinear to p, as this would give rise to a fixed line in ξ.

Recall now the above defined symplecton ζ. This is a fixed symplecton through p locally opposite ξ. Since all the lines through p in ζ are mapped to locally opposite lines by g, no line of ζ is fixed by θ. Consequently we can repeat the arguments of the previous paragraph with ζ in place of ξ and find that K is collinear to a unique point $w \in \zeta$, with $w \neq p$ necessarily symplectic to p. It follows that v is symplectic to w and so, by the point-symp relations (Axiom 2.4.5), v is close to ζ, again leading to a fixed line $v^{\perp} \cap \zeta$ in ζ, a contradiction. This proves the proposition. □

Now let $\mathscr{P}$ be the set of fixed points of θ and let $\mathscr{L}$ be the set of fixed symplecta of θ.

Theorem 6.3.4. *The point-line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$ is a Moufang generalised quadrangle.*

Proof. We start by showing that distinct fixed points are either symplectic or opposite. Indeed, if two fixed points were collinear, then the corresponding line would be fixed, contradicting Proposition 6.3.3. If they were special, then their centre would be fixed and we obtain two fixed lines, again the same contradiction. Hence distinct fixed points can only be symplectic or opposite. Dually, two distinct fixed symplecta either intersect in a unique (fixed) point, or are opposite.

Furthermore, we claim that for each fixed point x there exists an opposite fixed point y. This is trivial for $x = p$ as we can take $y = p'$, and for any fixed point x opposite p (as then we can take $y = p$). So we may assume $x \perp\!\!\!\perp p$. If $x \in \xi$, then we can take $y = \zeta \cap \xi'$ (these symplecta indeed intersect in a unique point inside the apartment Λ). If $x \notin \xi$, then the symplecton $\xi(p, x)$ intersects ξ exactly in p and so x is opposite $q_\xi \in \mathscr{O}_\xi$.

We now prove the main axiom for generalised quadrangles. Let x be a fixed point and ν a fixed symplecton not containing x. If x were close to ν, then $x^{\perp} \cap \nu$ would be a fixed line, contradicting Proposition 6.3.3. Hence x is far from ν and the unique point of $x^{\perp\!\!\!\perp} \cap \nu$ is fixed, as is the corresponding symplecton through it and x.

We conclude that $(\mathscr{P}, \mathscr{L})$ is a generalised quadrangle (a polar space of rank 2). Since ξ contains at least three fixed points (the points of $\mathscr{O}_\xi$), and through p there exist at least three fixed symplecta (the members of $\mathscr{O}_p$), we obtain a thick generalised quadrangle. Since no pair of distinct fixed points is collinear, Main Result 1 of [26] asserts that $(\mathscr{P}, \mathscr{L})$ is a Moufang quadrangle. □

6.4. **Identification of the fixed quadrangles.** In this subsection we finally identify the quadrangles Q fixed by the collineations constructed in the previous two. In the following theorem, we will determine their so-called Tits index, see [30].

Theorem 6.4.1. *The fix structure of θ in $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$, with $\mathbb{A}$ either a separable quadratic extension $\mathbb{L}$ of $\mathbb{K}$ or a quaternion division algebra $\mathbb{H}$ over $\mathbb{K}$, is a Moufang quadrangle of type D_5 or E_6, respectively. More exactly, it are Moufang quadrangles with Tits indices ${}^2\mathsf{D}^{(2)}_{5,2}$ and ${}^2\mathsf{E}^{16'}_{6,2}$, respectively in $\mathsf{F}_4(\mathbb{K}, \mathbb{L})$ and $\mathsf{F}_4(\mathbb{K}, \mathbb{H})$, respectively.*

Proof. First of all, if we restrict in the above construction $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ to $\mathsf{B}_{4,2}(\mathbb{K}, \mathbb{A})$ (by taking the intersection with a suitable extended equator geometry in the corresponding dual metasymplectic space), and consequently also $\mathscr{O}_p$ to the hyperbolic line of ξ_p through p_ξ and q_ξ, then we obtain a (Moufang) subquadrangle Q' fully embedded in $\mathsf{B}_{4,1}(\mathbb{K}, \mathbb{A})$.

