Subgeometries of (exceptional) Lie incidence geometries induced by maximal root subsystems

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Abstract

A maximal full rank subgroup of a simple group G of Lie type is a maximal subgroup H of Lie type that arises from a root subsystem of the same rank as the underlying root system. We investigate how the spherical building related to H sits in that related to G, where we concentrate on G being of exceptional type over an arbitrary field. We consider the long root subgeometries and other parapolar spaces related to G. We provide a general treatment of the simply laced case and give a detailed geometric study in all exceptional cases.

Contents

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1 Introduction

This paper grew out of a question asked by Sasha Ivanov to the second author whether the maximal subgroup $(\mathsf{PSL}_3(2) \times \mathsf{PSL}_3(2)) : 2$ has a geometric interpretation in the ambient group $\mathsf{F}_4(2)$. In other words, can one see the two projective planes of order 2 on which the said maximal subgroup acts in a natural way? This question puzzled us for a moment and, the answer not being clear at once, we started to investigate similar phenomena in the exceptional groups of Lie type, hoping they could teach us something about Sasha's question. The "similarity" was defined as "subgroups also arising from a maximal root subsystem". Eventually we obtained a rather general and complete answer, also yielding an answer to the original question. The present paper reports about this.

Interpreting (simple) groups of Lie type geometrically lies at the heart of Tits' theory of (spherical) buildings. The interaction between the group and the associated geometry has proved to be very fruitful both for geometric and group theoretic investigations. In this paper, we take this interaction one step further by interpreting certain subgroups of groups of Lie type geometrically inside the building of the ambient group. Some subgroups, like parabolic ones, have a standard and natural interpretation (namely, as the stabiliser of a residue). Some other famous examples also have a well known interpretation, think of classical groups inside each other, Dickson's group of type G_2 inside the classical group $\mathsf{PSO}_8(\mathbb{K})$, and the split groups of type F_4 as maximal subgroups of groups of type E_6. In this paper, we consider maximal subgroups of groups of Lie type which are also groups of Lie type themselves and on top have the same rank as the ambient group. We call these *maximal full rank subgroups*. The Borel-de Siebenthal theory says that such subgroups can be constructed in a uniform way using the underlying root system—basically the Dynkin type of the subgroup is given by adding the longest root to a fundamental system of roots and deleting an arbitrary fundamental root. What does not seem to be known is how these subgroups act on the ambient building; in particular if and how the building belonging to the subgroup sits in the ambient building. This is exactly the subject of the present paper. Since for the classical groups, this answer can be deduced from Aschbacher's list of classes of maximal subgroups of classical groups, see also the monograph of Kleidmann and Liebeck [11], we concentrate on the exceptional groups of Lie type.

The way we tackle this, is natural: we consider the long root subgroup geometry $\Gamma(G)$ of the exceptional group G of Lie type in question. Then $\Gamma(H)$, with H a maximal full rank subgroup, is naturally (and fully) embedded in $\Gamma(G)$. However, there is always, what we call, a *companion geometry* $\Gamma^*(H)$, also embedded in $\Gamma(G)$ as a kind of complement to $\Gamma(H)$. In the simply laced case, we provide a uniform way to determine the type of the geometry $\Gamma^*(H)$. It will turn out that it is always of *Jordan* type (basically meaning that it is a strong parapolar space).

Main Result. *The companion geometries of the maximal full rank subgroups of the Chevalley groups with associated simply laced Dynkin diagram are given in* Table 2.

In particular, with the (standard) notation of Section 2, this implies the following rather unexpected inclusions of irreducible Lie incidence geometries of the same rank.

Corollary to the Main Result.

(*i*) *The Grassmannian* $\mathsf{A}_{7,2}(\mathbb{K})$ *is a subgeometry of the minuscule geometry* $\mathsf{E}_{7,7}(\mathbb{K})$*;*
(*ii*) *the Grasmannian* $\mathsf{A}_{7,4}(\mathbb{K})$ *is a subgeometry of the long root geometry* $\mathsf{E}_{7,1}(\mathbb{K})$*;*
(*iii*) *the half spin geometry* $\mathsf{D}_{8,8}(\mathbb{K})$ *is a subgeometry of the long root geometry* $\mathsf{E}_{8,8}(\mathbb{K})$*;*
(*iv*) *the Grasmannian* $\mathsf{A}_{8,3}(\mathbb{K})$ *is a subgeometry of the long root geometry* $\mathsf{E}_{8,8}(\mathbb{K})$*;*
(*v*) *the long root geometry* $\mathsf{A}_{2,\{1,2\}}(\mathbb{K},\mathbb{A})$ *is a subgeomery of* $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$*;*
(*vi*) *the half spin geometry* $\mathsf{B}_{4,4}(\mathbb{K},\mathbb{A})$ *is a subgeometry of* $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$*.*

Note that $\Gamma(H)$ and $\Gamma^*(H)$ are *coupled* geometries, that is, each point of one geometry is uniquely (geometrically) defined by a corresponding object in the other geometry. This gives rise to some beautiful geometry showcasing the exceptionality of the exceptional geometries. We emphasize this by independent (from the Main Result above) constructions of the said subgeometries. Moreover, we also interpret the most interesting maximal full rank subgroups in the minuscule geometries of types E_6 and E_7 by constructing appropriate subgeometries of the latter. A key concept in both the long root subgroup geometries and the minuscule geometries is that of an *equator geometry*.

Since there is only one type of non-simply laced spherical buildings of exceptional type and rank at least 3, namely type F_4, and the complication of non-split buildings arises here, we did not feel the need to develop a general theory leading to a similar conclusion as in our Main Result above. Rather we directly construct the subgeometries corresponding to the maximal full rank subgroups in a combinatorial way. This, for instance, gives rise to a rather surprising inclusion of the long root subgroup geometry of the Cayley plane inside the short root metasymplectic space associated to the Cayley numbers (over an arbitrary field). We also treat type G_2, the Moufang hexagons.

All the constructions of the various coupled subgeometries in (exceptional) spherical buildings of type X_n yield non-thick buildings of type X_n the thick frame of which has the Dynkin type of the given maximal full rank subgroup. This is explained in some more detail in Section 6.

Outline of the paper—In Section 2 we introduce notation and the objects we will study. We assume the reader to be familiar with the basics of Tits buildings and (crystallographic) root systems. In Section 3 we prove our Main Result. Since we do this in a uniform way, this includes the classical types A_n and D_n. In Section 4 we provide geometric constructions of the subgeometries related to the maximal full rank subgroups in the exceptional simply laced cases. For each Dynkin type, we include a short introduction into the corresponding parapolar spaces with explicit concrete definitions of the various equator geometries that play a role (a general and rather abstract definition can be found in [22]). The non-simply laced case is treated in Section 5. Here we only provide geometric and combinatorial constructions. We discuss the application to non-thick buildings in Section 6.

2 Preliminaries

2.1 Lie incidence geometries

Definition 2.1. A *point-line geometry* $\Gamma = (X, \mathscr{L})$ is a bipartite graph with classes X and $\mathscr{L}$. In this paper, no two members of $\mathscr{L}$ are adjacent to exactly the same set of vertices in X and so we can identify each member of $\mathscr{L}$ with its set of neighbours in X. The set X is the set of *points* and $\mathscr{L}$ is the set of *lines*. Two points x, y are called *collinear*, in symbols $x \perp y$, if they are contained in a common line. The set of points collinear to a given point x is denoted by $x^\perp$. The (geometric) *distance* between two points is half of the graph distance in Γ.

A *partial linear space* is a point-line geometry for which there is at most one line through two points. Let $\Gamma = (X, \mathscr{L})$ be a partial linear space. Then a subset $M \subseteq X$ is called a *subspace* when every line of X that intersects M in at least two points, is contained in M. The subspace M is said to be *convex* when for any two points in M, any shortest path in Γ, as a graph, connecting these two points, is also contained in M. A *hyperplane* is a proper subspace that intersects each line nontrivially. A *singular* subspace is a subspace in which every pair of points is collinear.

Definition 2.2. (PS) A polar space is a partial linear space for which $x^\perp$ is a hyperplane for each point x.

(PPS) A parapolar space is a connected partial linear space such that each pair of either collinar points, or noncollinear points x, y with $|x^\perp \cap y^\perp| \geq 2$, is contained in a convex subspace isomorphic to a polar space.

With this definition, each polar space is a parapolar space. Sometimes it is required that a parapolar space is not a polar space, but for us this makes no difference as we only use the language and will always work with specific parapolar spaces. We note that parapolar spaces are *gamma spaces*, that is, given a point p and a line L, either all, exactly one, or no points on L are collinear to p.

Notation 2.3. Some notation that is used in the language of parapolar spaces is the following. Let x, y be two points. If $|x^\perp \cap y^\perp| = 1$, then we say that x and y are *special*, or that they are a *special pair*. We denote the unique member of $x^\perp \cap y^\perp$ by $[x, y]$. If $|x^\perp \cap y^\perp| \geq 2$, then we say that x and y are *symplectic*, or that they are a *symplectic pair* (some authors call such a pair *polar*). Finally, if x and y represent opposite simplices in the corresponding building, then we call them *opposite*.

If some maximal singular subspace of a polar space has finite dimension, then all maximal singular subspaces have the same dimension $r - 1$, and we say that the polar space has *rank* $r \geq 1$.

A convex subspace isomorphic to a polar space will be called a *symplecton*, or briefly, a *symp*. If the rank of all symplecta of a parapolar space are equal, say to $r \geq 2$, then r is called the *uniform symplectic rank* of the parapolar space.

Before we recall the standard procedure how spherical buildings give rise to point-line geometries, let us agree on some notation for some specific buildings. For an excellent introduction to buildings, we refer to [1].

Notation 2.4. (A) A Moufang building of type A_n, $n \geq 2$, is uniquely determined by an alternative division ring $\mathbb{D}$ and denoted $\mathsf{A}_n(\mathbb{D})$ (with the understanding that, in the associative case, points are parametrized by triples up to a right scalar factor).

(B) The norm of a quadratic alternative division algebra $\mathbb{A}$ over some field $\mathbb{K}$ is an anisotropic quadratic form Q. It can be used to define a quadric with equation

$$X_{-n}X_n + X_{-n+1}X_{n-1} + \cdots + X_{-1}X_1 = Q(X_0),$$

with $(X_{-n}, X_{-n+1}, \ldots, X_{-1}, X_0.X_1, \ldots, X_n) \in \mathbb{K}^n \times \mathbb{A} \times \mathbb{K}^n$. The corresponding building is denoted by $\mathsf{B}_n(\mathbb{K}, \mathbb{A})$.

(C) For an associative alternative division algebra $\mathbb{A}$ over some field $\mathbb{K}$, $\mathbb{A} \neq \mathbb{K}$, with standard involution $x \mapsto \overline{x}$, the pseudo-quadratic form $\overline{X}_{-n}X_n + \cdots + \overline{X}_{-1}X_1 \in \mathbb{K}$ in $2n$ variables defines a building which we denote by $\mathsf{C}_n(\mathbb{A}, \mathbb{K})$. If $\mathbb{A}$ is non-associative, then $\mathsf{C}_3(\mathbb{A}, \mathbb{K})$ is the building corresponding to the nonembeddable polar space of rank 3 with non-Desarguesian planes. If $\mathbb{A} = \mathbb{K}$, we set $\mathsf{C}_n(\mathbb{K}, \mathbb{K})$ equal to the building arising from the polar space corresponding to a non-degenerate alternating bilinear form in n variables over $\mathbb{K}$.

(D) A building of type D_n, $n \geq 4$, is determined by a (commutative) field $\mathbb{K}$ and denoted by $\mathsf{D}_n(\mathbb{K})$. For $n = 3$ we denote $\mathsf{D}_3(\mathbb{D}) = \mathsf{A}_3(\mathbb{D})$, for any associative division ring $\mathbb{D}$.

(E) A buildings of type E_n, $n \in \{6, 7, 8\}$ is uniquely determined by a (commutative field) $\mathbb{K}$ and denoted by $\mathsf{E}_n(\mathbb{K})$.

(F) A building of type F_4 is determined by a quadratic alternative division algebra $\mathbb{A}$ over some field $\mathbb{K}$ and denoted by $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$, where we assume that the residues of type $\{1, 2\}$ correspond to $\mathsf{A}_2(\mathbb{K})$ and the ones of type $\{3, 4\}$ to $\mathsf{A}_2(\mathbb{A})$.

(G) A Moufang hexagon is determined by a quadratic Jordan division algebra $\mathbb{J}$ over some field $\mathbb{K}$ and denoted $\mathsf{G}_2(\mathbb{K}, \mathbb{J})$. We assume that the panels of type 1 are coordinatized by $\mathbb{K}$, and those of type 2 by $\mathbb{J}$, see [20].

The thin building (or Coxeter complex) of type X_n is always denoted by $\mathsf{X}_n(1)$.

Definition 2.5. Let Δ be a (simplicial) spherical building of type X_n with corresponding Coxeter system (W, S), $|S| = n \geq 2$. Let J a nonempty subset of S. We define a point-line geometry $\Gamma = (X, \mathscr{L})$ as follows. The set X of points consists of all simplices of Δ of type J. A typical line consists of the set of simplices of type J whose union with a given simplex of cotype j, $j \in J$, is a chamber. If Δ is denoted by $\mathsf{X}_n(*)$, with $(*)$ representing one of the algebraic structures in Notation 2.4, then Γ is denoted by $\mathsf{X}_{n,J}(*)$. If $J = \{j\}$, then we also write $\mathsf{X}_{n,j}(*)$. In any case, we say that Γ is *of type* $\mathsf{X}_{n,J}$ and call it a J-Grassmannian geometry.

We number the elements of $S = \{s_1, s_2, \ldots, s_n\}$ using Bourbaki [4] labelling of the spherical Dynkin diagrams. For J as above, we usually only write the indices, that is, we view J as a set of natural numbers.

Lemma 2.6 (Proposition 11.4.10 of [5])**.** *Let Y be a simplex of type K of a spherical building $\Delta \cong \mathsf{X}_n(*)$ (as above). The points of $\mathsf{X}_{n,J}(*)$ that are incident with Y form a convex subspace of $\mathsf{X}_{n,J}(*)$ of type $\mathsf{Y}_{m,J\setminus K}$, where Y_m corresponds to the Dynkin diagram that is obtained by first deleting the nodes corresponding to K from the Dynkin diagram X_n, and then taking the connected components that contain at least one element of J.*

We will call such a subspace Y as in the previous lemma a *K-grammatical subspace*, inspired by [13]. Note that, if Y_m is disconnected, then the corresponding grammatical subspace is a direct product space (and not a disjoint union).

2.2 Long root subgroup geometries

Many things that follow are valid over an arbitrary Dynkin diagram. However, we will only apply things in the simply laced case. Hence we will not be concerned to much about the details in the general *split case*. We content ourselves with mentioning that in the simply laced case, all buildings are split, except for type A_n, in which case split corresponds to be defined over a commutative field. In the other cases, the buildings $\mathsf{B}_n(\mathbb{K},\mathbb{K})$, $\mathsf{C}_n(\mathbb{K},\mathbb{K})$ and $\mathsf{F}_4(\mathbb{K},\mathbb{K})$ are split.

Definition 2.7. Let Δ be a (split) spherical building with corresponding Coxeter system (W,S) and Dynkin diagram X_n. Let J be the set of nodes of X_n that are adjacent to the node extending X_n to an affine diagram (equivalently, in terms of the corresponding root system, the types corresponding to the roots of a fundamental system not perpendicular to the highest root). We say that the corresponding point-line geometry of type $\mathsf{X}_{n,J}$ is the *long root subgroup geometry of* Δ. (We usually omit the word "subgroup".)

Example 2.8 ([5])**.** Let Σ be a thin spherical building with Coxeter system (W,S) and corresponding irreducible root system ψ, not of type C_n. By fixing a fundamental chamber $\mathscr{C}$ of Σ, we fix a fundamental system of ψ and hence a highest root α_0: the unique long root that is contained in the closure of $\mathscr{C}$. The stabilizer of α_0 in W equals $\langle S\setminus J\rangle$ with J as in Definition 2.7. At the same time, the points of $\mathsf{X}_{n,J}(1)$ are the J-simplices of Σ, and hence the cosets of $\langle S\setminus J\rangle$ in W. We can hence find a bijection:

$$\text{Points of } \mathsf{X}_{n,J}(1) \to \text{ Long roots of } \psi : x = w\langle S\setminus J\rangle \mapsto \alpha_x = w\alpha_0.$$

This bijection has the following nice property:

$$\begin{aligned}
\langle \alpha_x, \alpha_y\rangle = 2 &\iff x \text{ and } y \text{ are equal},\\
\langle \alpha_x, \alpha_y\rangle = 1 &\iff x \text{ and } y \text{ are collinear},\\
\langle \alpha_x, \alpha_y\rangle = 0 &\iff x \text{ and } y \text{ are symplectic},\\
\langle \alpha_x, \alpha_y\rangle = -1 &\iff x \text{ and } y \text{ are special},\\
\langle \alpha_x, \alpha_y\rangle = -2 &\iff x \text{ and } y \text{ are opposite}.
\end{aligned}$$

Type C_n has some special features, which are not important for us in the present paper, so we exclude it.

