SUBGEOMETRIES ISOMORPHIC TO RESIDUES IN EXCEPTIONAL LIE INCIDENCE GEOMETRIES

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ABSTRACT. We show that a geometry isomorphic to a point residual in a Lie incidence geometry of exceptional type, either a strong parapolar space or the long root subgroup geometry, is a trace, that is, coincides with the set of points collinear to a given point p and not opposite to a given object opposite p. We also show uniqueness of the line residue in the long root subgroup geometry of type E_8.

1. INTRODUCTION

Subgeometries of a given geometric structure play a similar role in incidence geometry as subgroups of groups in group theory. A good knowledge of all subgeometries of a geometry Δ helps to understand Δ. It can also be used to characterise certain automorphisms of Δ by its fixed point structure. In this case the ideal situation is that a given subgeometry is unique up to a projectivity. The investigation and classification of all automorphisms of the exceptional spherical buildings that do not map any chamber to an opposite, prompted the authors of [8] to show that the long root subgroup geometries of types $\mathsf{E}_{7,1}$ and $\mathsf{E}_{6,2}$ admit projectively unique embeddings into the long root subgroup geometry of type $\mathsf{E}_{8,8}$. In the present paper we direct our attention to residual geometries, that is, the geometries isomorphic to a point (or line) residue in the exceptional Lie incidence geometries of type E. Our main aim is to investigate how the minuscule geometries $\mathsf{E}_{6,1}$ and $\mathsf{E}_{7,7}$ are sitting in the long root subgroup geometry $\mathsf{E}_{8,8}(\mathbb{K})$, and we show that this happens in a projectively unique way. We complete the job for the exceptional Lie incidence geometries of type E by showing uniqueness of full embeddings of geometries of types $\mathsf{D}_{5,5}$, $\mathsf{A}_{5,3}$, $\mathsf{D}_{6,6}$ and $\mathsf{E}_{6,1}$ into Lie incidence geometries of types $\mathsf{E}_{6,1}$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$ and $\mathsf{E}_{7,7}$, respectively. The analogous results for type F_4 uses different techniques and shall be done elsewhere. This is due to the fact that all buildings of type E are split, whereas there exists a variety of buildings of type F_4, ranging from split, over mixed, to non-split and even non-embeddable.

In order to state our main results, we need to explain what a *trace geometry* is. Given a Lie incidence geometry Δ and a point p thereof, there are objects which are opposite p in the building-theoretic sense. Select one such object τ (in practice, and in this paper, τ is either a point or a symplecton). Then the *trace geometry with respect to* (p,τ) is the subgeometry of Δ induced on the point set $p^\perp \cap \tau^{\not\equiv}$, where $p^\perp$ denotes the set of points collinear to p and $\tau^{\not\equiv}$ the set of points not opposite τ. The *trace geometry with respect to lines* is induced on the set of points collinear to (all points of) a given line L and not opposite any point of a given line M

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opposite L (here we assume that pairs of points can be opposite). Assuming familiarity with standard terminology about embedded geometries and Lie incidence geometries (see Section 2), we can now summarise all our results as follows.

Main Result. *Let $\mathbb{K}$ be a field and let $\mathsf{D}_{5,5}(\mathbb{K})$, $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$ be fully embedded in $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,7}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$ and $\mathsf{E}_{8.8}(\mathbb{K})$, respectively. Then the former is a trace geometry in the latter. If $\mathsf{E}_{6,1}(\mathbb{K})$ is fully embedded in $\mathsf{E}_{8,8}(\mathbb{K})$, then it is a para—that is, a proper convex subspace properly containing a symplecton—of an equator geometry, or, equivalently, a trace geometry with respect to two opposite lines.*

One would hope that the techniques developed in [8] to prove uniqueness of embedded long root subgroup geometries in the exceptional type case is applicable in the situation of the present paper. However, there is an essential difference. In [8], one must find two points p and q in the ambient geometry Δ such that the embedded geometry Ω coincides with the *equator geometry* $E(p,q)$ (see Section 2.3.2). The points p and q are not too far away from Ω and can be recognised with the point residuals. In the present situation, however, we must find a point p collinear to all points of Ω, which can also be done with the point-residuals, but, in the generic situation, we must also find a point which is special to all points of Ω. This can no longer be accomplished by considering residues. The technique that works here is to prove that there is a *companion* embedded geometry Ω^*, which is isomorphic either to an equator geometry—and then we apply the results of [8]—or to Ω—in which case we find a point collinear to all points of Ω^* and that is precisely the wanted second point.