An equation of $\mathsf{B}_{4,1}(\mathbb{K}, \mathbb{A})$ is given by

$$x_{-4}x_4 + x_{-3}x_3 + x_{-2}x_2 + x_{-1}x_1 = z_0\overline{z}_0 - bz'_0\overline{z}'_0,$$

where we view the underlying vector space as isomorphic to $\mathbb{K}^4 \oplus \mathbb{L}^{\frac{e}{2}} \oplus \mathbb{K}^4$ and $b = 0$ if $e = 2$. Also, recall that $z \mapsto \overline{z}$ is the Galois involution of the separable quadratic extension $\mathbb{L}/\mathbb{K}$ and recall also that this extension is given by the irreducible quadratic polynomial $x^2 - x + d$.

Given the fact that $\mathscr{O}_\xi$ is a point residual in this quadrangle Q', we see that the quadrangle Q' is obtained from $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$ by intersecting with the subspace of codimension 2 of $\mathsf{PG}(7+e,\mathbb{K})$ with equations

$$\left\{\begin{array}{rcl} x_{-3} &=& a(x_3+x_2),\\ x_{-2} &=& adx_2, \end{array}\right.$$

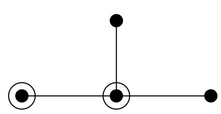
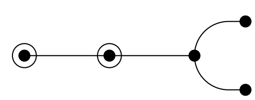
where a and b are as before, that is, $\mathsf{N}(z_1) - a\mathsf{N}(z_2) - b\mathsf{N}(z_3) = 0$ if, and only if, $z_1 = z_2 = z_3 = 0$, for all $z_1, z_2, z_3 \in \mathbb{L}$. This intersection has equations

$$\left\{\begin{array}{rcl} x_{-1}x_1 &=& z_0\overline{z}_0 - bz'_o\overline{z}'_0 - a(x_3^2+x_3x_2+dx_2^2),\\ x_{-3} &=& a(x_3+x_2),\\ x_{-2} &=& adx_2. \end{array}\right.$$

Splitting these equations, that are defined over $\mathbb{K}$, over $\mathbb{L}$, we see that Q' is obtained by Galois descent (more exactly, a Galois involution) from a hyperbolic quadric in $\mathsf{PG}(5+e,\mathbb{L})$, that is, a building of type $\mathsf{D}_{3+e/2}$. The Tits index of Q' as a Moufang quadrangle is hence

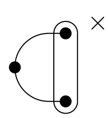
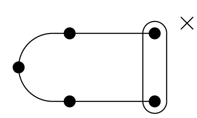
$$\begin{cases} {}^1\mathsf{D}_{4,2}^{(1)} & \text{if } e=2,\\ {}^2\mathsf{D}_{5,2}^{(1)} & \text{if } e=4. \end{cases}$$

Pictorially, these are

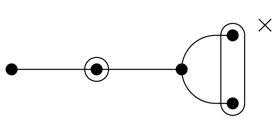
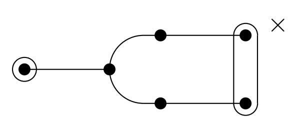
 and 

respectively.

Now Q is an extension of Q' and as such a wide Moufang quadrangle. Given the rules explained in Appendix C of [34], the Moufang quadrangle Q must be of type ${}^1\mathsf{D}_n$ or ${}^2\mathsf{D}_n$, for certain $n \geq 5$ if $e=2$, and in case $e=4$, there is no other possibility than type ${}^2\mathsf{E}_6$. Given the fact that $\mathscr{O}_\xi$ is a Hermitian variety in a projective space of dimension $1+e$ and hence has Tits index

${}^2\mathsf{A}_{3,1}^{(1)}$  $(e=2)$ and ${}^2\mathsf{A}_{5,1}^{(1)}$  $(e=4)$,

respectively, we can compare and glue the "anisotropic kernels" of the previous diagrams (that is, the uncircled nodes). It follows that the only possibilities for Q are the Tits indices

${}^2\mathsf{D}_{5,2}^{(2)}$  $(e=2)$ and ${}^2\mathsf{E}_{6,2}^{16'}$  $(e=4)$.