Lemma 2.9. *In any long root geometry of (spherical) type $\mathsf{X}_{n,J}$, two points p,q are either equal, collinear (notation: $p \perp q$), symplectic (notation $p \perp\!\!\!\perp q$), special (notation $p \bowtie q$) or opposite (notation: $p \equiv q$).*

Definition 2.10. Let x be a point of a long root geometry Γ. Let Σ be any apartment containing x. Then x corresponds to a root α_x of Σ with corresponding root group Z_{α_x}. Define $Z_x := Z_{\alpha_x}$. This definition is independent of the choice of Σ since, in the split case, every member of Z_{α_x} fixes each point collinear or symplectic to x, and so it fixes every chamber having a panel in the inside of any half apartment centred at x (see also Timmesfeld's theory [19]).

Define $G := \langle Z_x \,|\, x \in X \rangle$. Then $Z_x^g = Z_{x^g}$ for all $g \in G$.

Note that, in the above definition, the restriction to the split case is essential in the sense that we otherwise have to consider the center of the group Z_{α_x} for Z_x.

The next lemma follows from Timmesfeld's theory [19].

Lemma 2.11. *For any two points x, y of Γ, we have (for some commutative field $\mathbb{K}$),*

$$\begin{aligned}[Z_x, Z_y] = 1 &\iff x \text{ and } y \text{ are equal, collinear or symplectic},\\ [Z_x, Z_y] = Z_{[x,y]} &\iff x \text{ and } y \text{ are special},\\ \langle Z_x, Z_y\rangle \cong \mathsf{PSL}_2(\mathbb{K}) &\iff x \text{ and } y \text{ are opposite.}\end{aligned}$$

Geometrically, this means that

$$\begin{aligned}y^{Z_x} = \{y\} &\iff x \text{ and } y \text{ are equal, collinear or symplectic},\\ y^{Z_x} \cup \{[x,y]\} \text{ is a line} &\iff x \text{ and } y \text{ are special},\\ y^{Z_x} \cup \{x\} = x^{Z_y} \cup \{y\} &\iff x \text{ and } y \text{ are opposite.}\end{aligned}$$

In the last case, the set $y^{Z_x} \cup \{x\} = x^{Z_y} \cup \{y\}$ is sometimes called the *imaginary line* joining x and y, see [9]. A geometric definition is given at the end of Section 4.1.2.

2.3 Root subsystems

In this section, let ψ be an irreducible crystallographic root system with corresponding reflection group W. Moreover, let $\{\alpha_1, \cdots \alpha_n\}$ be a fundamental system of ψ, and let α_0 be the highest root of ψ with respect to $\{\alpha_1, \cdots \alpha_n\}$.

Definition 2.12. A subset ϕ of ψ is called a *root subsystem* of ψ when for every $\alpha \in \phi$, we have $-\alpha \in \phi$, and moreover for every $\alpha, \beta \in \phi$ with $\alpha + \beta \in \psi$, we have $\alpha + \beta \in \phi$ The subsystem ϕ is called *maximal* when there exists no subsystem ϕ' with $\phi \subset \phi' \subset \psi$.

Example 2.13. Let $i \in \{12, \ldots, n\}$ and let λ_i be the ith coefficient of α_0. Consider the map

$$\mathrm{pr}_i : \psi \to \mathbb{Z} : \alpha = \sum_{j=1}^{n} \beta_j \alpha_j \mapsto \beta_i.$$

Since α_0 is the highest root, we have $\mathrm{pr}_i(\psi) \subseteq [-\lambda_i, \lambda_i]$. Define

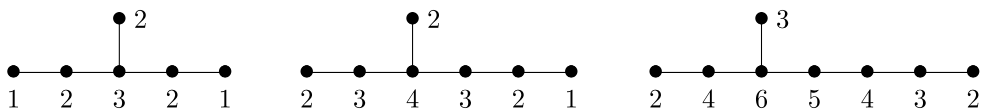
$$\phi_i := \{\alpha \in \psi \,|\, \mathrm{pr}_i(\alpha) = 0 \bmod \lambda_i\}.$$

This is a root subsystem of ψ with fundamental system $\{-\alpha_0, \alpha_1, \cdots, \hat{\alpha}_i, \cdots, \alpha_n\}$. Denote its reflection group with W_i. For $0 < j < \lambda_i$, define

$$\phi_i^j := \{\alpha \in \psi \,|\, \mathrm{pr}_i(\alpha) = j \bmod \lambda_i\}.$$

The reflection group W_i stabilizes these subsets ϕ_i^j and even acts transitively on the roots contained in it (see for example [14], Lemma 4.3).

In the simply laced case, the coefficients λ_i are all equal to 1 for type A_n; they are all equal to 2, except for the extremal nodes of the diagram, for type D_n, $n \geq 4$, and for types $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$, we display them on the diagram, with obvious notation:



The following lemma is contained in the so-called Borel–de Siebenthal theory [3].

Lemma 2.14 (Borel-de Siebenthal)**.** *The root subsystem ϕ_i of ψ of* Example 2.13 *is a maximal root subsystem if and only if λ_i is prime. All maximal root subsystems of ψ of rank n can, up to W-equivalence, be constructed like this.*

Let G be a group of Lie type with root system ψ. A maximal root subsystem as above gives rise to a subgroup H of Lie type of the same rank as G. A subgeometry of any Grassmannian corresponding to G on which H naturally acts as group of Lie type will be called a *full rank Lie subgeometry.*

3 Full rank Lie subgeometries of long root geometries

3.1 Finding the long root subgeometries

Convention 3.1. Let Δ be a building of type A_n (for $n \geq 2$), D_n (for $n \geq 4$) or E_n (for $n = 6, 7, 8$), and denote with Ω the long root geometry associated to Δ. The points of Ω are hence given by all simplices of Δ of type J, for some well defined J. Fix an apartment Σ of Δ, and denote with ψ the simplices of Δ of type J contained in Σ. Identifying ψ with a root system, as in Example 2.8, we can fix a fundamental system $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of ψ. Denote with α_0 the highest root of ψ with respect to Π. We continue with the notation introduced in Example 2.13.

Definition 3.2. For a subset ϕ of ψ, we define $\langle\phi\rangle$ to be the smallest subspace of Ω which contains ϕ. We define $\langle\langle\phi\rangle\rangle$ to be the smallest subspace of Ω which contains ϕ while being invariant under $G_\phi := \langle Z_\alpha \,|\, \alpha \in \phi\rangle$. Geometrically, $\langle\langle\phi\rangle\rangle$ is the smallest subspace containing ϕ which is closed under taking shortest paths between special points and imaginary lines through opposite points.

The following lemma is an immediate consequence of Timmesfeld's theory [19]. It establishes the "obvious" containments of long root geometries.

Lemma 3.3. *Let $i \in \{1, \dots, n\}$. Denote the irreducible components of the root system ϕ_i with $\phi_{i,1}, \dots, \phi_{i,r}$.*

The subspace $\langle\langle \phi_i \rangle\rangle$ is the disjoint union of the subspaces $\langle\langle \phi_{i,1} \rangle\rangle, \dots, \langle\langle \phi_{i,r} \rangle\rangle$, we will call these the irreducible components of $\langle\langle \phi_i \rangle\rangle$. Moreover, for $l \neq m \in \{1, \dots, r\}$, the following hold:

- *(i) The subspace $\langle\langle \phi_{i,l} \rangle\rangle$ is a long root geometry of the same type as type of the root system $\phi_{i,l}$.*
- *(ii) The group G_{ϕ_i} acts transitively on the points of $\langle\langle \phi_{i,l} \rangle\rangle$.*
- *(iii) Two points are collinear (symplectic, special or opposite) in $\langle\langle \phi_{i,l} \rangle\rangle$ if they are collinear (symplectic, special or opposite, respectively) in Ω.*
- *(iv) Every symp in $\langle\langle \phi_{i,l} \rangle\rangle$ is the intersection of a symp of Ω with the subspace $\langle\langle \phi_{i,l} \rangle\rangle$.*
- *(v) Every point x_l of $\langle\langle \phi_{i,l} \rangle\rangle$ is symplectic in Ω to every point x_m of $\langle\langle \phi_{i,m} \rangle\rangle$. The symplecton of Ω determined by x_l and x_m contains no other points of $\langle\langle \phi_i \rangle\rangle$ then x_l and x_m.*

Now in the rest of this section, we will determine the companion geometries. These will be the subspaces generated by the ϕ_i^j.

3.2 Finding the companion geometries

3.2.1 Nailing down the types

Definition 3.4. For $i \in \{1, \dots, n\}$, denote $\Omega_i := \langle\langle \phi_i \rangle\rangle$. Moreover, for $0 < j < \lambda_i$, denote $\Omega_i^j := \langle \phi_i^j \rangle$.

Lemma 3.5. *Let $i \in \{1, \dots, n\}$. The group G_{ϕ_i} stabilizes the subspaces Ω_i^j for $0 < j < \lambda_i$.*

Proof. As G_{ϕ_i} is generated by the groups Z_α with $\alpha \in \phi_i$, it suffices to prove that the latter stabilize Ω_i^j. To that end, take $\alpha \in \phi_i$ and $z \in Z_\alpha$.

We first prove that $(\phi_i^j)^z \subseteq \Omega_i^j$. Let $\beta \in \phi_i^j$ and $z \in Z_\alpha$. The only point of ϕ opposite α is $-\alpha$, which is contained in ϕ_i, so we know that α and β are not opposite. If α and β are collinear or symplectic, then z fixes β, by Lemma 2.11, in which case we can conclude that $\beta^z \in \Omega_i^j$. If α and β are special, then $\alpha + \beta \in \phi$, this hence also corresponds to a point of the geometry, which is the unique point of Ω collinear to both α and β. As $\mathrm{proj}_i(\alpha + \beta) = \mathrm{proj}_i(\alpha) + \mathrm{proj}_i(\beta)$, we obtain that $\alpha + \beta \in \phi_i^j \subseteq \Omega_i^j$. Using Lemma 2.11, we find that β^z is a point on the line through β and $\alpha + \beta$. As both β and $\alpha + \beta$ are contained in the subspace Ω_i^j, we know that β^z is, too. We conclude that $(\phi_i^j)^z \subseteq \Omega_i^j$.

Now note that $(\Omega_i^j)^z = \langle \phi_i^j \rangle^z$ is the smallest subspace that contains $(\phi_i^j)^z$. As Ω_i^j is a subspace, this proves that $(\Omega_i^j)^z \subseteq \Omega_i^j$. By repeating these arguments with z^{-1} instead of z, we conclude that $(\Omega_i^j)^z = \Omega_i^j$. □

Lemma 3.6. *Let $i \in \{1, \dots, n\}$. No point of Ω_i is opposite a point of Ω_i^j, for $0 < j < \lambda_i$.*

Proof. Take $\alpha \in \phi_i$. The points of Ω that are not opposite α form a subspace of Ω. As this subspace contains $\phi_{i,j}$, it also contains Ω_i^j, implying that α is not opposite any point of Ω_i^j.

Let y be any point of Ω_i. By Lemma 3.3, there is an element $g \in G_{\phi_i}$ for which $y^g \in \phi_i$. It follows from the previous paragraph that y^g is not opposite any point of Ω_i^j. We hence find that y is not opposite any point of $(\Omega_i^j)^{g^{-1}}$, which by Lemma 3.5 coincides with Ω_i^j. □

In order to determine the type of Ω_i^j, we try to interpret a generic point of it in Ω_i by looking at what it is collinear with in Ω_i. This is carried out in the next lemma. For the definition of a Jordan node, we refer to Section 4.1.1.

Lemma 3.7. *Let $i \in \{1, \ldots, n\}$, suppose that λ_i is prime and let $0 < j < \lambda_i$. Let $\alpha \in \phi_i^j$ and let Ω_i' be an irreducible component of Ω_i, say of rank m. The set S_α of points of Ω_i' that are collinear to α forms a nonempty $\{k\}$-grammatical subspace of Ω_i', for some k, as in* Table 1. *The possibilities for k correspond exactly to the Jordan nodes of the diagram.*

Type of Ω_i'	possibilities for k
$A_{m,\{1,m\}}$ (for $m \geq 1$)	$k \in \{1, \ldots, m\}$
$D_{m,2}$ (for $m \geq 4$)	$k \in \{1, m-1, m\}$
$E_{6,2}$	$k \in \{1, 6\}$
$E_{7,1}$	$k = 7$

Table 1: S_α is a $\{k\}$-grammatical subspace of Ω_i'

Proof. Suppose for a contradiction that S_α is empty. Let ϕ_i' be the set of roots of ϕ_i contained in Ω_i'. We claim that α is symplectic to all roots β of ϕ_i'. It follows from Lemma 3.6 that α is not opposite β, and, by assumption, α is not collinear to β. If α were special to β, then $-\beta \in \phi_i'$ would be collinear to α, contradicting our assumption that S_α is empty. We conclude that α is symplectic to β. The set of roots

$$\phi_i' \cup \{\gamma \in \psi \mid \langle \gamma, \phi_i' \rangle = 0\}$$

is a root subsystem of ψ, which contains α (because we just showed that it is perpendicular to ϕ_i') and ϕ_i (because the roots in Ω_i not contained in ϕ_i' are all perpendicular to ϕ_i' as they belong to different components). It however follows from Lemma 2.14 that ϕ_i is a maximal root subsystem of ψ, implying that $\psi = \phi_i' \cup \phi_i' \cup \{\gamma \in \psi \mid \langle \gamma, \phi_i' \rangle = 0\}$, a contradiction to the irreducibility of ψ. We conclude that S_α is not empty.

Let x and y be two points of S_α. As α is collinear to both x and y, we find that x and y are not opposite. Suppose for a contradiction that x and y would be special, then α is the unique point collinear to both x and y. As $x, y \in \Omega_i$, it follows from the definition of Ω_i that $y^{Z_x} \subseteq \Omega_i$. By Lemma 2.11, the set y^{Z_x} consists of the points on the line through y and α different from α. As Ω_i is moreover a subspace, this implies that $\alpha \in \Omega_i$, a contradiction. From this, we may conclude that any two points of S_α are either collinear or symplectic.

Next, we argue that S_α is a convex subspace of Ω_i'. As Ω is a parapolar space, it is clear that S_α is a subspace of Ω_i'. Let x and y be two noncollinear points of S_α. By the previous argument, we find that x and y are symplectic. Denote with ξ_i' the symplecton of Ω_i' determined by x and y, and by ξ the symplecton of Ω determined by x and y. We aim to prove that $\xi_i' \subseteq S_\alpha$. Suppose for a contradiction that there is some element $z \in \xi_i'$ not contained in S_α. As Ω_i' is a long root geometry, there is a point $w \in \Omega_i'$ which is symplectic to z but opposite to some point of ξ_i'. Using the fact that $\xi_i' = \Omega_i' \cap \xi$ and that Ω is a long root geometry, we find that w is opposite every point of ξ which is not collinear to z, in particular to α. But this implies that $\alpha \in \phi_i^j$ is opposite to $w \in \Omega_i$, a contradiction to Lemma 3.6.

It follows from [13] that every convex subspace of Ω_i' that contains no pair of special points, is automatically grammatical.

Recall from the first paragraph of this proof that $S_\alpha \cap \phi_i'$ is not empty. We claim that for every root $\beta \in S_\alpha \cap \phi_i'$, and every root $\gamma \in \phi_i'$ collinear to β, either $\gamma \in S_\alpha$ or $\beta - \gamma \in S_\alpha$. As $\langle \alpha, \beta \rangle = 1$, we find that $\langle \alpha, \beta - \gamma \rangle = 1 - \langle \alpha, \gamma \rangle$. Taking into account that α is neither equal to, nor opposite either γ or $\beta - \gamma$, we find that either $\langle \alpha, \gamma \rangle = 1$ or $\langle \alpha, \beta - \gamma \rangle = 1$, which indeed proves that α is either collinear to γ or to $\beta - \gamma$.

Now we observe that no K-grammatical subspace with $|K| > 1$ satisfies the property of the previous paragraph (which intuitively expresses that S_α is rather large). Hence $K = \{k\}$, $1 \le k \le m$.

If k is not as in Table 1, then we are in the cases D_m, $\mathsf{E}_6, \mathsf{E}_7$ or E_8 and it is easily checked that in a suitable residue the vertex corresponding to k defines the long root subgroup geometry of that residue, hence the geometry S_α contains special pairs, a contradiction. $\square$

The previous lemma already provides enough information about the companion geometry in some cases. For instance, the companion geometry of $\mathsf{A}_{1,1}(\mathbb{K}) \cup \mathsf{E}_{7,1}(\mathbb{K})$ in $\mathsf{E}_{8,8}(\mathbb{K})$ arising for $i = 8$ is $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{E}_{7,7}(\mathbb{K})$, since there is only one type of grammatical subspace in both $\mathsf{A}_1(\mathbb{K})$ and $\mathsf{E}_{7,1}(\mathbb{K})$. But in most cases, we do not know yet enough since there are too many choices for k in Table 1. So we have to further pin it down and limit the possibilities for k. That is exactly what we do in Lemma 3.9 below, using the global root system. First we note that heuristics and numbers already suffice to make right guesses.

Remark 3.8. Since we know the number of points of an apartment of a long root geometry (which is the number of roots), and we know the number of points of an apartment in each of the Jordan geometries (the latter are defined in Section 4.1.1), and each point belongs to either the long root subgeometry or a companion geometry, simple arithmetics can already lead to the right guesses, especially in the irreducible case. Let us give an example. Let $i = 2$ in case of E_8. There are 240 roots, 72 of which are taken by the long root geometry of $\mathsf{A}_8(\mathbb{K})$. There remain 168 roots. Apartments of type $\mathsf{A}_{8,1}, \mathsf{A}_{8,2}, \mathsf{A}_{8,3}$ and $\mathsf{A}_{8,4}$ have 9, $\binom{9}{2} = 72$, $\binom{9}{3} = 84$ and $\binom{9}{4} = 126$ points, respectively. The only way 168 can be written as a sum of these is as 84+84, leading to a coupled $\mathsf{A}_{8,3}$ and $\mathsf{A}_{8,5}$, using the heuristic that no duality class of $\mathsf{A}_8(\mathbb{K})$ plays a favourite role. Similar, but not completely identical, story for $i = 1$, in which case long root D_8 already accounts for 112 points/roots. The remaining 128 either give rise to eight copies of $\mathsf{D}_{8,1}(\mathbb{K})$ or one copy of $\mathsf{D}_{8,8}(\mathbb{K})$. The heuristic that large subgroups produce few orbits leads to $\mathsf{D}_{8,8}(\mathbb{K})$.