Note that along the way we also have to deal with similar embedding questions for some classical geometries.

The paper is organised as follows. In Section 2 we recall some definitions and list some properties of the exceptional Lie incidence geometries of type E. In Section 3, we show our Main Result for $\mathsf{D}_{5,5}(\mathbb{K})$ embedded in $\mathsf{E}_{6,1}(\mathbb{K})$. The strategy of the proof is to study the ways in which the skeleton graph of an apartment $\mathsf{D}_{5,5}(1)$ can be embedded in $\mathsf{E}_{6,1}(\mathbb{K})$. This avoids to have to first prove uniqueness of the full embedding of $\mathsf{A}_{4,2}(\mathbb{K})$ in $\mathsf{D}_{5,5}(\mathbb{K})$, which would be another valuable strategy, call it the *point residual strategy*. In Sections 4 to 7, we prove the rest of the first part of our Main Result using the point residual strategy. In Section 8 we prove an interesting consequence and in the final section we prove the second part of the Main Result.

2. Preliminaries, definitions and notation

2.1. **Point-line geometries.** For the purposes of this paper, a *point-line geometry*, which we shall usually denote by $\Delta = (X(\Delta), \mathscr{L}(\Delta))$, is a pair consisting of a *point set* $X(\Delta)$ and a set $\mathscr{L}(\Delta)$ of *lines*, which are subsets of $X(\Delta)$. Two points x, y in such a structure are called *collinear*, in symbols $x \perp y$, if they are contained in some line. We will exclusively be dealing with *partial linear spaces*, which are point-line geometries with the property that each pair of collinear points is contained in exactly one line. The set of points collinear to a given point x is denoted by $x^{\perp}$. A *subspace* Y is a set of points $Y \subseteq X$ with the property that, if a line has two points in common with Y, then it is completely contained in Y. A *geometric hyperplane* of Δ is a subspace which intersects each line. It is *proper* if it does not coincide with $X(\Delta)$.

The *collinearity graph* or *point graph* of Γ has as set of vertices the points of Γ, adjacent when collinear. The *distance* between two points is the distance in the collinearity graph. The *diameter* of Δ is the diameter of the collinearity graph. We say that Δ is *connected* if the collinearity graph is.

A *full subgeometry* $\Gamma' = (X', \mathscr{L}')$ of Γ is a geometry with $X' \subseteq X$ and $\mathscr{L}' \subseteq \mathscr{L}$. This implies that all points of Γ on a line of Γ' are points of Γ' and explains the adjective 'full'. Full subgeometries need not be subspaces.

Now a *polar space* is a thick point-line geometry in which the perp of every point is a proper geometric hyperplane; this definition is justified by [2]. This forces all singular subspaces to be projective spaces. In our case the polar spaces will have finite rank, that is, there is a natural number $r \geq 2$ such that all singular subspaces (which are projective spaces) have dimension $\leq r-1$, and there exist singular subspaces of dimension $r-1$. A prominent notion in polar geometry is *opposition*. Two singular subspaces U, W are *opposite* if no point of $U \cup W$ is collinear to all points of $U \cup W$. Opposite subspaces automatically have the same dimension. Opposite points are just non-collinear ones. The singular subspaces of dimension $r-1$ are called *generators*. It is easy to see that polar spaces satisfy the so-called *one-or-all axiom*: each point is collinear to either exactly one point or to all points of a given line.

A *convex* subspace of a point-line geometry is a subspace with the property that every shortest path in the collinearity graph between two points of the subspace is contained in the subspace. A convex subspace isomorphic to a polar space is a *symplecton*, or *symp* for short.