This concludes the proof of this theorem. □

Remark 6.4.2. In the previous section, we made certain choices that may have seem to be artificial, or at least, predestinated. For instance the use of twice the parameter a might seem to generate only a special case. However, the subquadrangle Q' is completely generic. Then, the arguments leading to the Tits indices only depend on Q' and the isomorphism class of the symplecta of Γ_4. Hence we know that at the end we must obtain the Moufang quadrangles of given type. Our choices now show that this is possible, and that we can obtain all of them this way. What is not proved by our method is that isomorphic fixed quadrangles are also isomorphic via an isomorphism of the metasymplectic space.

6.5. **Other domestic collineations.**

6.5.1. *Central elations.*

Lemma 6.5.1. *Let U_c be the root group with centre c of a metasymplectic space Γ_1. Let K be a line with exactly one point z collinear to c and all the other points special to c. Then U_c acts sharply transitively on $K \setminus \{z\}$. Consequently, U_c acts sharply transitively on $\mathscr{I}(c,x) \setminus \{c\}$, for each point x opposite c.*

Proof. Let k, k' be two points of $K \setminus \{z\}$. We apply the method outlined in Section 6.1 to prove that there exists a unique central collineation with centre c that maps k to k'. In particular, we need a chamber $\{p, L, \pi, \xi\}$ and two apartments Λ and Λ'. Let Λ be an apartment containing c, k and K. Such an apartment exists as we can extend the flags $\{c\}$ and $\{k, K\}$ to two chambers. Take now $p = c$ and let g be the identity on the residue of this point. Let L be the unique line through c intersecting K and note that L belongs to Λ. Let ξ be an arbitrary symplecton of Λ through L and let g' also be the identity on ξ. Denote by C a chamber of Λ extending the flag $\{c, L, \xi\}$ and let C^* be the locally opposite chamber through c in Λ. Denote by K^* the line opposite K in Λ (note that this line intersects the line of C^*) and denote by k^*, k'' the unique point collinear to k, k', respectively, and to a point of K^* (then k^* and k'' are opposite c). Now let C' be the projection of C^* on k''. Then C' is opposite C by Lemma 2.8.7 and we define Λ' to be the unique apartment through the chambers C and C'.

Now by Theorem 6.3.1, we get a unique collineation θ extending g and g' and mapping Λ to Λ'. As L is fixed under this collineation, and K is contained in Λ' as the unique line intersecting L and having a point collinear to k'', also K must be fixed under this collineation as the unique line locally opposite L through $L \cap K$. Hence k is mapped to k' and similarly also k^* is mapped to k''.

So there is only left to verify that θ is a central collineation with centre c. By Lemma 2.10.4 and the fact that k^* is mapped to k'', we see that $E(c, k^*)$ is stabilised under θ. As θ extends g, we see that it is in fact pointwise fixed. As θ also extends g', we see that it pointwise fixes ξ. Then Lemma 5.1.4(ii) ensures that θ is a central elation with centre c. $\square$

This theorem takes care of the Cases (Dom1), (Dom4)(M), (Dom14)(i) and (Dom14)(i') of the Main Result.

6.5.2. *(Weak) subbuildings.* Finally we prove existence for the Cases (Dom4)(K), (Dom4)(L) and (Dom14)(ii). All the corresponding collineations pointwise fix an apartment, which implies that we can always take $\Lambda = \Lambda'$, which simplifies the verification that the various local collineations have compatible actions.

Proposition 6.5.2. *Suppose we are in the separable case and $\dim_{\mathbb{K}} \mathbb{A} \leq 2$. Then there exists a collineation of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ with as set of fixed points exactly the union of an extended equator geometry and its tropics geometry.*