Lemma 3.9. *Let $i \in \{1, \ldots, n\}$, suppose that λ_i is prime and denote with $\phi_{i,1}, \ldots, \phi_{i,r}$ the connected components of ϕ_i. Let $0 < j < \lambda_i$ and let $\alpha \in \phi_i^j$. The set T_α of points of ϕ_i collinear to α is the union of k_l-grammatical subspaces of $\Omega_{i,l}$ for k_l as in* Table 2*, after possibly renumbering the components $\phi_{i,1}, \ldots, \phi_{i,r}$, and/or renumbering the nodes of the diagram of an individual component $\phi_{i,l}$ by applying a diagram automorphism.*

Type of ψ	i	Type of $\phi_i = \phi_{i,1}, \ldots, \phi_{i,r}$	$(k_1, \ldots, k_r)$
D_4	2	$\mathsf{A}_1 \cup \mathsf{A}_1 \cup \mathsf{A}_1 \cup \mathsf{A}_1$	$(1,1,1,1)$
D_m $(m \geq 5)$	2 or $m-2$	$\mathsf{A}_1 \cup \mathsf{A}_1 \cup \mathsf{D}_{m-2}$	$(1,1,1)$
	$2 < i < m-2$	$\mathsf{D}_i \cup \mathsf{D}_{m-i}$	$(1,1)$
E_6	$2, 3$ or 5	$\mathsf{A}_1 \cup \mathsf{A}_5$	$(1,3)$
	4	$\mathsf{A}_2 \cup \mathsf{A}_2 \cup \mathsf{A}_2$	$(1,1,1)$
E_7	1 or 6	$\mathsf{A}_1 \cup \mathsf{D}_6$	$(1,6)$
	2	A_7	(4)
	3 or 5	$\mathsf{A}_2 \cup \mathsf{A}_5$	$(1,2)$
E_8	8	$\mathsf{A}_1 \cup \mathsf{E}_7$	$(1,7)$
	7	$\mathsf{A}_2 \cup \mathsf{E}_6$	$(1,1)$
	5	$\mathsf{A}_4 \cup \mathsf{A}_4$	$(1,2)$
	1	D_8	(8)
	2	A_8	(3)

Table 2: α is collinear to the union of k_l-components of $\Omega_{i,l}$ $(l = 1, \ldots, r)$

Proof. We start by making two observations regarding ψ, ϕ_i and Ω_i.

1. Let β_1, β_2 be two symplectic roots of ϕ_i, both contained in T_α. Then, by calculating their dot product, we see that the roots $\alpha - \beta_1$ and $\alpha - \beta_2$ are also symplectic. Denote with ξ_i the symplecton in Ω_i determined by β_1 and β_2, and with ζ the symplecton in Ω determined by $\alpha - \beta_1$ and $\alpha - \beta_2$. Then a straight forward calculation using the dot product yields

$$\{\alpha\} \cup \{\alpha - \beta \mid \beta \in \xi_i \cap \phi_i\} \cup \{\gamma \mid \gamma \in \phi_i \cap T_\alpha \text{ with } \langle \beta_1, \gamma \rangle = \langle \beta_2, \gamma \rangle = 0\} \subseteq \zeta.$$

2. Let $M_1, M_2 \subseteq T_\alpha \cap \phi_i$ be two sets of mutually collinear roots for which $\langle M_1, M_2 \rangle = 0$, that is, each root in M_1 is symplectic to each root in M_2. Then, again an easy calculation with dot products, implies that

$$\{\alpha\} \cup M_1 \cup \{\alpha - \beta \mid \beta \in M_2\}$$

forms a set of mutually collinear roots.

In all cases, these two observations suffice to prove the lemma. We work out three explicit examples when ψ has type E_8, all other cases are completely similar.

- *Let* $i = 5$. Note that $\phi_i = \phi_{5,1} \cup \phi_{5,2}$ has type $\mathsf{A}_4 \cup \mathsf{A}_4$. By Lemma 3.7, we know that T_α is the union of a $\{k_1\}$-grammatical subspace of $\phi_{5,1}$ and a $\{k_2\}$-grammatical subspace of $\phi_{5,2}$. After possibly applying diagram morphisms on the diagrams of $\phi_{i,1}$ and $\phi_{i,2}$, we find that $k_1, k_2 \in \{1, 2\}$. Let $l = 1, 2$. Denote with $S_{\alpha,l}$ the points of $\Omega_{5,l}$ collinear to α. If $k_l = 1$ (or 2), then $S_{\alpha,l}$ is a point-line geometry of type $\mathsf{A}_{3,1}$ (or $\mathsf{A}_{1,1} \times \mathsf{A}_{2,1}$, respectively). First suppose that $k_1 = k_2 = 1$. Then both $S_{\alpha,1}$ and $S_{\alpha,2}$ consist of 4 mutually collinear roots. By applying Argument 2 above to these two sets, we find 9 mutually collinear roots in ψ, a contradiction. Without loss of generality, we can hence assume that $k_2 = 2$. We find roots β_1 and β_2 of $\phi_{5,2}$ that are symplectic. By Argument 1 above, we find that $\alpha - \beta_1$ and $\alpha - \beta_2$ are roots of ψ that are symplectic, and that all roots of $\phi_{5,1}$ must be contained in the symplecton of Ω determined by these two points. This implies that all points of $\phi_{5,1}$ must be contained in one common symplecton, from which we obtain that $k_1 = 1$.

- *Let* $i = 1$. Then ϕ_1 has type D_8. It follows from Lemma 3.7 that T_α is a $\{k\}$-grammatical subspace of Ω_1 for $k \in \{1, 8\}$ (after possibly renumbering the diagram by applying a diagram morphism). Suppose that $k = 1$. Then T_α is a point-line geometry of type $\mathsf{D}_{7,1}$. Choose two symplectic roots of $T_\alpha \cap \phi_1$. It follows from Argument 2 above that there is a symplecton of Ω that contains both α and T_α, implying that Ω contains a symplecton of rank at least 8, a contradiction. We hence conclude that $k = 8$.

- *Let* $i = 2$. Then ϕ_2 has type A_8. Again by Lemma 3.7, we find that T_α is a $\{k\}$-grammatical subspace of Ω_2, for some $k \in \{1, 2, 3, 4\}$. If $k = 1$, then T_α is a point-line geometry of type $\mathsf{A}_{7,1}$, implying that $T_\alpha \cap \phi_2$ contains 8 mutually collinear roots, a contradiction (as ψ does not contain 9 mutually collinear roots). If $k = 2$, then by applying Observation 2 above to $T_\alpha \cap \phi_2$, we obtain that the collinearity graph of ψ should admit two 8-cliques with just 6 points in common, while two distinct 8-cliques of ψ have at most 5 points in common. Suppose that $k = 4$, then T_α is a point line geometry of type $\mathsf{A}_{3,1} \times \mathsf{A}_{5,1}$. Let β_1 and β_2 be two symplectic roots of T_α. By Observation 1 above, the set

$$\{\gamma \in T_\alpha \cap \phi_2 \,|\, \langle \beta_1, \gamma \rangle = \langle \beta_2, \gamma \rangle = 0\}$$

  would have to be contained in a symplecton of ϕ'. One, however, again easily verifies that this is not the case. We conclude that $k = 3$. □

Remark 3.10. The sets T_α we obtain in Lemma 3.9 are maximal in the following sense. Take $g \in G_{\phi_i}$, then either $T_\alpha^g = T_\alpha$ or there exists some point in T_α^g which is opposite some point of T_α.

Lemma 3.9 determines the types of the various companion geometries. It remains to prove that the companion geometries are well defined and really embedded geometries, that is, the line set determined by the given type coincides with the line set as a subspace of Ω.

3.2.2 Well-definedness of the companion geometries

Lemma 3.11. *Let $i \in \{1, \dots, n\}$ and suppose that λ_i is prime. Let $0 < j < \lambda_i$ and let $\alpha \in \phi_i^j$. There is a root $\beta \in \phi_i^j$ such that the points of Ω_i collinear to both α and β are not contained in a common symplecton of Ω_i.*

Proof. Denote with T_α the points of Ω_i collinear to α.

Let $\gamma \in T_\alpha \cap \phi_i$. As $\text{proj}_i(\alpha - \gamma) = \text{proj}_i(\alpha) - \text{proj}_i(\gamma)$, we find $\alpha - \gamma \in \phi_i^j$. This root $\beta := \alpha - \gamma$ is collinear to all roots of ϕ_i that are collinear to α and symplectic to γ. By Lemma 3.9, we know what $T_\alpha \cap \phi_i$ looks like, and in all cases, we can pick a root $\gamma \in T_\alpha \cap \phi_i$ such that the roots of $T_\alpha \cap \phi_i$ that are symplectic to γ are not contained in a common symplecton of Ω_i. ☐

Lemma 3.12. *Let $i \in \{1, \dots, n\}$ and $0 < j < \lambda_i$. The group G_{ϕ_i} acts transitively on the points of Ω_i^j. Moreover, no two points of Ω_i^j are collinear to the same subset of Ω_i.*

Proof. Denote $G := G_{\phi_i}$, and let α a root in ϕ_i^j. We first prove that G acts transitively on Ω_i^j, that is, $\alpha^G = \Omega_i^j$. Note that it follows from Lemma 3.5 that $\alpha^G \subseteq \Omega_i^j$. We prove the other inclusion. The group W_i from Example 2.13 acts transitively on ϕ_i^j. For $\beta, \gamma \in \phi_i^j$ and $\gamma \in \phi_i^j$, one finds elements u in $\langle Z_\beta, Z_{-\beta} \rangle \leq G$ such that $\gamma^{s_\beta} = \gamma^u$. From this, we can already conclude that $\phi_i^j \subseteq \alpha^G$. In order to prove that Ω_i^j is contained in α^G, it hence suffices to prove that α^G is a subspace.

Let x and y be any two collinear points in α^G. We aim to prove that the line L through x and y is fully contained in α^G. Without loss of generality, we may assume that $x = \alpha$. Let g be an element of G which maps α to y, and let T_α be the set of points in Ω_i collinear to α. We distinguish two different cases.

1. $T_\alpha^g \neq T_\alpha$. In this case, it follows from Remark 3.10 that there exist points $p \in T_\alpha$ and $q \in T_\alpha^g$ such that p and q are opposite. The point p is then special to y, with $\alpha = [p, y]$. The group $Z_p \leq G$ acts transitively on the points of $L \setminus y$, implying that L is contained in α^G.

2. $T_\alpha^g = T_\alpha$. We try to obtain a contradiction. Let $\beta \in \phi_i^j$ be a root as in Lemma 3.11 (it is collinear to α, collinear to at least two points of T_α and there is no symplecton of Ω_i that contains all roots collinear to both α and β.) As both y and β are collinear to all roots collinear to α and β, we find that y and β are collinear or symplectic. If they were symplectic, the symplecton of Ω determined by y and β would contain all points of Ω_i collinear to both y and β, which are precisely the points of Ω_i collinear to α and β. We have however chosen β in such a way that no such symplecton exists. We conclude that y and β are collinear. Now consider the root $\alpha - \beta$, which exists because α and β are collinear. It is contained in ϕ_i (by just considering proj_i), is collinear to α (and hence also to y) and special to β. But then both α and y are collinear to β and $\alpha - \beta$, a contradiction to the fact that β and $\alpha - \beta$ are special.

We conclude that G acts transitively on Ω_i^j. The argument above then automatically also implies that no two points of Ω_i^j are collinear to the same set of points of Ω_i^j. ☐

Now we still have to verify that the sets of points of Ω_i^j that correspond to the lines of the K-Grassmannian as given by Lemma 3.9, and with K corresponding to the array $(k_1, \ldots, k_r)$ as in Table 2, are precisely the lines of Ω completely contained in it.

A *pencil* of ℓ-grammatical subspaces is a set of grammatical subspaces defining a line in the corresponding ℓ-Grassmannian geometry.

Proposition 3.13. *The lines of Ω_i^j correspond to pencils of grammatical subspaces of Ω_i.*

Proof. Let x and y in Ω_i^j and let T_x and T_y be the grammatical subspaces of Ω_i collinear to x and y, respectively. By Remark 3.10, there is a point $p \in T_x$ opposite to some point $q \in T_y$. First suppose $x \perp y$. The group Z_p fixes all points of T_y collinear or symplectic to p and acts transitively on points of $xy \setminus \{x\}$, Now using the fact that Ω is a gamma space, we find that points of T_y collinear or symplectic to p are collinear to xy, and hence contained in T_x. This shows that every symplecton contained in T_y contains at least one point of T_x. This is enough to conclude that the intersection is large enough so that the grammatical subspaces T_x and T_y belong to the same pencil, as can be verified case-by-case.

Now assume x and y are not collinear, but $T_x \cap T_y$ is large, in particular contains at least a point, so that p and y are special. Then similarly as above, the action of Z_p, which stabilizes the pencil P of grammatical subspaces defined by T_x and T_y, shows that each member of P is defined by a unique point of the line containing y and $[p, y]$. Hence x belongs to that line, and since $x \perp p$, we see that $x = [p, y]$, implying that x is collinear to y. □

Taking Lemma 3.9 and Proposition 3.13 together, we obtain the Main Result mentioned in the introduction.

4 Some geometric constructions

In the previous sections, we saw which types of full rank Lie geometries embed in the long root geometries of exceptional type in the simply laced case. This also provided a recipe of how to construct them. In this section, we will phrase these constructions purely geometrically, mostly in terms of so-called *equator geometries*. These are subgeometries of Lie incidence geometries arising from two opposite flags by considering the points "in the middle", or "on the equator", where the two flags play the role of the poles.

Moreover, we will also construct most of the full rank Lie subgeometries inside more popular Lie incidence geometries than the long root ones, in casu, the minuscule geometries $\mathsf{E}_{6,1}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$ of types E_6 and E_7, whose natural representation lives in projective space of dimension 26 and 55, which we call the *Schläfli* and the *Gosset* varieties, respectively, since they can be constructed using the corresponding graphs. For type E_8, the smallest dimension corresponds to the long root geometry (adjoint representation).

In the next section, we will then treat the non-simply laced cases. Also there, more popular geometries exist. For type G_2, the dual hexagon is more popular since in the spit case is it simply the split Cayley hexagon, which lives on a parabolic quadric in

6-dimensional projective space; for type F_4, the *dual* of the long root geometry in the split case arises from intersecting the Schläfli variety with a hyperplane; it lives in 25-dimensional projective space.

4.1 Inside the long root subgroup geometries

4.1.1 Some conventions

We first introduce some terminology for nodes of the exceptional Dynkin diagrams. The node corresponding to the fundamental root not perpendicular to the longest root will be called the *polar node*. The unique node adjacent to it is the *subpolar node*. Every node in the orbit of the node corresponding to the longest root in the extended Dynkin diagram under the symmetry group of the extended diagram is called a *Jordan node*. The latter can be defined in the same way for classical Dynkin diagrams, too. For Coxeter diagrams, the Jordan nodes are those that are Jordan nodes in some Dynkin diagram underlying the Coxeter diagram. Here is a table with the Jordan nodes thus defined:

Coxeter type	Jordan nodes
A_n	$1, 2, \ldots, n$
$\mathsf{B}_n/\mathsf{C}_n$	$1, n$
D_n	$1, n-1, n$
E_6	$1, 6$
E_7	7
$\mathsf{E}_8/\mathsf{F}_4/\mathsf{G}_2$	none

Not coincidently, the diagrams having no Jordan nodes are precisely those that do not extend to another spherical diagram. Jordan nodes can also be defined as those corresponding to the fundamental roots where the coefficient of the highest root in its expression as a linear combination of fundamental roots, is equal to 1. Also, by [13], the Jordan nodes of X_n are precisely those nodes i for which the Lie incidence geometry of type $\mathsf{X}_{n,i}$ is strong, that is, has no special pairs and this is equivalent to all convex subspaces to correspond to residues of the underlying building, and, in the simply laced case, to apartments to generate the geometry. The Lie incidence geometry corresponding to a Jordan node will be called a *Jordan (Lie incidence) geometry*. It follows from the previous sections (cf. Lemma 3.9 combined with Proposition 3.13) that the maximal full rank Lie subgeometries embed in the ambient long root geometry as a coupled union of a long root geometry with one or more Jordan Lie incidence geometries. Also, the Lie incidence geometries of type E_6 and E_7 that we called "more popular" in the introduction to the current section are the Jordan ones for these types (and they are also known as the minuscule geometries).

A maximal full rank Lie subgeometry is *of Dynkin cotype* i if its Coxeter type is the residue of vertex i (in Bourbaki labellng) in the extended Dynkin diagram.