Now a parapolar space is a connected point-line geometry which is not a polar space, such that two points at distance 2 either have a unique common neighbour in the collinearity graph—and then we call these two points *special*—or are contained in a symplecton—the two points are called *symplectic*—and every line is contained in a symp. A parapolar space without special pairs is called *strong*. A symplecton through two noncollinear points x, y is unique and denoted by $\xi(x,y)$. The set of symps of a parapolar space Δ is denoted by $\Xi(\Delta)$. Parapolar spaces found their birth in Section 3 of [4].

The parapolar spaces we will encounter all have the rather peculiar property that all symps have the same rank, which is then called the (uniform) *symplectic rank* of the parapolar space. In contrast, the maximal singular subspaces (which will be projective spaces) will not all have the same dimension. The *singular ranks* of a parapolar space with only projective spaces as singular subspaces (which is automatic if the symplectic rank is at least 3) are the dimensions of the maximal singular subspaces. In general, a singular subspace which is a projective space of (projective) dimension d will be called a *(singular) d-space* for short.

Now let $\Delta = (X(\Delta), \mathscr{L}(\Delta))$ be a parapolar space all of whose symps have rank at least 3. Let $x \in X$. Then we define the geometry $\Delta_x = (X(\Delta_x), \mathscr{L}(\Delta_x))$ as the geometry with point set the set of lines through x, and the lines are the planar line pencils with vertex x, that is, the set of lines through x in a plane through x, and call it the *residue at x*, or the *point residual at x*.

In the present paper we will exclusively deal with Lie incidence geometries, which are projective, polar and parapolar spaces arising from spherical buildings. Assuming the basics of Tits' theory of spherical buildings, we introduce these now briefly.

2.2. **Lie incidence geometries.** Let Δ be an irreducible thick spherical building. Let n be its rank, let I be its type set and let $i \in I$. Then we define a point-line geometry Δ as follows. The point set $X(\Delta)$ is just the set of vertices of Δ of type i; a typical line of Δ is the set of vertices of type i completing a given panel of cotype i to a chamber. The geometry Δ is called a *Lie incidence geometry*. For instance, if Δ has type A_n, $n \geq 2$, and $i = 1$ (we use Bourbaki labelling of the vertices of the Coxeter or Dynkin diagrams), then Δ is the point-line geometry of a projective space of dimension n, and if $n \geq 3$, it is defined over some skew field $\mathbb{K}$, in which case we denote it by $\mathsf{PG}(n, \mathbb{K})$. If X_n is the Coxeter type of Δ and Δ is defined using $i \in I$ as above, then we say that Δ has *type* $\mathsf{X}_{n,i}$. Another example: Geometries of type $\mathsf{B}_{n,1}$ and $\mathsf{D}_{n,1}$ are polar spaces. Geometries of type $\mathsf{D}_{n,n}$ are more specifically called *half spin geometries*

Buildings of type $\mathsf{A},\mathsf{D},\mathsf{E}$ are uniquely defined by their underlying field $\mathbb{K}$ (or skew field in the case of A), provided the rank is at least 3. We denote the corresponding building of type X_n by $\mathsf{X}_n(\mathbb{K})$, and the corresponding Lie incidence geometries of type $\mathsf{X}_{n,i}$ by $\mathsf{X}_{n,i}(\mathbb{K})$.