Proof. Let $\Lambda = \Lambda'$ be an apartment of Γ_4, let p and q be two opposite points of Λ and let ξ be a symplecton of Λ through p. Let g be the identity on the residue of p. Let p' be the point of $\xi \cap \Lambda$ opposite p. Then we can choose a basis for ξ such that ξ is the symplectic polar space corresponding to the alternating form

$$x_{-3}y_3 + x_3y_{-3} + x_{-2}y_2 + x_2y_{-2} + x_{-1}y_1 + x_1y_{-1}$$

if $\mathbb{A} = \mathbb{K}$, and the polar space given by

$$\overline{x}_{-3}x_3 + \overline{x}_{-2}x_2 + \overline{x}_{-1}x_1 \in \mathbb{K}$$

if $\mathbb{A} \neq \mathbb{K}$, but in both cases $p = \langle e_{-1}\rangle$ and $p' = \langle e_1\rangle$. Let then g' be a collineation acting on this polar space, with respect to the ordering $x_{-1}, x_1, x_{-2}, x_2, x_{-3}, x_3$ of the coordinates, by the matrix

$$\begin{pmatrix} aI_2 & 0 \\ 0 & I_4 \end{pmatrix},$$

with $a = -1$ if $\mathbb{K} = \mathbb{A}$ and $a \in \mathbb{A} \setminus \{1\}$ with $a\overline{a} = 1$ otherwise. These collineations are clearly compatible, as they act both trivial on their common domain. It is also clear that their union is compatible with the identity in Λ. Hence there exists a unique collineation θ extending g and g' and fixing Λ.

We only have to check that θ pointwise fixes an extended equator geometry. Indeed $\{p,q\} \cup E(p,q)$ is pointwise fixed as θ extends g and $\Lambda = \Lambda'$. Also, as θ extends g', the hyperbolic line $h(p,p')$ is pointwise fixed. Now Corollary 3.2.3 implies that $\widehat{E}(p,q)$ is pointwise fixed. Consequently also its tropics geometry is fixed. Since g' is nontrivial, and $\widehat{E}(p,q) \cup \widehat{T}(p,q)$ is a hyperplane, the proposition follows (see also the last paragraph in Case (c) of the proof of Theorem 5.4.3). □

The next proposition also uses the identification between quadrangles from Subsection 6.2.5. Note that not all the details are worked out here, as these are similar and even easier than the ones in the previous subsections.

Proposition 6.5.3. *Suppose that $\mathbb{A}$ is a separable quadratic extension of $\mathbb{K}$ or a quaternion division algebra over $\mathbb{K}$. Then there exists a collineation of $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ with as fix structure a metasymplectic space canonically isomorphic to $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{K})$ or $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{L})$ (where $\mathbb{L}$ is a subalgebra of $\mathbb{A}$ of dimension 2 over $\mathbb{K}$ fixed under some automorphism of $\mathbb{A}$), respectively.*

Proof. Let $\Lambda = \Lambda'$ be an apartment of Γ_1, let p a point of Λ and let ξ be a symplecton of Λ through p.

We first assume that $\mathbb{A}$ is a separable quadratic extension of $\mathbb{K}$. Then let g be the collineation acting on the residue of p ($\cong \mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$) by the identity matrix and the standard involution as field automorphism. Let now g' be the collineation given by the trivial field automorphism and the matrix (with respect to the ordering of the coordinates $x_{-1}, x_1, x_{-2}, x_2, x_{-3}, x_3, x_0, x_0'$ of the defining equation $x_{-1}x_1 + x_{-2}x_2 + x_{-3}x_3 = x_0^2 + x_0x_0' + d{x_0'}^2$)

$$\begin{pmatrix} I_6 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix},$$

where I_6 denotes the 6×6 identity matrix. By identifying p with p_{-1} in this last polar space, one verifies easily that the restriction of g' to the residue Q_p of p (in ξ) is the only nontrivial collineation of the quadrangle Q_p fixing the subquadrangle Q_p' over $\mathbb{K}$. As also g' acts nontrivial on the dual of this quadrangle and fixes this subquadrangle, they must have the same action on this Q_p.

Assume now that $\mathbb{A} = \mathbb{H}$ is a quaternion division algebra over $\mathbb{K}$. Denote by Q_ξ a copy of a point residue in a symplecton in $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{H})$ and by Q'_ξ a copy of a point residue in a symplecton in $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{L})$, with $\mathbb{L}$ in $\mathbb{H}$ canonically as usual. Similarly for Q_p and Q'_p in $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{H})$ and $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{L})$, respectively.