If the Coxeter type of a maximal full rank Lie incidence subgeometry is reducible, then the irreducible components might appear either as factors of a Cartesian product geometry, or as a perpendicular union of independent geometries. This perpendicularity is given by the perpendicularity of the corresponding roots. Hence, if the underlying long root geometry is a parapolar space, subgeometries are perpendicular precisely when all points of one subgeometry are symplectic to all points of the other(s). In case of type G_2, a generalized hexagon, a point x and a line L are perpendicular precisely when they are not incident and not at maximal distance, and we also write $x \perp\!\!\!\perp L$. A similar thing happens for type F_4, where the short roots can be thought of as corresponding to the symps. Then a point x and a symp ξ are perpendicular, denoted as $x \perp\!\!\!\perp \xi$, precisely when x is close to ξ (cf. Fact 5.1). Note that not all points of ξ are symplectic to x, hence there is danger of confusion with the usual meaning of the notation $\perp\!\!\!\perp$; we shall therefore only use that symbol for a perpendicular point-symp pair when it is absolutely clear from the context that it concerns a relation between points and symps, and not between mutual point sets.

4.1.2 Some basic properties of long root subgroup geometries

We state as facts some basic properties shared by all long root subgroup geometries.

Fact 4.1. *If $a \perp b \perp c \perp d$ is a path in Δ, then $a \bowtie c$ and $b \bowtie d$ if and only if a is opposite d.*

Fact 4.2. *For each point p and each symp ξ, there is at least one point $q \in \xi$ symplectic to p; that point q is unique if and only if ξ contains some point opposite p. In this case, all points of $q^\perp \cap \xi \setminus \{q\}$ are special to p and all points of $\xi \setminus q^\perp$ are opposite p.*

For two opposite points p, q, we denote with $R(p, q)$ the set of lines containing collinear points to p and to q. Likewise, for two opposite lines L, M, we let $R(L, M)$ be the set of points having collinear points in both L and M.

Fact 4.3. *Let Δ be a long root geometry of exceptional type E over the field $\mathbb{K}$, or a Lie incidence geometry isomorphic to $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$, for some quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$, or a Moufang hexagon defined over the field $\mathbb{K}$. Then, for each pair of opposite points p, q, the set of points $R(L, M)$, with $L, M \in R(p, q)$ opposite, is independent of the choice of $L, M \in R(p, q)$. The stabilizer of $R(L, M)$ inside the little projective group of Δ contains $\mathsf{PSL}_2(\mathbb{K})$.*

Also, $R(L, M) = \{p, q\}^{\perp\!\!\!\perp\perp\!\!\!\perp}$, the set of points symplectic to all points that are symplectic to both p and q.

The set $R(L, M)$ is called an *imaginary line* and denoted $I(p, q)$. It is uniquely determined by each pair of its points.

4.1.3 The Dynkin cotype corresponds to the polar node

This type of maximal full rank Lie subgeometries has a canonical geometric description, valid for all long root geometries of exceptional type E over the field $\mathbb{K}$, or a Lie incidence geometry isomorphic to $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$, for some quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$, or a Moufang hexagon defined over the field $\mathbb{K}$. Let Δ be such a geometry. Let p, q

be two opposite points of Δ. The set $p^{\perp\!\!\!\perp} \cap q^{\perp\!\!\!\perp}$ is called an *equator set*. It is empty for Moufang hexagons, and it does not contain lines for type $\mathsf{F}_{4,1}$. In the other cases we endow it with the induces lines and call this the *equator geometry (with poles p, q)*, denoted by $E(p, q)$. For type $\mathsf{F}_{4,1}$, we endow it with the intersections with symplecta that share at least two points with it, and also call it the *equator geometry (with poles p, q)*, denoted by $E(p, q)$. In the nonempty case, $E(p, q)$ is the long root subgroup geometry Ω corresponding to the residue of a vertex of type the polar node. Any pair of points of $I(p, q)$ can serve as poles. Hence the corresponding maximal full rank Lie subgeometry is $\mathsf{A}_{1,1}(\mathbb{K}) \times \Omega$. Its companion geometry is defined as follows. For each point $x \in I(p, q)$, let $R(x)$ be the set of points collinear to x and at distance 2 (in the collinearity graph; otherwise said, special to) from every member of $I(p, q) \setminus \{x\}$. Note that $R(x)$, endowed with all lines completely contained in it, is a Lie incidence geometry Ω' corresponding to the point residual building at x and related to the subpolar node. The union of all $R(x)$ for x ranging over $I(p, q)$ is a product geometry $L \times \Omega'$, where L is any member of $R(p, q)$; in fact the point set $L \times \Omega'$ is also the union of all members of $R(p, q)$. We call this product geometry the *subequator geometry*.

4.1.4 The Dynkin cotype corresponds to the subpolar node

The long root subgeometries—In this case, the maximal full rank Lie subgeometry is the direct product of $\Omega_1 := \mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ with another (long root) Lie incidence geometry, say Ω_2. The component Ω_1 is obtained by taking the *special closure* of two opposite lines, that is, the smallest subspace containing the two opposite lines and closed under taking the centre of a pair of special points contained in the subspace. Let p, q be two opposite points in this geometry Ω_1, and let L, M be the lines in this geometry belonging to (p, q), and let $p \perp x \in L$, $q \perp y \in M$. Then Ω_2 is the intersection $E(p, q) \cap E(x, y)$. Inside $E(p, q)$, it can easily be checked that this coincides with the equator geometry, appropriately defined (see below for each of the separate cases), of a pair of opposite objects of $E(p, q)$ corresponding to the lines through p. Let us briefly work this out for the E-cases.

In $\mathsf{E}_{6,2}(\mathbb{K})$, points have type 2 and lines have type 4. Here, $E(p, q)$ is $\mathsf{A}_{5,\{1,5\}}(\mathbb{K})$, and type 4 elements of the building correspond to Segre subgeometries of type $(2, 2)$, that is, product spaces of two planes. Considering a pair Γ, Γ' of these, the equator geometry $E(\Gamma, \Gamma')$ is the geometry induced by the set of points collinear to a plane of Γ and to one of Γ'. In the underlying projective space $\mathsf{PG}(5, \mathbb{K})$ we obtain the set of point-hyperplane pairs having their point inside a fixed plane π and having their hyperplane through a disjoint plane π', or vice versa. This is the union of two long root geometries isomorphic to $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$.

In $\mathsf{E}_{7,1}(\mathbb{K})$, points have type 1 and lines type 3. Here, $E(p, q)$ is $\mathsf{D}_{6,2}(\mathbb{K})$ and type 3 elements correspond to convex subgeometries of type $\mathsf{A}_{5,2}$. Considering a pair Γ, Γ' of these, the equator geometry $E(\Gamma, \Gamma')$ is the geometry induced by the set of points collinear to a(n automatically non-maximal) singular subspace of dimension 3 of Γ and to one of Γ'. In the underling polar space, it is the set of lines intersecting each of two opposite maximal singular subspaces in a point.

In $\mathsf{E}_{8,8}(\mathbb{K})$ finally, points have type 8 and lines type 7. Here, $E(p,q)$ is $\mathsf{E}_{7,1}(\mathbb{K})$ and type 7 elements of $\mathsf{E}_{8,8}(\mathbb{K})$ have type 7 in $\mathsf{E}_{7,1}(\mathbb{K})$ and correspond to convex subgeometries of type $\mathsf{E}_{6,1}$. Considering a pair Γ, Γ' of these, the equator geometry $E(\Gamma, \Gamma')$ is the geometry induced by the set of points collinear to a(n automatically maximal) singular subspace of dimension 5 of Γ and to one of Γ'.

The companion geometries—We now describe the general construction of the companion geometries from the long root subgeometry $\Omega_1 \times \Omega_2$ (see the previous paragraph). The following also holds in a sort of degenerate form for type F_4, and it is worked out in detail in §5.2.4. For type E, proofs are similar (and simpler, in fact) and so we just give the construction.

Consider two opposite points p, q of Ω_1 and let $p \perp p_1 \perp q_1 \perp q \perp q_2 \perp p_2 \perp p$ be the unique hexagon in Ω_1 thus defined. For each plane π_1 through p, p_1, there exist unique planes π_2 and π_3 containing q, q_1 and p_2, q_2, respectively, such that π_1, π_2, π_3 intersect a common plane π in three respective points. Explicitly, the intersection point $a_1 := \pi_1 \cap \pi$ is given by the unique point of π_1 not opposite both q and q_2. The point $\pi_2 \cap \pi$ is defined as the unique point a_2 collinear to both a_1 and q, and, likewise, $\pi_3 \cap \pi$ is the unique point a_3 collinear to both a_2 and p_2, or a_1 and q_2. Note that $a_3 \in E(p,q)$. The points a_2 and a_3 thus defined also determine π_2 and π_3, respectively. By varying π_1, the plane π describes the maximal planes of the geometry $\pi \times \Omega_3$, where Ω_3 is the residual geometry of the line pp_1. We call $\pi \times \Omega_3$ the *half subequator intersection geometry* for further reference in our tables.

One can do the same with the line pp_2 to obtain the second companion geometry, isomorphic to $\pi \times \Omega_3$. One checks that a direct way to obtain this final companion is to collect the centres of all special pairs contained in $\pi \times \Omega_3$. However, this is not a very geometrically transparent construction. For the sake of easy reference, we call this the *centre geometry*, but we do not insist on it further.

Now we take a look at the individual exceptional simply laced cases and relate the general constructions so far to some specific constructions.

4.2 Case of type E_6

4.2.1 Table of maximal full rank Lie subgeometries

	Type	Isomorphism class	Comments
2	$\mathsf{A}_1 \times \mathsf{A}_5$	$\mathsf{A}_{1,1}(\mathbb{K}) \perp\!\!\!\perp \mathsf{A}_{5,\{1,5\}}(\mathbb{K}) \cup$ $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{5,3}(\mathbb{K})$	Imaginary line & its equator in $\mathsf{E}_{6,2}$ Subequator in $\mathsf{E}_{6,2}$
		$\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{5,1}(\mathbb{K}) \cup$ $\mathsf{A}_{5,2}(\mathbb{K})$	$(1,5)$-Segre geometry in $\mathsf{E}_{6,1}$ Equator of previous in $\mathsf{E}_{6,1}$
4	$\mathsf{A}_2 \times \mathsf{A}_2 \times \mathsf{A}_2$	$\mathsf{A}_{2,\{1,2\}}(\mathbb{K}) \perp\!\!\!\perp \mathsf{A}_{2,\{1,2\}}(\mathbb{K}) \perp\!\!\!\perp \mathsf{A}_{2,\{1.2\}}(\mathbb{K}) \cup$ $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K}) \cup$ $\mathsf{A}_{2,2}(\mathbb{K}) \times \mathsf{A}_{2,2}(\mathbb{K}) \times \mathsf{A}_{2,2}(\mathbb{K})$	Equator intersection in $\mathsf{E}_{6,2}$ Half subequator intersection in $\mathsf{E}_{6,2}$ Centre geometry of previous
		$(\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})) \cup (\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})) \cup$ $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$	Coupled Segre geometries in $\mathsf{E}_{6,1}$ Equator of previous in $\mathsf{E}_{6,1}$

4.2.2 Trivia about the minuscule geometry $\mathsf{E}_{6,1}(\mathbb{K})$

The minuscule geometry of type E_6 over the field $\mathbb{K}$ is the Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$. It is a parapolar space of constant symplectic rank 5 with the characterizing property that each point residual is isomorphic to the half spin geometry $\mathsf{D}_{5,5}(\mathbb{K})$. The maximal singular subspaces have projective dimensions 4 and 5; the non-maximal singular subspaces of dimension 4 are usually called $4'$-spaces. The singular 5-spaces correspond to vertices of type 2 of the corresponding building and two such 5-spaces are opposite (as vertices of the spherical building) if and only if the collinearity relation defines a bijection, and hence an isomorphism, between the two 5-spaces.

For a point x and a 5-space U, we say that x and U are *close* if $x^\perp \cap U$ is a 3-space. There are only two other possibilities, namely, $x \in U$ and $|x^\perp \cap U| = 1$.

4.2.3 Case $\mathsf{A}_1 \times \mathsf{A}_5$

Proposition 4.4 of [8] implies the following construction of the full rank subgeometry of Dynkin cotype 2.

Construction 4.4 (Dynkin cotype 2 for E_6)**.** Let W, W' be opposite 5-spaces of $\mathsf{E}_{6,1}(\mathbb{K})$. Let $\mathscr{L}_1$ be the set of lines intersecting $W \cup W'$ in precisely two points (hence each of W and W' in exactly one point). Then for each point x on each member of $\mathscr{L}_1$ there exists a unique 5-space W_x intersecting all members of $\mathscr{L}_1$, and the collection of all such intersection points is precisely W_x; if $x \notin W \cup W'$, then W_x is opposite both W and W'. Hence the union of all members of $\mathscr{L}_1$ induces in $\mathsf{E}_{6,1}(\mathbb{K})$ a Segre geometry $\mathscr{S}(W, W')$ of type $(5, 1)$, the product geometry $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{5,1}(\mathbb{K})$ of a projective line with a projective 5-space.

The set of points x such that both $x^\perp \cap W$ and $x^\perp \cap W'$ are 3-spaces, together with all lines entirely contained in it, forms a Lie incidence geometry $E(W, W')$ isomorphic to $\mathsf{A}_{5,2}(\mathbb{K})$, called the *equator geometry (with poles* W, W'*)*. Each point of $E(W, W')$ is collinear to a 3-space of each 5-space of $\mathscr{S}(W, W')$ and hence every pair of 5-spaces of $\mathscr{S}(W, W')$ can serve as pair of poles of $E(W, W')$.

We note that, performing the above construction to a skeleton of W (inducing a skeleton in W', we obtain all the points of an apartment. By [2, 6], this generates $\mathsf{E}_{6,1}(\mathbb{K})$. Hence $\mathscr{S}(W, W') \cup E(W, W')$ generates $\mathsf{E}_{6,1}(\mathbb{K})$. In the universal embedding of $\mathsf{E}_{6,1}(\mathbb{K})$, the Segre geometry $\mathscr{S}(W, W')$ spans an 11-dimensional space, whereas $E(W, W')$ is (universally) embedded in a complementary subspace of dimension 14.

4.2.4 Case $\mathsf{A}_2 \times \mathsf{A}_2 \times \mathsf{A}_2$

Also this case is realized by a construction already in the literature. Indeed, the following can be extracted from §1.5.6 of [8], in particular Remark 5.27 therein. Set $\Delta := \mathsf{E}_{6,1}(\mathbb{K})$.

Construction 4.5 (Dynkin cotype 4 for E_6)**.** Let π and π' be two opposite planes in Δ. This means that the collinearity relation between them is empty. Let U_1 and U_2 be two distinct singular 5-spaces of Δ containing π. Then there exist unique 5-spaces U_1' and U_2' containing π' such that some planes $\pi_i \subseteq U_i$ and $\pi_i' \subseteq U_i'$ span a singular 5-space U_i'', $i = 1, 2$. Then the set $E(\pi, \pi')$ of points of Δ collinear to some line in each of the planes π, π', π_i, π_i', $i = 1, 2$, is the point set of a fully embedded geometry isomorphic to $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ (the line set is just the induced one). Moreover, the set $\Pi(\pi, \pi')$ of 5-spaces close to each point of $E(\pi, \pi')$, is the point set of a non-thick generalized hexagon, which in $\mathsf{E}_{6,2}(\mathbb{K})$ corresponds to a standard (and uniquely) embedded $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$.

Again, the set $E(\pi, \pi')$, together with the union of all 5-spaces belonging to $\Pi(\pi, \pi')$, generates Δ. In the universal embedding of Δ in $\mathsf{PG}(26, \mathbb{K})$, the set $E(\pi, \pi')$ spans an 8-space and the union of all 5-spaces in $\Pi(\pi, \pi')$ spans a 17-dimensional subspace. Now, the set of planes in $\Pi(\pi, \pi')$ contained in at least two 5-space of $\Pi(\pi, \pi')$ form a bipartite graph under the collinearity relation. The planes of each class form again a Segre geometry; hence we obtain two coupled Segre geometries isomorphic to $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$.

The set $\Pi(\pi, \pi')$ also corresponds to the set Σ of symps of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$ obtained in Construction 5.9, viewing $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$ as a full subgeometry of Δ (and then indeed the 5-spaces of Δ fully contained in $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$ correspond to the symplecta of the latter, see e.g. [7]). This provides yet another way to define $\Pi(\pi, \pi')$ and consequently $E(\pi, \pi')$, using the tight connection between $\mathsf{E}_{6,1}(\mathbb{K})$ and $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$.