In he present paper we are most interested in parapolar spaces of exceptional type. More exactly, the Lie incidence geometries $\mathsf{E}_{6,1}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$, which are sometimes called the *minuscule geometries* of types E_6 and E_7, respectively, and the Lie incidence geometries $\mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$ and $\mathsf{E}_{8,8}(\mathbb{K})$, which are also called the *long root subgroup geometries* ot type E. We gather the most important properties of these in Section 2.3. Prominent subgeometries that we will also need are $\mathsf{A}_{4,2}(\mathbb{K})$, $\mathsf{A}_{5,2}(\mathbb{K})$, $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{D}_{5,5}(\mathbb{K})$ and $\mathsf{D}_{6,6}(\mathbb{K})$. The first three are well known Grassmannians of projective spaces. The latter two are so-called *half spin geometries* arising from (nondegenerate) hyperbolic quadrics in $\mathsf{PG}(9,\mathbb{K})$ and $\mathsf{PG}(11,\mathbb{K})$, respectively, by taking one system of generators as points, and a typical line is then the set of generators of that system though a given singular subspace of dimension 2 and 3, respectively. The properties of these Lie incidence geometries that we will need are easily deduced from the hyperbolic quadric. We explicitly note that $\mathsf{D}_{n,n}(\mathbb{K})$, $n \geq 5$, has singular ranks 3 and $n-1$. Nonmaximal 3-spaces will be called $3'$-spaces.

Objects in a Lie incidence geometry Δ will be called *opposite* if they are opposite in the building-theoretic sense. They will be called *locally opposite* (with respect to a point p) if they are opposite in Δ_x (of course this requires that the two objects correspond to flags containing or incident with x). Opposite objects a and b are denoted $x \equiv b$; the symbol $a^{\equiv}$ means the set of objects opposite a and $a^{\not\equiv}$ is the set of objects of the type of opposite objects, not opposite a.

Lie incidence geometries admit natural full embeddings in projective spaces. The natural embeddings of $\mathsf{D}_{5,5}(\mathbb{K})$, $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$, $\mathsf{E}_{7,7}(\mathbb{K})$, $\mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$ and $\mathsf{E}_{8.8}(\mathbb{K})$ occur in $\mathsf{PG}(15,\mathbb{K})$, $\mathsf{PG}(19,\mathbb{K})$, $\mathsf{PG}(26,\mathbb{K})$, $\mathsf{PG}(31,\mathbb{K})$, $\mathsf{PG}(55,\mathbb{K})$, $\mathsf{PG}(77,\mathbb{K})$, $\mathsf{PG}(127,\mathbb{K})$ and $\mathsf{PG}(247,\mathbb{K})$, respectively. Moreover the natural embeddings of $\mathsf{D}_{5,5}(\mathbb{K})$, $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$ are known to be *absolutely universal*, that is, every other full embedding is isomorphic to a projection of the natural one from some subspace onto some complementary subspace.

Finally, we need some terminology concerning embedding. Let Ω and Δ be two polar or parapolar spaces. We say that Ω is *(fully) embedded in* Δ if Ω is isomorphic to a (full) subgeometry of Δ. Usually we identify Ω with the isomorphic subgeometry of Δ, talking about points of Δ that are also points of Ω. If both are polar spaces or strong parapolar spaces, and Ω is embedded in Δ, then we call the embedding *isometric* if the distance between two points of Ω either measured in Ω, or measured in Δ, is the same. If Ω is a polar space or a strong parapolar space of diameter at most 3 and Δ is a nonstrong parapolar space of diameter at most 3, then the embedding is called *isometric* if symplectic points of Ω are also symplectic in Δ, and points at distance 3 in Ω are special in Δ. A graph that is isomorphic to a (non-full) subgeometry of Δ is called *laxly embedded* (to distinguish it from the full embeddings). A isometric lax embedding of a graph of diameter 2 into a (para)polar space is defined in the obvious way. The graphs we will encounter are the *skeletons of apartments*, that is, the vertices are the vertices of certain type, say i, of an apartment of a spherical building of type X_n, adjacent when contained in adjacent chambers. Hinting at the heuristic that apartments are buildings over the field of order 1, we denote such apartment by $\mathsf{X}_{n,i}(1)$. We will only use this for $\mathsf{D}_{5,5}(1)$.

2.3. **Some parapolar spaces of exceptional type.** The below properties are taken from [8], where it is noted that they follow either in a standard way from the the Coxeter diagram, or from a representation of an apartment of the corresponding building as can be found in [11]. Most properties can also be found in Chapters 14 to 18 of [10]. For the long root subgroup geometries we also refer to [3].