We use the notation of Subsection 6.2.5. In the ambient projective space of Q_ξ we consider the collineation induced by $u \mapsto u^\sigma$, with $\sigma : (z_0, z'_0) \mapsto (z_0, cz'_0)$, with $1 \neq c \in \mathbb{L}$ and $c\overline{c} = 1$. It is clear that this defines a linear collineation (with 4×4 identity matrix in the middle) and that it preserves Q_ξ (as $N(u) = N(u^\sigma)$); (x_4, x_5, x_6, x_7) transforms to $(x_4, x_5, c_1x_6 - dc_2x_7, c_2x_6 + (c_1 + c_2)x_7)$, with $c = c_1 + ic_2$. Hence, more precisely, it maps the point (ℓ, u, ℓ') to the point (ℓ, u^σ, ℓ'), and hence the line $[u, \ell, u']$ to the line $[u^\sigma, \ell, u'^\sigma]$. The fix structure in Q_ξ is hence precisely the quadrangle Q'_ξ.

It suffices now to exhibit a collineation in Q_p that maps the point (u, ℓ, u') to $(u^\sigma, \ell, u'^\sigma)$. This is obtained by the following collineation of $\mathsf{PG}(3, \mathbb{H})$:

$$\begin{pmatrix} z_{-2} \\ z_2 \\ z_{-3} \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma\gamma^{-\sigma} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma\gamma^{-\sigma} \end{pmatrix} \cdot \begin{pmatrix} z_{-2} \\ z_2 \\ z_{-3} \\ z_3 \end{pmatrix}^\sigma,$$

which can easily be shown to stabilise Q_p and act as desired.

Extending these collineations to the whole residue of p and ξ, we get also some g and g', respectively, in this case.

In both cases the union of g and g' is clearly compatible with the identity on Λ. So we obtain a collineation extending g and g' and fixing Λ. This must be a nontrivial collineation of the desired form, by similar arguments as in the second last paragraph of Case (b) in the proof of Theorem 5.4.2. □

References

[1] P. Abramenko & K. Brown, Automorphisms of non-spherical buildings have unbounded displacement, *Innov. Incid. Geom.* **10** (2009), 1–13.

[2] F. Buekenhout & E.E. Shult, On the foundations of polar geometry, *Geom. Dedicata* **3** (1974), 155–170.

[3] I. Cardinali, L. Giuzzi & A. Pasini, Nearly all subspaces of a classical polar space arise from its universal embedding, *Lin. Alg. Appl.* **627** (2021), 287–207.

[4] A.M. Cohen, An axiom system for metasymplectic spaces, *Geom. Dedicata* **12** (1982), 417–433.

[5] A.M. Cohen & E.E. Shult. Affine polar spaces, *Geom. Dedicata* **35** (1990), 43-76.

[6] B. De Bruyn & H. Van Maldeghem, Non-embeddable polar spaces. *Mu\ňster J. Math.* **7** (2014), 557–588.

[7] A. De Schepper, N. S. N. Sastry & H. Van Maldeghem, Split buildings of type F_4 in buildings of type E_6, *Abh. Math. Sem. Univ. Hamburg* **88** (2018), 97–160.

[8] P. Jansen, J. Schillewaert & H. Van Maldeghem, Embedded metasymplectic spaces of type $\mathsf{F}_{4,4}$, submitted.

[9] A. Kasikova, Simple connectedness of hyperplanes in point-line geometries related to buildings, *J. Combin. Theory Ser. A*, **118** (2011), 641–671.

[10] A. Kasikova & H. Van Maldeghem, Vertex opposition in spherical buildings, *Des. Codes Cryptogr.* **68** (2013), 285–318.

[11] B. Leeb, *A characterization of irreducible symmetric spaces and Euclidean buildings of higher rank by their asymptotic geometry,* Bonner Mathematische Schriften **326**, Universität Bonn, 2000.

[12] K. McCrimmon, Pre-book on *alternative algebras*, unpublished, `https://mysite.science.uottawa.ca/neher/papers/alternative/`.

[13] B. Mühlherr, H. Petersson & R. M. Weiss, *Descent in Buildings*, Annals of Mathematics Studies **190**, Princeton University Press, 2015.