4.3 Case of type E_7

4.3.1 Table of maximal full rank Lie subgeometries

	Type	Isomorphism class	Comments
1	$\mathsf{A}_1 \times \mathsf{D}_6$	$\mathsf{A}_{1,1}(\mathbb{K}) \perp\!\!\!\perp \mathsf{D}_{6,2}(\mathbb{K}) \cup$	Imaginary line & its equator in $\mathsf{E}_{7,1}$
		$\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{D}_{6,6}(\mathbb{K})$	Subequator in $\mathsf{E}_{7,1}$
		$\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{D}_{6,1} \cup$	Product space line times symp in $\mathsf{E}_{7,7}(\mathbb{K})$
		$\mathsf{D}_{6,6}(\mathbb{K})$	Equator of previous in $\mathsf{E}_{7,7}$
2	A_7	$\mathsf{A}_{7,2}(\mathbb{K}) \cup \mathsf{A}_{7,6}(\mathbb{K})$	Merged poles & equators from A_6 in $\mathsf{E}_{7,7}$
		$\mathsf{A}_{7,4}(\mathbb{K}) \cup$	Symps of previous are points in $\mathsf{E}_{7,1}$
		$\mathsf{A}_{7,\{1,7\}}(\mathbb{K})$	Centre geometry of previous; $\mathsf{A}_{7,\{1,7\}} \leq \mathsf{E}_{7,1}$
3	$\mathsf{A}_2 \times \mathsf{A}_5$	$\mathsf{A}_{2,\{1.2\}}(\mathbb{K}) \perp\!\!\!\perp \mathsf{A}_{5,\{1,5\}}(\mathbb{K}) \cup$	Equator intersection in $\mathsf{E}_{7,1}$
		$\mathsf{A}_{2,2}(\mathbb{K}) \times \mathsf{A}_{5,2}(\mathbb{K}) \cup$	Half subequator intersection in $\mathsf{E}_{7,1}$
		$\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{5,4}(\mathbb{K})$	Centre geometry of previous
		$\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{5,1}(\mathbb{K}) \cup$	Product space in $\mathsf{E}_{7,7}$
		$\mathsf{A}_{2,2}(\mathbb{K}) \times \mathsf{A}_{5,5}(\mathbb{K}) \cup$	Coupled to previous in $\mathsf{E}_{7,7}$
		$\mathsf{A}_{5,3}(\mathbb{K})$	Equator intersection in $\mathsf{E}_{7,7}$
4	$\mathsf{A}_1 \times \mathsf{A}_3 \times \mathsf{A}_3$		$\leq \mathsf{A}_1 \times \mathsf{D}_6$ (not maximal)

4.3.2 Trivia about the minuscule geometry of type E_7

The minuscule geometry of type E_7 over the field $\mathbb{K}$ is the Lie incidence geometry $\mathsf{E}_{7,7}(\mathbb{K})$. It is a parapolar space of constant symplectic rank 6 with the characterizing property that each point residual is isomorphic to the minuscule geometry $\mathsf{E}_{6,1}(\mathbb{K})$. The maximal singular subspaces have projective dimensions 5 and 6; the non-maximal singular subspaces of dimension 5 are usually called $5'$-spaces. The singular 6-spaces correspond to vertices of type 2 of the corresponding building. The vertices of type 1 correspond to the symps. Two such symps are opposite if and only if the collinearity relation defines a bijection, and hence an isomorphism, between the two symplecta. Symps are called *adjacent* if they intersect in a 5-space.

For a point x and a symp ξ, we say that x and ξ are *close* if $x^\perp \cap U$ is a 5-space. There are only two other possibilities, namely, $x \in \xi$ and $|x^\perp \cap \xi| = 1$.

Fact 4.6. *Two 6-spaces are opposite if and only if being symplectic induces a duality between them.*

Fact 4.7. *For a point p and a 6-space W, the only possibilities for $p^\perp \cap W$ are $\emptyset$, a line, a 4-space and W itself (the latter if and only if $p \in W$).*

Fact 4.8. *A 4-space is contained in a unique 6-space and a unique maximal 5-space.*

Fact 4.9. *For opposite points p, q, the map $p^\perp \cap q^{\perp\!\!\perp} \to p^{\perp\!\!\perp} \cap q^\perp : x \mapsto x^\perp \cap q^\perp$ induces a duality between geometries isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$.*

4.3.3 Case $\mathsf{A}_1 \times \mathsf{D}_6$

Construction inside the minuscule geometry—This case is very similar to the case of Dynkin cotype 2 for E_6. Sections 3.3 and 4.3 of [8] yield the following construction.

Construction 4.10 (Dynkin cotype 1 for E_7)**.** Consider two opposite symps ξ, ξ' in $\mathsf{E}_{7,7}(\mathbb{K})$. Let $\mathscr{L}_1$ be the set of lines intersecting $\xi \cup \xi'$ in precisely two points (hence each of ξ and ξ' in exactly one point), and for a point $x \in \xi$, let $\beta(x)$ be the unique collinear point in ξ'. Then for each point x on each member of $\mathscr{L}_1$ there exists a unique symp ξ_x intersecting all members of $\mathscr{L}_1$, and the collection of all such intersection points is precisely ξ_x; if $x \notin \xi \cup \xi'$, then ξ_x is opposite both ξ and ξ'. Hence the union of all members of $\mathscr{L}_1$ induces in $\mathsf{E}_{7,7}(\mathbb{K})$ a product geometry $L \times \xi$, with $L \in \mathscr{L}_1$, of a projective line with a polar space, isomorphic to $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{D}_{6,1}(\mathbb{K})$.

The set of points x of $\mathsf{E}_{7,7}(\mathbb{K})$ such that both $x^\perp \cap \xi$ and $x^\perp \cap \xi'$ are $5'$-spaces, together with all lines entirely contained in it, forms a Lie incidence geometry $E(\xi, \xi')$ isomorphic to $\mathsf{D}_{6,6}(\mathbb{K})$, called the *equator geometry (with poles ξ, ξ')*. Each point of $E(\xi, \xi')$ is collinear to a $5'$-space of each symp of $L \times \xi$ of rank 6, and hence every pair of rank 6 symps of $L \times \xi$ can serve as pair of poles of $E(\xi, \xi')$.

We again note that, performing the above construction to a skeleton of ξ (inducing a skeleton in ξ'), we obtain the point set of an apartment of the corresponding building. By [2, 6], this generates $\mathsf{E}_{7,7}(\mathbb{K})$. Hence $L \times \xi \cup E(\xi, \xi')$ generates $\mathsf{E}_{7,7}(\mathbb{K})$. In the universal embedding of $\mathsf{E}_{7,7}(\mathbb{K})$ in $\mathsf{PG}(55, \mathbb{K})$, the product geometry $L \times \xi$ spans an 23-dimensional space, whereas $E(\xi, \xi')$ is (universally) embedded in a complementary subspace of dimension 31.

It is shown in [8] that the only way in which $\mathsf{D}_{6,6}(\mathbb{K})$ is fully embedded in $\mathsf{E}_{7,7}(\mathbb{K})$ is as an equator geometry like above. In the point residual of $E(\xi,\xi')$, one sees the residue of $\mathsf{D}_{6,6}(\mathbb{K})$, which is $\mathsf{A}_{5,2}(\mathbb{K})$, and a bunch of mutually opposite 5-spaces (coming from the $5'$-spaces in $L\times\xi$ to which the point is collinear) forming a Segre geometry of type $(5,1)$. This is exactly Construction 4.4.

Derived constructions in the long root geometry—We can now also go to $\mathsf{E}_{7,1}(\mathbb{K})$ as follows. The points of $\mathsf{E}_{7,1}(\mathbb{K})$ are the symps of $\mathsf{E}_{7,7}(\mathbb{K})$. Taking the symps of rank 6 of $L\times\xi$, we obtain an imaginary line of $\mathsf{E}_{7,1}(\mathbb{K})$. The corresponding equator geometry can be obtained in two different ways:

(*i*) It corresponds to the collection of symps of $\mathsf{E}_{7,7}(\mathbb{K})$ generated by the symps of $E(\xi,\xi')$;

(*ii*) it also corresponds to the collection of symps generated by the lines K and $\beta(K)$, with K running through the set of lines of ξ.

The corresponding subequator geometry is constructed as the set of symps generated by a point of $E(\xi,\xi')$ and any non-collinear point of $L\times\xi$.

4.3.4 Case $\mathsf{A}_2\times\mathsf{A}_5$

Construction inside the minuscule geometry—It is shown in Proposition 5.31 of [8] that the geometry $\mathsf{A}_{5,3}(\mathbb{K})$ has a unique full embedding Γ in $\mathsf{E}_{7,7}(\mathbb{K})$, and it arises from six symps $\xi_1,\ldots,\xi_6$, with $\xi_i\cap\xi_{i+1}=W_{i,i+1}$ a 5-space (subscripts modulo 6), and ξ_i opposite ξ_{i+3} (again subscripts modulo 6), as the intersection of the equator geometries $E(\xi_i,\xi_{i+3})$, $i=1,2,3$. Now, the fact that opposite symps define a product space isomorphic to $\mathsf{A}_{1,1}(\mathbb{K})\times\mathsf{D}_{6,1}(\mathbb{K})$, implies that the 5-spaces $W_{i,i+1}$ and $W_{i+2,i+3}$ are contained in a unique Segre geometry (fully embedded geometry isomorphic to $\mathsf{A}_{1,1}(\mathbb{K})\times\mathsf{A}_{5,1}(\mathbb{K})$), call it $\mathscr{S}(W_{i,i+1},W_{i+2,i+3})$. Let $x\in W_{1,2}$ be arbitrary. Let $x'\in W_{3,4}$ and $x''\in W_{5,6}$ be collinear with x. If x' were not collinear to x'', then the symp defined by x and the unique point x_0 of $W_{3,4}$ collinear to x'' would contain x,x',x'' and hence at least a line M of $W_{6,1}$, implying that $x'\in\xi_4$ would be collinear to at least two points of ξ_1, namely x and a point of M, contradicting the fact that ξ_1 and ξ_4 are opposite.

Hence each point $x\in W_{12}$ is contained in a unique plane π_x intersecting $\mathscr{S}(W_{1,2},W_{3,4})$ in a line, and the same for $\mathscr{S}(W_{3,4},W_{5,6})$ and $\mathscr{S}(W_{1,2},W_{5,6})$. A routine argument shows that every singular 5-space $W'_{3,4}$ of $\mathscr{S}(W_{1,2},W_{3,4})$ is contained in a symp ξ'_3 together with $W_{2,3}$. There is also a unique symp ξ'_4 in the product space defined by ξ_1 and ξ_4 containing $W'_{3,4}$. Then ξ'_4 contains a unique 5-space $W'_{4,5}$ that also belongs to $\mathscr{S}(W_{4,5},W_{6,1})$ and is contained in a symp ξ'_5 together with $W_{5,6}$. Now suppose $W'_{3,4}\neq W_{1,2}$. Then clearly the symps $\xi_1,\xi_2,\xi'_3,\xi'_4,\xi'_5,\xi_6$ define the same intersection Γ of equator geometries, that is,

$$E(\xi_1,\xi_2)\cap E(\xi_3,\xi_4)\cap E(\xi_5,\xi_6)=E(\xi_1,\xi_2)\cap E(\xi'_3,\xi'_4)\cap E(\xi'_5,\xi_6).$$

Consequently, the Segre geometry $\mathscr{S}(W'_{3,4},W_{5,6})$ is contained in the union Φ of planes π_x, with x ranging over $W_{1,2}$. Varying $W'_{3,4}$, we find that Φ is a product space $\pi_x\times W_{1,2}$, for arbitrary $x\in W_{1,2}$. Similarly, we find a product space Φ' using $W_{2,3},W_{4,5}$ and $W_{6,1}$. Then Γ is defined by each "hexagon" of symps generated by respective 5-spaces of Φ and

Φ'. In fact, the incidence graph on these symps and 5-spaces is the incidence graph of a non-thick generalized hexagon, which in $\mathsf{E}_{7,1}(\mathbb{K})$ defines a fully embedded $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$.

Now, a point of Φ is collinear to a subgeometry of Φ' isomorphic to $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{4,1}(\mathbb{K})$. Hence points of Φ correspond to lines of the maximal planes of Φ', and to hyperplanes of the maximal 5-spaces Φ'. This explains why Φ' is written as $\mathsf{A}_{2,2}(\mathbb{K}) \times \mathsf{A}_{5,5}(\mathbb{K})$.

Derived constructions in the long root geometry—We already derived the standard $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$. The symps of Γ define a set of symps of $\mathsf{E}_{7,1}(\mathbb{K})$, which gives rise to an embedded $\mathsf{A}_{5,\{1,5\}}(\mathbb{K})$. Finally, let $x \in W_{1,2}$ again. Select a line $L \subseteq W_{1,2}$ containing x, and a line $L' \subseteq \pi_x$ containing x. We see that L and L' define a unique symp $\xi(L, L')$, which in fact depends on a line of a 5-space and a line of a plane. The set of all such symps, using Φ, will form a geometry $\mathsf{A}_{2,2}(\mathbb{K}) \times \mathsf{A}_{5,2}(\mathbb{K})$. In Φ' symps relate to the dual of the components, as explained above, whence the geometry $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{5,4}(\mathbb{K})$ as coupled geometry in $\mathsf{E}_{7,1}(\mathbb{K})$.

4.3.5 Case A_7

This is an interesting, because irreducible, case.

Construction inside the minuscule geometry—We start off with a pair of opposite 6-spaces, say W, W'. Let $E(W, W')$ be the set of points x of $\mathsf{E}_{7,7}(\mathbb{K})$ such that $x^\perp \cap W$ is a line and $x^\perp \cap W'$ is a subspace of dimension 4. Similarly, $E(W', W)$ is the set of points y of $\mathsf{E}_{7,7}(\mathbb{K})$ such that $y^\perp \cap W$ is a subspace of dimension 4 and $y^\perp \cap E'$ is a line. Our goal is to show that W and $E(W, W')$ (and symmetrically W' and $E(W', W)$) generate a subgeometry of $\mathsf{E}_{7,7}(\mathbb{K})$ isomorphic to $\mathsf{A}_{7,2}(\mathbb{K})$.

Lemma 4.11. *The set $E(W, W')$, endowed with all the lines of $\mathsf{E}_{7,7}(\mathbb{K})$ entirely contained in it, is a Lie incidence geometry isomorphic to $\mathsf{A}_{6,2}(\mathbb{K})$.*

Proof. Consider an arbitrary 4-space V' in W' and let U' be the unique maximal 5-space containing V'. We claim that there is a unique point $u \in U'$ with $u^\perp \cap W \neq \emptyset$, and that for such u holds that $u^\perp \cap W$ is a line. First assume for a contradiction that there are two points $u_1, u_2 \in U$ with $u_i^\perp \cap W \neq \emptyset$, $i = 1, 2$. If $u_1^\perp \cap u_2^\perp \cap W \neq \emptyset$, then a point in W collinear to u_1 and u_2 is also collinear to $\langle u_1, u_2 \rangle \cap V' \subseteq W'$, contradicting the fact that W and W' are opposite. Hence every $y \in u_1^\perp \cap W$ is symplectic to u_2 and $\xi(u_2, y)$ contains u_1, u_2 and $(u_1^\perp \cup u_2^\perp) \cap W$. Since the latter is at least a line, by assumption, the point $\langle u_1, u_2 \rangle \cap V'$ is collinear to at least one point of W, a contradiction again. Hence at most one point u in U' has the property that $u^\perp \cap W$ is nonempty.

Since being symplectic induces a duality between W and W', there is a unique line $L \subseteq W$ all points of which are symplectic to all points of V'. Select $x \in L$ arbitrary. Select $x' \in W'$ opposite x. By Fact 4.9, there is a point $u_x \perp x$ collinear to V'. Uniqueness of U' yields $u_x \in U'$. By the previous paragraph, $u_x = u_y =: u$ for distinct $x, y \in L$. Since every point of U' is symplectic with every point of $u^\perp \cap W$, it follows that $u^\perp \cap W = L$. The claim is proved.

Now from our proof follows that for each line L in W, there is a point u with $u^\perp \cap W = L$ and $u^\perp \cap W'$ a 4-space; just take for the latter $L^{\perp\!\!\perp} \cap W'$ and apply the proof. Uniqess also follows from that proof.

Hence $E(W,W')$ is in natural bijective correspondence to the set of lines of W, hence to $\mathsf{A}_{6,2}(\mathbb{K})$. It is now routine to check that this bijection is an isomorphism, i.e., maps lines to lines. Indeed, let first K be a line entirely contained in $E(W,W')$. Pick distinct $x,y\in K$. Considering any point in $(x^\perp\cap W)\setminus y^\perp$, we obtain a symp ξ containing K and the span S of $L_x := x^\perp\cap W$ and $L_y := y^\perp\cap W$. If S has dimension 3, then x is collinear to a plane of $S\subseteq W$, a contradiction. Hence L_x and L_y intersect in some point p_K, and $p_K\perp K$. Now in ξ we see that K corresponds to a full line pencil in $\langle L_x,L_y\rangle$. Conversely, let L_1,L_2 be two intersecting lines in W. If the points $u_1,u_2\in E(W,W')$ with $u_i^\perp\cap W = L_i$, $i=1,2$, are not collinear, then they are symplectic and the symp they determine contains a plane of W and a plane of W' contradicting the fact that W does not contains any point collinear to any point of W'. Hence $u_1\perp u_2$ and the first part shows that the planar line pencil determined by L_1 and L_2 corresponds to the line $\langle u_1,u_2\rangle$. □

We call $E(W,W')$ a *directed equator geometry* for further reference.

Proposition 4.12. *The* 6*-space* W *and* $E(W,W')$ *(and symmetrically* W' *and* $E(W',W)$*) generate a subgeometry of* $\mathsf{E}_{7,7}(\mathbb{K})$ *isomorphic to* $\mathsf{A}_{7,2}(\mathbb{K})$*.*

Proof. We use the technique of Section 5.1 of [22]. In the Lie incidence geometry $\mathsf{A}_{7,2}(\mathbb{K})$ absolutely embedded in $\mathsf{PG}(27,\mathbb{K})$ we select a singular subspace W of dimension 6 and an opposite geometry Γ isomorphic to $\mathsf{A}_{6,2}(\mathbb{K})$ (these correspond to a point and a hyperplane not containing that point, respectively, of the underlying geometry $\mathsf{A}_{7,1}(\mathbb{K})\cong\mathsf{PG}(7,\mathbb{K})$). It is easy to see that every point of $\mathsf{A}_{7,2}(\mathbb{K})$ not in W and not in Γ lies on a unique line of $\mathsf{A}_{7,2}(\mathbb{K})$ joining a point of W with one of Γ. Hence the union of the planes intersecting Γ in a point x and W in a line L, is $\mathsf{A}_{7,2}(\mathbb{K})$. The map $x\mapsto L$ induces an isomorphism from the geometry Γ to the line Grassmannian of W, preserving cross-ratio, i.e., the isomorphism is linear. It is now clear, by composing with a linear collineation of W, which is possible since W and $\langle\Gamma\rangle$ are complementary subspaces in $\mathsf{PG}(27,\mathbb{K})$—of dimensions 6 and 20, respectively—that every such linear isomorphism comes from an ambient $\mathsf{A}_{7,2}(\mathbb{K})$.