2.3.1. *Minuscule geometries of types* E_6 *and* E_7. The Lie incidence geometry $\Delta \cong \mathsf{E}_{6,1}(\mathbb{K})$, for any field $\mathbb{K}$, has the following properties.

(1) The point residuals are isomorphic to $\mathsf{D}_{5,5}(\mathbb{K})$.
(2) The symps of Δ are isomorphic to $\mathsf{D}_{5,1}(\mathbb{K})$, that is, to the polar spaces arising from hyperbolic quadrics in $\mathsf{PG}(9,\mathbb{K})$.
(3) The singular ranks of Δ are 4 and 5. Nonmaximal singular subspaces of dimension 4 are called $4'$-spaces.
(4) The diameter of Δ is equal to 2 and Δ is strong.
(5) A point p not contained in a given symp ξ is collinear either to no points of ξ, or to all points of a $4'$-space contained in ξ. In the first case p is called *far* from ξ, in the latter case *close*. Here, "far" is a synonym for "opposite".
(6) Two symps meet either in a unique point or in a 4-space; in the latter case the symps are called *adjacent*. It follows that a $4'$-space is contained in a unique *symp*.
(7) The geometry with point set the set of symps of Δ, where a typical line is the set of symps containing a given 4-space, is isomorphic to Δ, and is for clarity denoted by $\mathsf{E}_{6,6}(\mathbb{K})$.
(8) A 3-space is contained in a unique 4-space and a unique 5-space (which intersect exactly in the given 3-space).
(9) The set of points not opposite a given symp ξ, that is $\xi^{\not\equiv}$, is a geometric hyperplane of Δ. For a given point p and opposite symp ξ, the set $p^\perp \cap \xi^{\not\equiv}$ is a subspace isomorphic to $\mathsf{D}_{5,5}(\mathbb{K})$, called a *trace geometry*.

The Lie incidence geometry $\Delta \cong \mathsf{E}_{7,7}(\mathbb{K})$, for any field $\mathbb{K}$, has the following properties.

(10) The point residuals are isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$.
(11) The symps of Δ are isomorphic to $\mathsf{D}_{6,1}(\mathbb{K})$, that is, to the polar spaces arising from hyperbolic quadrics in $\mathsf{PG}(11,\mathbb{K})$.
(12) The singular ranks of Δ are 5 and 6. Nonmaximal singular subspaces of dimension 5 are called $5'$-spaces.
(13) The diameter of Δ is equal to 3 and Δ is strong.
(14) A point p not contained in a given symp ξ is collinear either to exactly one point q of ξ, or to all points of a $5'$-space U contained in ξ. In the former case p is opposite each point of ξ which is at distance 2 from q. IN the latter case, if $p' \notin \xi$ is collinear to all points of a $5'$ space U', then $p \equiv p'$ if and only if $U \cap U' = \emptyset$.
(15) Two symps which share a point meet in a line or in a 5-space.
(16) The geometry with point set the set of symps of Δ, where a typical line is the set of symps containing a given 5-space, is isomorphic to $\mathsf{E}_{7,1}(\mathbb{K})$.
(17) A 4-space is contained in a unique 5-space and a unique 6-space (which intersect exactly in the given 4-space).
(18) The set of points not opposite a given point q, that is $q^{\not\equiv}$, is a geometric hyperplane of Δ. For another given point p opposite q, the set $p^\perp \cap q^{\not\equiv}$ is a subspace isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$, called a *trace geometry*.

2.3.2. *Long root subgroup geometries of exceptional type* E. The long root subgroup geometries of type E have a number of common properties. We begin with stating these.

Let Δ be a Lie incidence geometry isomorphic to either $\mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$, or $\mathsf{E}_{8,8}(\mathbb{K})$, for some field $\mathbb{K}$.