[14] B. Mühlherr & H. Van Maldeghem, Exceptional Moufang quadrangles of type F_4, *Canad. J. Math.* **51** (1999), 347–371.

[15] Y. Neyt, J. Parkinson, H. Van Maldeghem & M. Victoor, Automorphisms and opposition in spherical buildings of exceptional type, IV. Buildings of type E_7, in progress,

[16] J. Parkinson & H. Van Maldeghem, Opposition diagrams for automorphisms of large spherical buildings, *J. Combin. Theory Ser.* A **162** (2019), 118–166.

[17] J. Parkinson & H. Van Maldeghem, Opposition diagrams for automorphisms of small spherical buildings, *Innov. Incid. Geom.* **17** (2019), 141–188.

[18] J. Parkinson & H. Van Maldeghem, Automorphisms and opposition in spherical buildings of exceptional type, I, *Canad. J. Math.* **74** (2022), 1517–1578.

[19] J. Parkinson & H. Van Maldeghem, Automorphisms and opposition in spherical buildings of classical type, submitted.

[20] J. Parkinson & H. Van Maldeghem, Automorphisms and opposition in spherical buildings of exceptional type, V. Buildings of type E_8, in progress,

[21] A. Pasini & H. Van Maldeghem, An essay on Freudenthal-Tits polar spaces, *J. Algebra*, to appear.

[22] S. Petit & H. Van Maldeghem, Generalized hexagons embedded in metasymplectic spaces, *J. Korean Math. Soc.* **60** (2023), 907–929.

[23] J. Roth & H. Van Maldeghem, Mixed relations for buildings of type F_4, *Innov. Incid. Geom.* **20** (2023), 543–578.

[24] E. E. Shult, *Points and Lines, Characterizing the Classical Geometries*. Universitext, Springer-Verlag, Berlin Heidelberg, 2011.

[25] A. Steinbach, Groups of Lie type generated by long root elements in $F_4(K)$, *J. Algebra* **255** (2002), 463–488.

[26] K. Struyve, Quadrangles embedded in metasymplectic spaces, *European J. Combin.* **31** (2010), 1579–1585.

[27] B. Temmermans, J. A. Thas & H. Van Maldeghem, Domesticity in projective spaces, *Innov. Incid. Geom.* **12** (2011), 141–149.

[28] B. Temmermans, J. A. Thas & H. Van Maldeghem, Collineations of polar spaces with restricted displacements, *Des. Codes Cryptogr.* **64** (2012), 61–80.

[29] F. G. Timmesfeld, *Abstract Root Subgroups and Groups of Lie Type*, Monographs in Mathematics **95**, Birkhäuser Verlag, Basel, 2001.

[30] J. Tits, Classification of simple algebraic groups, **in** *Algebraic groups and discontinuous subgroups*, Proc. Summer Mathematical Inst., Boulder, July 5–August 6, 1965, *Proc. Symp. Pure Math.* **9**, Amer. Math. Soc., Providence, RI (1966), 33–62.

[31] J. Tits, *Buildings of Spherical Type and Finite BN-Pairs*, Springer Lecture Notes Series **386**, Springer-Verlag, 1974.

[32] J. Tits, Twin buildings and groups of Kac-Moody type, *London Math. Soc. Lecture Note Ser.* **165** (Proceedings of a conference on Groups, Combinatorics and Geometry, ed. M. Liebeck and J. Saxl, Durham 1990), Cambridge University Press (1992), 249–286.

[33] J. Tits & R. Weiss, *Moufang Polygons*, Springer Monographs in Mathemetics, Springer, 2002

[34] H. Van Maldeghem, *Generalized Polygons*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 1998.

[35] H. Van Maldeghem, The geometry of the Freudenthal-Tits Magic Square, **in** *Algebraic Combinatorics and the Monster Group* (ed. A.A. Ivanov), *London Math. Soc. Lect. Notes Ser.* **487** (2023), 289–326.

[36] R. A. Wilson, A quaternionic construction of E_7, *Proc. Amer. Math. Soc.* **142** (2014), 867–80.

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