Hence, in order to derive the assertion from Lemma 4.11, we only still have to check whether, in the absolutely universal embedding of $\mathsf{E}_{7,7}(\mathbb{K})$ in $\mathsf{PG}(55,\mathbb{K})$, the subspaces generated by W and $E(W,W')$ are disjoint. To that aim, we choose a basis in W, take the corresponding basis of W' (and note that every base point of W is opposite a unique base point of W'; moreover, these bases generate opposite flags of type $\{1,2,3,4,5\}$. The points of $E(W,W')$ collinear with lines generated by base points define an apartment in $E(W,W')$, and likewise in $E(W',W)$. It follows that we can extend the opposite flags to opposite chambers and that we obtain the points of an apartment of the underlying building of type E_7. Now, by [2, 6], this apartment generates $\mathsf{E}_{7,7}(\mathbb{K})$. Hence W,W', $E(W,W')$ and $E(W',W)$ generate $\mathsf{E}_{7,7}(\mathbb{K})$, and so they generate $\mathsf{PG}(55,\mathbb{K})$. But the universal embeddings of W,W', $E(W,W')$ and $E(W',W)$ happen in projective subspaces of dimensions $6,6,21$ and 21, respectively. Hence these subspaces are disjoint, as otherwise they do not generate a space of dimension 55. □

Hence the subspaces Δ and Δ' generated by W and $E(W,W')$, and by W' and $E(W',W)$, respectively, define subgeometries isomorphic to $\mathsf{A}_{7,2}(\mathbb{K})$. Clearly, a point of one is collinear to a symp of the other (indeed, we may now take for W any 6-space in Δ and perform the construction. Then we consider a point of W and see that it is collinear to a subgeometry of $E(W', W)$ isomorphic to $\mathsf{A}_{5,2}(\mathbb{K})$, and to nothing in W'). Hence we may view one as $\mathsf{A}_{7,2}(\mathbb{K})$ and the other as $\mathsf{A}_{7,6}(\mathbb{K})$.

Considering the point residual at some point of W, we also see that, in the residue, we get inside Δ a residue isomorphic to $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{5,1}(\mathbb{K})$, and from Δ we get $\mathsf{A}_{5,2}(\mathbb{K})$, as noticed in the previous paragraph. Hence in the point residual we again recover Construction 4.4.

Derived constructions in the long root geometry—If we consider Δ and Δ' as the 2- and 6-Grassmannian, respectively, of the same 7-dimensional projective space, then one checks that collinearity between Δ and Δ' induces a duality of that projective space. Hence symps correspond to symps under that duality, because they are objects of symmetric type 4 in both $\mathsf{A}_{7,2}(\mathbb{K})$ and $\mathsf{A}_{7,6}(\mathbb{K})$. Each such corresponding pair of symps spans a symp of $\mathsf{E}_{7,7}(\mathbb{K})$, and the set of these symps forms the points set in $\mathsf{E}_{7,1}(\mathbb{K})$ of an embedded geometry Ω isomorphic to $\mathsf{A}_{7,4}(\mathbb{K})$. To get to the long root geometry, one notices that a pair (x, y) of points of Ω at distance 3 in Ω corresponds to a pair of 3-space of $\mathsf{PG}(7, \mathbb{K})$ intersecting in a point u and generating a hyperplane H, with $u \in H$. However, one also checks that in $\mathsf{E}_{7,1}(\mathbb{K})$, the pair $\{x, y\}$ is special, and so defines a unique point $p_{x,y}$ of $\mathsf{E}_{7,1}(\mathbb{K})$. It now so happens—but we shall not prove this—that the point $p_{x,y}$ only depends on u and H. Hence we obtain a set of points bijective with the point set of $\mathsf{A}_{7,\{1,7\}}(\mathbb{K})$, and actually, one can show that, endowed with the lines contained in it, it actually is isomorphic to $\mathsf{A}_{7,\{1,7\}}(\mathbb{K})$. This way, we constructed the full rank subgeometries of Dynkin cotype 2 in the long root geometry of type E_7 only using the minuscule geometry $\mathsf{E}_{7,7}(\mathbb{K})$, which is much more accessible.

4.4 Case of type E_8

4.4.1 Table of maximal full rank Lie subgeometries

	Type	Isomorphism class	Comments
1	D_8	$\mathsf{D}_{8,8}(\mathbb{K})\cup$ $\mathsf{D}_{8,2}(\mathbb{K})$	Merged trace geometries in $\mathsf{E}_{8,8}$ Centre geometry of previous; $\mathsf{D}_{8,2} \le \mathsf{E}_{8,8}$
2	A_8	$\mathsf{A}_{8,3}(\mathbb{K})\cup\mathsf{A}_{8,6}(\mathbb{K})\cup$ $\mathsf{A}_{8,\{1,8\}}(\mathbb{K})$	Merged trace geometries in $\mathsf{E}_{8,8}$ Centre geometry of previous; $\mathsf{A}_{8,\{1,8\}} \le \mathsf{E}_{8,8}$
3	$\mathsf{A}_1\times\mathsf{A}_7$		$\le \mathsf{A}_1 \times \mathsf{E}_7$ (not maximal)
4	$\mathsf{A}_1\times\mathsf{A}_2\times\mathsf{A}_5$		$\le \mathsf{A}_1 \times \mathsf{E}_7$ (not maximal)
5	$\mathsf{A}_4\times\mathsf{A}_4$	$\mathsf{A}_{4,\{1,4\}}(\mathbb{K}) \perp\!\!\!\perp \mathsf{A}_{4,\{1,4\}}(\mathbb{K})\cup$ $\mathsf{A}_{4,1}(\mathbb{K})\times\mathsf{A}_{4,2}(\mathbb{K}) \cup \mathsf{A}_{4,4}(\mathbb{K})\times\mathsf{A}_{4,3}(\mathbb{K})\cup$ $\mathsf{A}_{4,2}(\mathbb{K})\times\mathsf{A}_{4,4}(\mathbb{K}) \cup \mathsf{A}_{4,3}(\mathbb{K})\times\mathsf{A}_{4,1}(\mathbb{K})$	Orthogonal $\mathsf{A}_{4,\{1,4\}}$ pair Directed half equators Directed half equators
6	$\mathsf{A}_3\times\mathsf{D}_5$		$\le \mathsf{D}_8$ (not maximal)
7	$\mathsf{A}_2\times\mathsf{E}_6$	$\mathsf{A}_{2,\{1.2\}}(\mathbb{K}) \perp\!\!\!\perp \mathsf{E}_{6,2}(\mathbb{K})\cup$ $\mathsf{A}_{2,1}(\mathbb{K})\times\mathsf{E}_{6,1}(\mathbb{K})\cup$ $\mathsf{A}_{2,2}(\mathbb{K})\times\mathsf{E}_{6,6}(\mathbb{K})$	Equator intersection in $\mathsf{E}_{8,8}$ Half subequator intersection in $\mathsf{E}_{8,8}$ Centre geometry of previous
8	$\mathsf{A}_1\times\mathsf{E}_7$	$\mathsf{A}_{1,1}(\mathbb{K}) \perp\!\!\!\perp \mathsf{E}_{7,1}(\mathbb{K})\cup$ $\mathsf{A}_{1,1}(\mathbb{K})\times\mathsf{E}_{7,7}(\mathbb{K})$	Imaginary line & its equator in $\mathsf{E}_{8,8}$ Subequator in $\mathsf{E}_{8,8}$

There is no minuscule or Jordan geometry in this case. We content ourselves with mentioning some geometric connection between the mutual companion geometries, sometimes describing them from scratch using the diagrams in [22, §7]. Note that the cases $\mathsf{A}_1 \times \mathsf{E}_7$ and $\mathsf{A}_2 \times \mathsf{E}_6$ are explained above as equator geometry and subequator geometry, and intersection of two equator geometries and intersection of half subequator geometries, respectively.

4.4.2 Case A_8

The following discussion is suggested by the second last diagram in §7.3 of [22]. Detailed proofs would be rather technical, though also straightforward.

Embeddings of the Jordan geometries—Consider two opposite singular subspaces of dimension 7, say U, U' in $\Delta := \mathsf{E}_{8,8}(\mathbb{K})$. Each point of U is special to all points of a hyperplane of U' and opposite the others. Hence the centre geometry $\Omega_{1,7}$ (with point set all centres of the special pairs from $U \cup U'$ and line set induced from Δ) is isomorphic to $\mathsf{A}_{7,\{1,7\}}(\mathbb{K})$. Now note that a point outside U is collinear either to the empty subset, a point, a plane, or a 5-space of U. Also, a singular 5-space is contained in a unique maximal 7-space and in a unique maximal 6-space. For each 5-space $W \subseteq U$, the unique maximal 6-space V containing W contains a unique point p_W that is symplectic to at least one point of U', and then it is symplectic to all points of a line $L' \in U'$ (and L' is the unique line in U' all points of which are special to all points of W); moreover $p_W^\perp \cap L'^\perp$ is a 5-space Z_W. The collection of points p_W when W ranges over all 5-spaces of U describes a so-called *trace geometry* Ω_6 isomorphic to $\mathsf{A}_{7,6}(\mathbb{K})$ when endowed with the lines of Δ it contains; the union of all Z_W for W ranging over all 5-spaces of U defines a trace geometry Ω_5 isomorphic to $\mathsf{A}_{7,5}(\mathbb{K})$. Now, just like in the first part of the proof of Proposition 4.12, the union $\Omega_5 \cup \Omega_6$ together with all lines joining a point of Ω_5 with a point of Ω_6 defines a geometry $\Omega_{5,6}$ isomorphic to $\mathsf{A}_{8,6}(\mathbb{K})$. Reversing the roles of U and U', we also find trace geometries Ω_2 and Ω_3 isomorphic to $\mathsf{A}_{7,2}(\mathbb{K})$ and $\mathsf{A}_{7,3}(\mathbb{K})$, respectively, which merge into a geometry $\Omega_{2,3}$ isomorphic to $\mathsf{A}_{8,3}(\mathbb{K})$. One then checks (and the notation for the subscripts was chosen as such) that a point of $\Omega_{5,6}$, which corresponds to a 5-space Y of $\mathsf{PG}(8,\mathbb{K})$, is collinear to all points of $\Omega_{2,3}$ that correspond to a plane of $\mathsf{PG}(8,\mathbb{K})$ contained in Y. This describes the coupling between $\Omega_{2,3}$ and $\Omega_{5,6}$.

Embeddings of the long root geometry—Now, the singular subspaces U, U' together with the centre geometry $\Omega_{1,7}$ do not generate a geometry isomorphic to the long root $\mathsf{A}_{8,\{1,8\}}(\mathbb{K})$; the dimension is one to short. However, there is another geometric way in which we can recover that long root geometry: A point p of $\Omega_{2,3}$ corresponds to a plane π of $\mathsf{PG}(8,\mathbb{K})$; a point q of $\Omega_{5,6}$ corresponds to a 5-space Π of $\mathsf{PG}(8,\mathbb{K})$. If π and Π intersect in a unique point of $\mathsf{PG}(8,\mathbb{K})$, then p and q are special; moreover the centre c only depends on the point-hyperplane pair $(\pi\cap\Pi, \langle\pi,\Pi\rangle)$. The set of all centres endowed with all induced lines is exactly the long root $\mathsf{A}_{8,\{1,8\}}(\mathbb{K})$. In fact, the set of points of $\Omega_{2,3}$ corresponding to planes of $\mathsf{PG}(8,\mathbb{K})$ that contain $\pi\cap\Pi$ and are contained in $\langle\pi,\Pi\rangle$, is the point set of a directed equator geometry of $\mathsf{E}_{7,7}(\mathbb{K})$, realized precisely in the point residual at c.

4.4.3 Case D_8

This paragraph is suggested by the third last diagram in §7.3 of [22]. As in the previous subsection, we omit the proofs, but the interested reader can fill them in.

Embedding of the Jordan geometry—Let Δ again be the geometry $\mathsf{E}_{8,8}(\mathbb{K})$. Consider two opposite symplecta ξ and ξ'. Each point x of one of these is symplectic to exactly one point $\beta(x)$ of the other (and so $\beta(\beta(x)) = x$). Curiously, the image under β of a 6-subspace that is a maximal subspace in Δ is a 6-space that is not a maximal subspace in Δ, and vice versa. Let U be a 6-space of ξ that is contained in a unique 7-space W_U of Δ. Then W_U contains a unique point x_U that is collinear to a 6-space W'_U contained in a symp ξ_U intersecting ξ' in a 6-space, which turns out to be $\beta(U)$. The collection of all x_U, for U ranging over all 6-spaces of ξ that are not maximal in Δ, endowed with the lines induced from Δ, is a geometry Ω_7 isomorphic to $\mathsf{D}_{7,7}(\mathbb{K})$. The union of all W'_U, for U again ranging over all 6-spaces of ξ that are not maximal in Δ, endowed with the lines induced from Δ, is a geometry Ω_6 isomorphic to $\mathsf{D}_{7,6}(\mathbb{K})$. The 6 in the index emphasizes the fact that collinearity between Ω_7 and Ω_6 defines an isomorphism that maps points of Ω_7 to singular 6-spaces of Ω_6, and so, in the common underlying polar space $\mathsf{D}_{7,1}(\mathbb{K})$, maximal 6-spaces of one system correspond to maximal subspaces of the other. Hence it now follows from Proposition 5.3 of [22] that $\Omega_6\cup\Omega_7$, together with all joining lines, constitutes a geometry Ω_{67} isomorphic to $\mathsf{D}_{8,8}(\mathbb{K})$.

Embedding of the long root geometry—Now any pair of points of Ω_{67} that corresponds to a pair of maximal singular subspaces of the underlying quadric $\mathsf{D}_{8,1}(\mathbb{K})$ intersecting in a line L, is special. The collection of such centres p_L (and indeed one can show that p_L only depends on L) is exactly the point set of the long root geometry $\mathsf{D}_{8,2}(\mathbb{K})$. In fact, fixing the line L of the underlying quadric $\mathsf{D}_{8,1}(\mathbb{K})$, the set of points of Ω_{67} that correspond to maximal singular subspaces of $\mathsf{D}_{8,1}(\mathbb{K})$ that contain L, is clearly the point set of a para Ω'_{67} of Ω_{67} isomorphic to $\mathsf{D}_{6,6}(\mathbb{K})$. Such a geometry embeds in Δ as the intersection of a(n equator) subgeometry $\mathsf{E}_{7,1}(\mathbb{K})$ with the point residual at p_L.

4.4.4 Case $\mathsf{A}_4 \times \mathsf{A}_4$

We do not know a direct way to construct the Jordan component here, but instead, we describe how to get from the long root component to its Jordan companion.

So let $\Omega_1 \cup \Omega_2 \cong \mathsf{A}_{4,\{1,4\}}(\mathbb{K}) \cup \mathsf{A}_{4,\{1,4\}}(\mathbb{K})$ be a long root subgroup subgeometry of $\mathsf{A}_{8,8}(\mathbb{K})$, with $\Omega_1 \perp\!\!\!\perp \Omega_2$. We define four subsets of points that we will call *directed half equators*. First we must fix a common underlying projective space $\mathsf{PG}(4,\mathbb{K})$ for Ω_1 and Ω_2. We do this as follows.

Choose an arbitrary underlying $\mathsf{PG}(4,\mathbb{K})$ for Ω_1. Select an arbitrary pair p, q of opposite points of Ω_1. Let Σ and Σ' be the two singular 3-spaces of Ω_1 through p, and without loss of generality we may assume that Σ corresponds to hyperplane of $\mathsf{PG}(4,\mathbb{K})$, that is, the points of Σ correspond to the point-hyperplane pairs of $\mathsf{PG}(4,\mathbb{K})$ with fixed hyperplane. Then Ω_2 is contained in $E(p,q) = p^{\perp\!\!\!\perp} \cap q^{\perp\!\!\!\perp}$ as follows. The subspaces Σ and Σ' correspond in $E(p,q)$ to opposite maximal singular 4-spaces U and U'. Then Ω_2 consists of the centres of all special pairs $\{x, x'\}$, with $x \in U$ and $x' \in U'$. The maximal singular 3-spaces of Ω_2 are given by the centres of the pairs $\{x, x'\}$ for fixed x and varying x', and for fixed x' and varying x. Now, we arrange the connection with $\mathsf{PG}(4,\mathbb{K})$ so that the maximal singular 3-spaces corresponding to fixed $x' \in U'$ correspond to hyperplanes of $\mathsf{PG}(4,\mathbb{K})$.

Now that we fixed the underlying projective space for both Ω_1 and Ω_2, we can speak about subspaces of type ℓ of them, meaning, the set of points corresponding to a residue of a vertex of type ℓ in the building naturally associated to $\mathsf{PG}(4,\mathbb{K})$ (and points have type 1, lines type 2, planes type 3 and 3-spaces type 4). Let $\{i,j\} = \{1,2\}$, let $k \in \{1,4\}$ and $\ell \in \{2,3\}$. Then define $E_k^\ell(\Omega_i, \Omega_j)$ as the set of points of Δ collinear to a subspace of type k of Ω_i and at the same time collinear to a subspace of type ℓ of Ω_j, with induced line set. This way we obtain eight geometries, but, with the aid of the representations of the apartments displayed in Section 7 of [22], one can check that these geometries are empty for $(i,j,k,\ell) \in \{(1,2,1,3), (1,2,4,2), (2,1,1,2), (2,1,4,3)\}$. The other geometries are all isomorphic to the Cartesian product of $\mathsf{PG}(4,\mathbb{K})$ with its line Grassmannian. Taking into account the types inherited from our fixed underlying $\mathsf{PG}(4,\mathbb{K})$, we set $E_k^\ell(\Omega_1,\Omega_2) = \mathsf{A}_{4,k}(\mathbb{K}) \times \mathsf{A}_{4,\ell}(\mathbb{K})$, and likewise $E_k^\ell(\Omega_2,\Omega_1) = \mathsf{A}_{4,\ell}(\mathbb{K}) \times \mathsf{A}_{4,k}(\mathbb{K})$. This provides the geometries mentioned in the above table. Remark that the indices now reflect the fact that the quotient of the full automorphism group of $\Omega_1 \cup \Omega_2$ by the type-preserving one is cyclic of order 4. Indeed, if we interchange Ω_1 with Ω_2, then in order to get the indices of the companion geometries right, we have to apply a duality to exactly one of Ω_1 or Ω_2. Applying the same map twice, we obtain dualities in both Ω_1 and Ω_2.

5 Buildings of exceptional types F_4 and G_2

In this section we construct, in a geometric and individual way, the maximal full rank Lie subgeometries of exceptional type corresponding to an irreducible non-simply laced Dynkin diagram; these correspond to the types F_4 and G_2.

5.1 Case of type G_2

In this low rank case, there are exactly two maximal root subsystems: one of type A_2 and one of type $\mathsf{A}_1 \times \mathsf{A}_1$.

5.1.1 Table of maximal full rank Lie subgeometries

Here is a table of maximal full rank Lie subgeometries of $\mathsf{G}_{2,1}(\mathbb{K},\mathbb{J})$ and $\mathsf{G}_{2,2}(\mathbb{K},\mathbb{J})$, with $\mathbb{J}$ a quadratic Jordan division algebra over $\mathbb{K}$.

	Type	Isomorphism class	Description
1	$\mathsf{A}_1 \times \mathsf{A}_1$	$\mathsf{A}_{1,1}(\mathbb{K}) \perp\!\!\perp$ $\mathsf{A}_{1,1}(\mathbb{J})$	Imaginary line in $\mathsf{G}_{2,1}$ Imaginary line in $\mathsf{G}_{2,2}$
2	A_2	$\mathsf{A}_{2,1}(\mathbb{K}) \cup \mathsf{A}_{2,2}(\mathbb{K})$ $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$	Ideal non-thick subhexagon in $\mathsf{G}_{2,2}$ $\mathsf{A}_{2,\{1,2\}} \leq \mathsf{G}_{2,1}$

5.1.2 Trivia about the Moufang hexagons $\mathsf{G}_{2,1}(\mathbb{K},\mathbb{J})$ and $\mathsf{G}_{2,2}(\mathbb{K},\mathbb{J})$

The Moufang hexagons $\mathsf{G}_{2,1}(\mathbb{K},\mathbb{J})$ and $\mathsf{G}_{2,2}(\mathbb{K},\mathbb{J})$ are dual to each other. Both hexagons Γ are *distance*-3 *regular*, that is, denoting the set of elements of Γ at distance i (in the incidence graph) from a certain element x, be it point or line, by $\Gamma_i(x)$, for each pair $\{x,y\}$ of opposite points, and each pair $\{L,M\}$ of opposite lines with $L,M \in \Gamma_3(x)\cap\Gamma_3(y)$, each point of $\Gamma_3(L)\cap\Gamma_3(M)$ is at distance 3 from each line of $\Gamma_3(L)\cap\Gamma_3(M)$. It follows that $(\Gamma_3(L)\cap\Gamma_3(M)) \cup (\bigcup(\Gamma_3(x)\cap\Gamma_3(y))$ is the point set of a non-thick subhexagon with set of ideal/thick points precisely $\Gamma_3(L)\cap\Gamma_3(M)$, and set of full/thick lines precisely $\Gamma_3(x)\cap\Gamma_3(y)$.

Also, according to [17], the hexagons $\mathsf{G}_{2,2}(\mathbb{K},\mathbb{J})$ *have ideal lines*, that is, with the terminology of [21], they are distance-2 regular. This is equivalent to the following condition: for each point x of the hexagon Γ, and each pair of points y,z opposite x, the sets $\Gamma_2(x)\cap\Gamma_4(y)$ and $\Gamma_2(x)\cap\Gamma_4(z)$ are either equal or intersect in al most one point, see [21]. It follows that every pair of opposite points is contained in a unique ideal subhexagon with two points per line (an ideal non-thick subhexagon). Interpreting the lines as edges of a graph, this subhexagon is the incidence graph of a projective plane Π. The corresponding ideal subhexagon is denoted 2Π and the dual by $(2\Pi)^*$.

5.1.3 Case A_2

Here the maximal full rank Lie subgeometry of $\mathsf{G}_{2,2}(\mathbb{K},\mathbb{J})$ is an ideal non-thick subhexagon, isomorphic to $2\mathsf{PG}(2,\mathbb{K})$. In $\mathsf{G}_{2,1}(\mathbb{K},\mathbb{J})$, it is just the dual, hence a non-thick full subhexagon isomorphic to $(2\mathsf{PG}(2,\mathbb{K}))^*$.

5.1.4 Case $\mathsf{A}_1 \times \mathsf{A}_1$

Here, the maximal full rank Lie subgeometry in both $\mathsf{G}_{2,1}(\mathbb{K},\mathbb{J})$ and $\mathsf{G}_{2,2}(\mathbb{K},\mathbb{J})$ is the non-thick subhexagon related to the distance-3 property described above. The set of thick points admits $\mathsf{PSL}_2(\mathbb{K})$ or $\mathsf{PSL}_2(\mathbb{A})$ and the set of thick lines admits independently $\mathsf{PSL}_2(\mathbb{A})$ or $\mathsf{PSL}_2(\mathbb{K})$, respectively, since central elations in $\mathsf{G}_{2,1}(\mathbb{K},\mathbb{J})$ with centre one of the thick points of the subhexagon stabilizes each thick line of it.

5.2 Case of type F_4

Type F_4 is again special in that there exist non-split buildings of relative type F_4, whereas this is not the case for types $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$.

5.2.1 Table of maximal full rank Lie subgeometries

Here is a table of maximal full rank Lie subgeometries of $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ and $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$, with $\mathbb{A}$ a quadratic alternative divison algebra over $\mathbb{K}$.

	Type	Isomorphism class	Comments
1	$\mathsf{A}_1\times\mathsf{C}_3$	$\mathsf{A}_{1,1}(\mathbb{K}) \perp\!\!\!\perp \mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})\cup$ $\mathsf{A}_{1,1}(\mathbb{K})\times\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$	Imaginary line & its equator in $\mathsf{F}_{4,1}$ Subequator in $\mathsf{F}_{4,1}$
		$\mathsf{A}_{1,1}(\mathbb{K})\times\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})\cup$ $\mathsf{C}_{3,2}(\mathbb{A},\mathbb{K})$	Symp times a line in $\mathsf{F}_{4,4}$ Symp equator in $\mathsf{F}_{4,4}$
2	$\mathsf{A}_2\times\mathsf{A}_2$	$\mathsf{A}_{2,\{1.2\}}(\mathbb{K}) \perp\!\!\!\perp$ $\mathsf{A}_{2,\{1,2\}}(\mathbb{A})$	Non-thick hexagon in $\mathsf{F}_{4,1}$ Non-thick hexagon in $\mathsf{F}_{4,4}$
3	$\mathsf{A}_1\times\mathsf{A}_3$		$\leq \mathsf{B}_4$ (not maximal)
4	B_4	$\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})\cup$ $\mathsf{B}_{4,4}(\mathbb{K},\mathbb{A})$	Extended equator in $\mathsf{F}_{4,4}$ Tropics geometry in $\mathsf{F}_{4,4}$
		$\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A})$	$\mathsf{B}_{4,2}(\mathbb{K},\mathbb{A}) \leq \mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$

5.2.2 Trivia about the metasymplectic spaces $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ and $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$

Set briefly $\Gamma_i := \mathsf{F}_{4,\mathsf{i}}(\mathbb{K},\mathbb{A})$, for $i \in \{1,4\}$. Note that Γ_1 is the long root subgroup geometry, and Γ_4 is often called te *short root subgroup geometry.*

Fact 5.1. *Let x be a point and ξ a symplecton of Γ_i. Then precisely one of the following situations occurs.*

(0) $x \in \xi$;

(1) *the set of points of ξ collinear with x is a line L. Every point y of $\xi \setminus L$ which is collinear with each point of L is symplectic to x and $\xi(x,y)$ contains L. Every other point z of ξ (i.e., every point z of ξ collinear with a unique point z' of L) is special to x and $\mathfrak{c}(x,z) = z' \in L$. We say that x and ξ are close;*

(2) *there is a unique point u of ξ symplectic to x and $\xi \cap \xi(x,u) = \{u\}$. All points v of ξ collinear with u are special to x and $\mathfrak{c}(x,v) \notin \xi$. All points of ξ not collinear with u are opposite x. We say that x and ξ are far.*

Fact 5.2. *The intersection of two symplecta ξ and ζ is either empty, or a point, or a plane and each of these occurs.*

(1) *If $\xi \cap \zeta$ is a point x, then every point in $\xi \setminus x^\perp$ is far from ζ.*
(2) *If $\xi \cap \zeta$ is a plane π, then points $x \in \xi$ and $y \in \zeta$ are special to each other if and only if $x^\perp \cap \pi \neq y^perp \cap \pi$.*

Fact 5.3. *Let x be a point and L a line. Then exactly one of the following occurs.*

(1) *$x \in L$;*
(2) *$x \perp L$;*
(3) *$x \perp p \in L$ for exactly one point p, and $x \perp\!\!\!\perp q$ for all $q \in L \setminus \{p\}$;*
(4) *$x \bowtie p \in L$ for exactly one point p, and x is opposite q for all $q \in L \setminus \{q\}$;*
(5) *$x \perp p \in L$ for exactly one point p, and $x \bowtie q$ for all $q \in L \setminus \{p\}$, with evidently $\mathfrak{c}(x,q) = p$;*
(6) *$x \perp\!\!\!\perp p \in L$ for exactly one point p, and $x \bowtie q$ for all $q \in L \setminus \{p\}$, with $\mathfrak{c}(x,q) = a \perp L$, for a unique point a (independent of q);*
(7) *$x \bowtie p$, for every $p \in L$. In this case there exists a unique line M such that $p \mapsto \mathfrak{c}(x,p)$ is a bijection from L to M.*

5.2.3 Case B_4

We now define the equator and extended equator geometries, see also [10], Proposition 6.26, and [7], Section 4.2.

Definition 5.4 (Equator Geometry)**.** Let p, q be two opposite points of Γ_i. Let $\mathscr{S}_p$ denote the family of symplecta containing p. Then, by Fact 5.1, each member of $\mathscr{S}_p$ contains a unique point which is symplectic to q. The set of all such points is called the *equator geometry of the pair* $\{p, q\}$. It is usually denoted by $E(p,q)$. Using Fact 5.1(2), it is easy to see that $E(p,q) = p^{\perp\!\!\!\perp} \cap q^{\perp\!\!\!\perp}$ and hence this definition is symmetric in p, q.

The following was proved in Proposition 6.26 of [10] for $\Gamma_4 = \mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$, but the proof remains valid for $\Gamma_4 = \mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$, with $\mathbb{A}$ any quadratic alternative division algebra. The reason is the following. In a polar space $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ (and we now use the symbol $\perp$ for collinearity in this polar space), taking two opposite lines L, M yields a set $L^\perp \cap M^\perp$ which coincides with $\{x,y\}^{\perp\perp}$, for each pair $\{x,y\}$ in $L^\perp \cap M^\perp$. We call such a set a *hyperbolic line* and denote it by $h(x,y)$.

Proposition 5.5. *Let p, q be two opposite points of Γ_4. Then, for any symplectic pair $\{u,v\}$ of points of $E(p,q)$, the hyperbolic line $h(u,v)$ is contained in $E(p,q)$. The geometry of points and hyperbolic lines of $E(p,q)$ is the point-line geometry of a polar space, which we also denote by $E(p,q)$, isomorphic to any point residual of Γ. A natural isomorphism from $E(p,q)$ to $\mathrm{Res}_{\Gamma_4}(p)$ is induced by the map $\varphi_{p,q}$ that sends a point $x \in E(p,q)$ to the symplecton $\xi(x,p)$.* $\square$

Note that, by Lemma 4.2.4 of [7], if p, q are opposite points of Γ_i, and $x, y \in E(p,q)$, then either $x = y$, or $\{x, y\}$ is a symplectic pair, or x is opposite y.

We now define the extended equator geometry for opposite points p, q in Γ_4. It provides a construction of a full rank subgeometry of Dynkin cotype 4.

Construction 5.6 (Dynkin cotype 4 for F_4)**.** Let p, q be two opposite points of Γ_4. Then define the point set

$$\widehat{E}(p,q) = \bigcup\{E(x,y) : x, y \in E(p,q), x \text{ opposite } y\}.$$

The set $\widehat{E}(p,q)$, endowed with all the hyperbolic lines in it, is called the *extended equator geometry* for p, q. Note that p, q and $E(p,q)$ are contained in $\widehat{E}(p,q)$.

The following proposition, proved in [15], establishes a maximal full rank Lie subgeometry of Dynkin cotype 4 and of type $\mathsf{B}_{4,1}$ inside $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$.

Proposition 5.7. *The extended equator geometry $\widetilde{E}(p,q)$, endowed with the hyperbolic lines contained in it, is a polar space isomorphic to $\mathsf{B}_{4,1}(\mathbb{K}, \mathbb{A})$.*

The proof of the following proposition is more or less similar to the one for $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$ in [7]. A complete proof is contained in [12].

Proposition 5.8. (1) *If a point is collinear to at least two points of $\widetilde{E}$, then it is collinear to precisely all points of a hyperbolic solid.*

(2) *For every hyperbolic solid Σ in $\widetilde{E}$, there exists a unique point $\beta(\Sigma)$ collinear to all points of Σ.*

(3) *For every hyperbolic plane π in $\widetilde{E}$, the set $\{\beta(\Sigma) \mid \pi \subseteq \Sigma$ is a hyperbolic solid in $\widetilde{E}\}$ is a line of Γ_4.*

(4) *Two hyperbolic solids Σ_1 and Σ_2 of $\widetilde{E}$ share a unique point x if and only if $\beta(\Sigma_1)$ and $\beta(\Sigma_2)$ form a special pair of points of Γ_4, and in this case $\mathfrak{c}(\beta(\Sigma_1), \beta(\Sigma_2)) = x$.*

(5) *Two hyperbolic solids Σ_1 and Σ_2 of $\widetilde{E}$ are disjoint if and only if $\beta(\Sigma_1)$ and $\beta(\Sigma_2)$ are opposite points of Γ_4.*

(6) *The set $\widehat{T}(p,q)$ of points $\beta(\Sigma)$, with Σ ranging through all hyperbolic solids of $\widehat{E}$, with all induced lines, is isomorphic to the dual polar space $\mathsf{B}_{4,4}(\mathbb{K}, \mathbb{A})$ corresponding to the polar space $\mathsf{B}_{4,1}(\mathbb{K}, \mathbb{A})$.*

The geometry induced on $\widehat{T}(p,q)$ is called the *tropics geometry*. Hence, for Dynkin type 4, we have a pair of coupled Lie incidence geometries $\mathsf{B}_{4,1}(\mathbb{K}, \mathbb{A})$ and $\mathsf{B}_{4,4}(\mathbb{K}, \mathbb{A})$ fully embedded in $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$.

5.2.4 Case $\mathsf{A}_2 \times \mathsf{A}_2$

The long root geometry $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{K})$ is fully embedded in the geometry $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$. Hence the latter contains a fully embedded $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$. Call it Γ. In this subsection we construct a full subgeometry Γ' of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ isomorphic to $\mathsf{A}_{2,\{1,2\}}(\mathbb{A})$, pointwise fixed under the little projective group of Γ.

Construction 5.9 (Dynkin cotype 2 for F_4)**.** The hexagon Γ has a natural partition $\mathscr{L}_1 \cup \mathscr{L}_2$ of its line set such that two distinct lines belong to the same partition class if and only if they contain collinear points. Each of $\mathscr{L}_1$ and $\mathscr{L}_2$ is the point set of a projective plane $\mathsf{PG}(2,\mathbb{K})$ the incidence graph is given by the graph with vertices the lines of $\mathscr{L}_1 \cup \mathscr{L}_2$, adjacent when intersecting in a unique point.

We construct Γ' in $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ as a geometry with point set a set of planes and line set a set of symplecta. To that aim, we let $p_0 \perp p_1 \perp \cdots \perp p_5 \perp p_0$ be an ordinary hexagon in Γ. Also, let π_{01} be an arbitrary plane containing the line $\langle p_0, p_1\rangle$. We may also assume, with loss of generality, that $\langle p_0, p_1\rangle \in \mathscr{L}_1$ and that $\mathscr{L}_1$ is the point set of $\mathsf{PG}(2,\mathbb{K})$.

Since no point collinear to p_1 is symplectic to p_4, which is opposite p_1, there is a unique line $L_0 \ni p_0$ in π_{01} all points of which are special to p_4. Likewise, there is a unique line $L_1 \ni p_1$ in π_{01} all points of which are special to p_3. Set $q_{01} = L_0 \cap L_1$. Since p_5 is special to p_1, the centre q_{45} of the special pair $\{p_4, q_{01}\}$ differs from p_5. By Fact 5.3, the points p_4, p_5 and q_{45} span a plane π_{45}. Since $p_4 \perp q_{45} \perp q_{01} \perp p_1$, we have $p_1 \bowtie q_{45}$, and so every point of the line $L_5 := \langle p_5, q_{45}\rangle$ is special to p_1.

Let q_{23} be the centre of the special pair $\{q_{01}, p_3\}$. If q_{23} were equal to q_{23}, then $p_0 \perp q_{01} \perp q_{23} = q_{45} \perp p_4$, with $p_0 \bowtie q_{23}$ and $q_{01} \bowtie p_4$, implies by Fact 4.1 that p_0 would be opposite p_4, a contradiction. Hence Fact 5.3 yields a plane α containing q_{01}, q_{23} and q_{45}. Also, Since $\{p_3, q_{01}\}$ is a special pair with centre $q_{23} \neq p_2$, and $\{p_3, p_1\}$ is special with centre p_2, the points p_2, p_3 and q_{23} span a plane π_{23}.

Since the centres of the special pairs $\{p_3, x\}$, with $x \in L_1$, all on the line $L_2 := \langle p_2, q_{23}\rangle$, and the lines L_1 and L_2 are obviously opposite in the symp $\xi(p_1, q_{23})$, it follows that π_{23} is the unique plane through $\langle p_2, p_3\rangle$ containing a point collinear to some point of π_{01}. Likewise, π_{45} is the unique plane through $\langle p_4, p_5\rangle$ containing a point collinear to some point of π_{01}. We now also see that π_{23} is the unique plane through $\langle p_2, p_3\rangle$ containing a point collinear to some point of π_{45} and vice versa.

Now let $p'_0 \in \langle p_0, p_5\rangle \setminus \{p_0, p_5\}$ be arbitrary. There is a unique path $p'_0 \perp p'_1 \perp p'_2 \in \langle p_2, p_3\rangle$. Considering the hexagon $p'_0 \perp p'_1 \perp p'_2 \perp p_3 \perp p_4 \perp p_5 \perp p'_0$, the foregoing paragraph implies that there exists a unique plane π'_{01} through $\langle p'_0, p'_1\rangle$ containing a point q'_{01} collinear to both q_{45} and q_{23}. Considering the hexagon $p'_0 \perp p_0 \perp p_1 \perp p_2 \perp p'_2 \perp p'_1 \perp p'_0$, we likewise conclude that there exists a unique plane π''_{01} through $\langle p'_0, p'_1\rangle$ containing a point q''_{01} collinear to both q_{01} and q_{23}. By the foregoing and the fact that q_{23} appears twice in our conclusions, we see that $q'_{01} = q''_{01}$ and $\pi'_{01} = \pi''_{01}$. Moreover, since the maximal singular subspaces of $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ are planes, we deduce $q'_{01} \in \alpha$.

Obviously, the point q'_{01} is the unique point of α collinear to p'_0 (if p'_0 were collinear to a line of α, then that line would intersect $\langle q_{01}, q_{23}\rangle$ in a point y distinct from q_{01}—because q_{01} is not collinear to p_5—and then p'_0 would be at distance 2 from the unique point of $\langle p_1, p_2\rangle \setminus \{p_1\}$ collinear to y, a contradiction to the fact that p_1 is the unique point of $\langle p_1, p_2\rangle$ at distance ≤ 2 from p'_0). Hence $q'_{01} \in \langle q_{01}, q_{45}\rangle$ (this happens inside the symplecton $\xi(q_{01}, p_5)$, which also contains p_0 and q_{45}).

Similarly, every line L of Γ intersecting $\langle p'_1, p'_2\rangle$ is contained in a unique plane π containing a point q of α, and that point is contained in $\langle q_{23}, q'_{01}\rangle$. Since this exhausts all lines $L \in \mathscr{L}_1$, it follows that the mapping $L \mapsto q$ is an isomorphism from $\mathsf{PG}(2,\mathbb{K})$ to α. Varying π_{01}, we obtain a set Π_1 of planes α containing, for each $L \in \mathscr{L}_1$, a point collinear to L. Similarly, there exists a set Π_2 of planes containing, for each $L \in \mathscr{L}_2$, a point collinear to L, and for each plane π through any member of $\mathscr{L}_2$, there exists $\beta \in \Pi_2$ intersecting π. For any plane π though a member of $\mathscr{L}_i$, we denote by $\Lambda_i(\pi)$ the unique member of Π_i intersecting π in a point.

Now let π_{01} be as above, and let π_{12} be a plane containing $\langle p_1, p_2\rangle$. Let q_{12} be the unique point of π_{12} special to both p_4 and p_5. Then $q_{12} \in \Lambda(\pi_{12})$. Suppose that π_{01} and π_{12} are not locally opposite. Then there is some plane α_1 through p_1 intersecting both π_{01} and π_{12} in respective lines M_1 and L_1'. We claim that $q_{01} \in L_1' = L_1$ and $q_{12} \in M_1$. Indeed, set $z = L_0 \cap L_1'$. Then z is collinear to some point on $\langle p_2, q_{12}\rangle$, and hence z is close to $\xi(q_{12}, p_3)$. It follows from Fact 4.1 that z is not opposite p_3, but the only point of L_0 not opposite p_3 is q_{01}. Hence $z = q_{01}$ and $L_1 = L_1'$. Similarly, $q_{12} \in M_1$. The claim is proved. Hence $q_{01} \perp q_{12}$.

Next we claim, still assuming that π_{01} and π_{12} are not locally opposite, that π_{12} and π_{23} are not locally opposite. Indeed, we observe that $q_{12} \bowtie q_{45}$ implies that q_{12} is opposite p_4 (since $p_4 \bowtie q_{01}$ and $p_4 \perp q_{45} \perp q_{01} \perp q_{12}$ and use Fact 4.1), a contradiction as p_4 is collinear to some point of $\Lambda_2(\pi_{12})$. Similarly q_{12} is not special to q_{23}. Now Fact 5.3 implies that $q_{12} \perp u \in \langle q_{23}, q_{45}\rangle$. If $u \neq q_{23}$, then we may assume without loss of generality that $q_{01} \perp Q_{45}$, leading to $p_2 \perp q_{23} \perp q_{45} \perp q_{12} \perp p_2$, contradicting $p_2 \bowtie q_{45}$. Hence $q_{12} \perp q_{23}$ and the claim is proved. Going on like this, it is clear that no plane π_1 through some member K_1 of $\mathscr{L}_1$ with $\Lambda(\pi_1) = \Lambda(\pi_{01})$ is locally opposite the plane π_2 through some member K_2 of $\mathscr{L}_2$ with $\Lambda(\pi_2) = \Lambda(\pi_{12})$ and $|K_1 \cap K_2| = 1$. It then also follows from our arguments that every point of $\Lambda(\pi_{01})$ is collinear to a unique line of $\Lambda(\pi_{12})$, implying that these two planes are contained in a unique symp $\xi(\pi_{01}, \pi_{12})$, in which they are opposite, since they are clearly disjoint.

We now claim that the map $\pi_{12} \mapsto \xi(\pi_{01}, \pi_{12})$ is a bijection from the set of planes through $\langle p_1, p_2\rangle$ not locally opposite π_{01} to the set of symps containing $\alpha := \Lambda(\pi_{01})$. This mapping is clearly injective, as otherwise the symp which is the image of at least two planes would contain every member of $\mathscr{L}_2$, a contradiction. We now show that it is surjective. So let ξ be any symp through α. Then $\xi \cap \xi(p_2, q_{01})$ is a plane β, by Fact 5.2 as q_{01} and q_{23} already belong to that intersection. Set $q_{12}' = p_1 \perp \cap p_2^{\perp} \cap \beta$. Then $\pi_{12}' = \langle p_1, p_2, q_{12}'\rangle$ is a plane which is not locally opposite π_{01}, as $\pi_{12}' \ni q_{12}' \perp q_{01} \in \pi_{01}$. Hence $\alpha' := \Lambda(\pi_{12}')$ is contained in a symp ζ together with α. It is easy to see that $q_{12}' \in \Lambda(\pi_{12})$, using the fact that it is collinear to both q_{01} and q_{23}. So $\zeta = \xi(q_{12}', q_{45})$ must coincide with $\Lambda(\pi_{12}')$ and the claim is proved.

Finally we claim that the graph with vertices the planes that contain either $\langle p_0, p_1\rangle$ or $\langle p_1, p_2\rangle$, adjacent when locally not opposite, is the incidence graph of a projective plane isomorphic to $\mathsf{PG}(2, \mathbb{A})$. Indeed, that projective plane can be thought of as having point set the set of planes of $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ containing $\langle p_0, p_1\rangle$, and lines are given by sets of such planes contained in a common symp through $\langle p_0, p_1\rangle$. It is now easy to see that the planes through $\langle p_0, p_1\rangle$ of a symp ξ are all locally not opposite the unique plane γ containing p_2 and intersecting ξ in a line (existing by Fact 5.1). In the residue of p_1, one also sees that no plane through $\langle p_0, p_1\rangle$ outside ξ is locally not opposite γ. This proves out last claim.

Now the set Σ of symps containing a member of Π_1 and a member of Π_2 clearly corresponds to a full embedding Γ' of the double $2\mathsf{PG}(2, \mathbb{A})$ in $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ where points of $2\mathsf{PG}(2, \mathbb{A})$ at mutual distance 2 are special in $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ (since the plane of Π_1 in a symp belonging to Σ is disjoint from the plane of Π_2 in that symp).

Clearly, the central elation of $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ with centre p_0 stabilizes all members of $\Pi_1 \cup \Pi_2$. Also, clearly the little projective group of Γ' acts on Γ' in the standard way, while fixing Γ pointwise.

6 Application to non-thick spherical buildings

It is well-known that every weak spherical building, say of type X_n gives rise to a unique thick spherical building of a different type Y_m. Scharlau [18] shows that the types Y_m given X_n are determined by the types of the Coxeter groups generated by reflections in the Coxeter group of type X_n. In particular all types of maximal full rank Lie incidence subgeometries qualify. Our constructions in the previous section provide very concrete examples of weak buildings of exceptional type, given as geometries rather than simplicial complexes or chamber systems, and also more concrete than in Rees' paper [16]. The recipe to do this is very simple: one considers the components of the geometries and replaces each line between components by the thin line consisting of the two points that were joined by the line. If types allow, one can take any geometry of the given type, and not only the one inside the thick building (for instance for type A_2 one can take any projective plane).

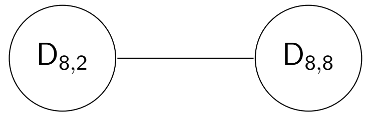
The examples related to G_2 are just multiples of generalized polygons, as in [21, §1.6]. We now explicitly consider the four irreducible types for the other exceptional cases. These will be given by a diagram showing their decomposition. The rules to read such a diagram are essentially the same as [22, §7], but updated to the thick case. There is an arbitrary underlying building Δ of type X_n. Each balloon represents a Lie incidence geometry related to Δ, and for balloons joined by an edge, a point of one balloon forms a thin line with a point of the other balloon if the corresponding objects of Δ are incident, or, equivalently, their union forms a simplex or flag.

For type E_7, we have the irreducible type A_7. It can be given as $\mathsf{E}_{7,7}$ geometry, or as $\mathsf{E}_{7,1}$ geometry.

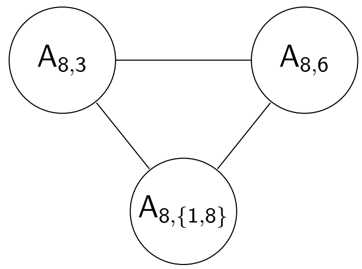
As $\mathsf{E}_{7,7}$ geometry:

And as $\mathsf{E}_{7,1}$ geometry:

For type E_8, we have the irreducible types D_8 and A_8. First type D_8:



And now type A_8:



For type F_4, we have the irreducible type B_4, and we can consider any polar space of rank 4. We represent its point set by $\mathsf{B}^*_{4,1}$ and the corresponding dual polar space by $\mathsf{B}_{4,4}$. We have the following diagram:

All other, reducible, cases can be derived from the previous tables. One particular case might be more involved, and that is the case of $\mathsf{A}_2 \times \mathsf{A}_2$ in F_4, because in this case the subgeometry lies simultaneously in $\mathsf{F}_{4,1}$ and $\mathsf{F}_{4,4}$. We now describe in an explicit way a weak building of type F_4 with underling thick building the cartesian product of two arbitrary projective planes π and π', and we give it in terms of a non-thick long root geometry $\Delta = (X, \mathscr{L})$ of type $\mathsf{F}_{4,1}$.

Let $\Omega = (Z, \mathscr{M})$ be the thick-lined generalized hexagon of which the point set Z is the set of point-line pairs of π, and $\mathscr{M}$ can be identified with the union of the point set $\mathscr{P}(\pi)$ of π and its line set $\mathscr{L}(\pi)$. For each point x of π', let π_x be a copy of π with isomorphism $\beta_x : \pi \to \pi_x$, and likewise, for each line L of π', let π_L be a copy of the dual π^* of π with corresponding isomorphism $\beta_L : \pi^* \to \pi_L$. Then the point set X of Δ is the disjoint union of Z and all π_x and π_L, for x and L ranging through the point and line set of π', respectively.

The lines are all members of $\mathscr{M}$, all lines of each plane π_x and π_L, x and L as above, and all the lines $\{x, y\}$ of size 2, where

(i) $x \in Z$ and $y = \beta_z(M)$, for arbitrary point z of π', with $x \in M \in \mathscr{P}(\pi)$, or an arbitrary line z of π', with $x \in M \in \mathscr{L}((\pi)$; or

(ii) $x \in \pi_z$ for some point z of π' and $y \in \beta_L(\beta^{-1}(x))$, for some line L of π' containing z.

Interchanging the roles of π and π' in the above construction results in going to the corresponding geometry of type $\mathsf{F}_{4,4}$.

References

[1] P. Abramenko & K. Brown, *Buildings: Theory and Applications,* Graduate Texts in Mathematics **248**, Springer, 2008.

[2] R. J. Blok & A. E. Brouwer, Spanning point-line geometries in buildings of spherical type, *J. Geom.* **62** (1998), 26–35.

[3] A. Borel & J. de Siebenthal, Les sous-groupes fermés de rang maximum des groupes de Lie clos, *Comm. Math. Helvetici* **23** (1949), 200–221.

[4] N. Bourbaki, *Lie groups and Lie algebras*, Chapters 4–6, *Elements of Mathematics*, Springer-Verlag, 2002.

[5] F. Buekenhout and A. M. Cohen, *Diagram Geometry Related to Classical Groups and Buildings*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge **57**, Springer, 2013.

[6] B. N. Cooperstein & E. E. Shult, Frames and bases of Lie incidence geometries, *J. Geom.* **60**, (1997), 17–46.

[7] A. De Schepper, N. S. N. Sastry & H. Van Maldeghem, Split buildings of type F_4 in buildings of type E_6, *Abh. Math. Sem. Univ. Hamburg* **88** (2018), 97–160.

[8] A. De Schepper & H. Van Maldeghem, On inclusions of exceptional long root geometries of type E, *Innov. Incidence Geom.* **20** (2023), 247–293.

[9] P. Jansen & H. Van Maldeghem, Imaginary geometries, to appea in *Mod. Math. Meth.*

[10] A. Kasikova & H. Van Maldeghem, Vertex opposition in spherical buildings, *Des. Codes Cryptogr.* **68** (2013), 285–318.

[11] P. Kleidman & M. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, Cambridge University Press, *London. Math. Soc. Lect. Notes Ser.* **129**, Cambridge1990.

[12] L. Lambrecht & H. Van Maldeghem, Automorphisms and opposition in spherical buildings of exceptional type, III. Metasymplectic spaces, submitted.

[13] J. Meulewaeter and H. Van Maldeghem, Convex subspaces of Lie incidence geometries, *Combinatorial Theory* **2** (2022), # 13, 28pp.

[14] T. Oshima, A classification of subsystems of a root system. https://arxiv.org/pdf/math/0611904.pdf

[15] S. Petit & H. Van Maldeghem, Generalized hexagons embedded in metasymplectic spaces, *J. Korean Math. Soc.* **60** (2023), 907–929.

[16] S. Rees, Weak buildings of spherical type, *Geom. Dedicata* **27** (1988), 15–47.

[17] M. A. Ronan, A geometric characterization of Moufang hexagons, *Invent. Math.* **57** (1980), 227–262.

[18] R. Scharlau, A structure theorem for weak buildings of spherical type, *Geom. Dedicata* **24** (1987), 77–84.

[19] F.G. Timmesfeld, *Abstract root subgroups and simple groups of Lie type*, Monographs in Mathematics **95**, Birkhäuser, 2001.

[20] J. Tits & R. Weiss, *Moufang Polygons*, Springer Monographs in Mathematics, Springer, 2002

[21] H. Van Maldeghem, *Generalized Polygons*, Modern Birkhäuser Classics, Birkhäuser, 1998.

[22] H. Van Maldeghem & M. Victoor, Combinatorial and geometric constructions of spherical buildings, **in** Surveys in Combinatorics 2019, Cambridge University Press (ed. A. Lo et al.), *London Math. Soc. Lect. Notes Ser.* **456** (2019), 237–265.