(1) The diameter Δ is 3. Points at distance 3 are opposite.
(2) For a sequence $p \perp a \perp b \perp q$ we have $p \equiv q$ if and only if $\{p,b\}$ and $\{q,a\}$ are both special pairs.
(3) A point collinear to at least one point of a given symplecton not containing that points, is collinear to either a line or a d'-space of the symp, where $d+1$ is the rank of the symplecton.
(4) If the points p,q are collinear to exactly a line L,M, respectively, of a symp ξ, then $p \equiv q$ if and only if L and M are opposite in the polar space ξ. Consequently, if $p^\perp \cap \xi = L \in \mathscr{L}(\Delta)$, and $r \in \xi$, then $\{p,r\}$ is a special pair if and only if $r^\perp \cap L$ is a unique point.
(5) The set of points not opposite a given point q, that is $q^{\not\equiv}$, is a geometric hyperplane of Δ. For another given point p opposite q, the set $p^\perp \cap q^{\not\equiv}$ is a subspace isomorphic to the point residual at p and called a *trace geometry*.
(6) If p and q are opposite points, then each symp through p contains a unique point symplectic to q.

The *equator geometry* $E(p,q)$ *(with poles* $p,q \in X(\Delta)$ is a full subgeometry consisting of the points symplectic to both p and q, and with induces line set. It is shown in [8] that it is a subspace and a geometry isomorphic to the long root subgroup geometry related to the point residual at p. For instance, if $\Delta \cong \mathsf{E}_{8,8}(\mathbb{K})$, the equator geometry is isomorphic to $\mathsf{E}_{7,1}(\mathbb{K})$. By Section 2F of [8], there is a set of points $\mathscr{C}$ such that every pair of points are poles of $E(p,q)$ (and no other point appears in a pair of poles for $E(p,q)$). The set $\mathscr{C}$ is called an *imaginary line* (and the notation $\mathscr{C}$ comes from the fact that it constitutes a conic in the standard embedding).

A trace geometry with respect to lines is defined in the introduction.

The Lie incidence geometry $\mathsf{E}_{7,1}(\mathbb{K})$ contains convex full subgeometries which are also subspaces, isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$. These are called *paras*.

2.4. **Three lemmas.** We recall the following result from [6].

Lemma 2.1 (Lemma 3.20 of [6])**.** *If a polar space is fully embedded in a parapolar space, then either it is contained in a singular subspace, or it is isometrically embedded in a symp.*

We will also need the following lemma.

Lemma 2.2. *A subspace of* $\mathsf{PG}(2n-1,\mathbb{K})$ *meeting every generator of a hyperbolic quadric* Q *isomorphic to* $\mathsf{D}_{n,1}(\mathbb{K})$ *has at least dimension n.*

Proof. Clearly, the result is true for $n = 2$. So assume $n \geq 3$.

Let T be a subspace of $\mathsf{PG}(2n-1,\mathbb{K})$ of dimension $n-1$ and suppose for a contradiction that T intersects every generator of Q. Clearly T is not contained in the span of every point perp as these have trivial global intersection. Let $p \in Q$ be a point with T not contained in $\langle p^\perp \rangle$. Then $T \cap \langle p^\perp \rangle$ defines a subspace of dimension $n-2$ in the quotient space $\langle p^\perp \rangle / \{p\} \cong \mathsf{PG}(2n-3,\mathbb{K})$, intersecting every generator of the hyperbolic quadric $p^\perp/\{p\}$ isomorphic to $\mathsf{D}_{n-1,1}(\mathbb{K})$ in at least a point. Repeating this argument over and over again, we eventually are reduced to the case $n=2$ of the lemma, which we already discussed, and which yields the desired contradiction. □

We also recall Lemma 2.3 of [8].

Lemma 2.3. *Let Ψ and Ψ' be connected point-line geometries with Ψ fully embedded in Ψ', such that for each point $p \in X(\Psi)$, each member of $\mathscr{L}(\Psi')$ containing p also belongs to $\mathscr{L}(\Psi)$. Then Ψ and Ψ' coincide.*

3. The uniqueness of $\mathsf{D}_{5,5}(\mathbb{K})$ in $\mathsf{E}_{6,1}(\mathbb{K})$

In this section, we set $\Omega := \mathsf{D}_{5,5}(\mathbb{K})$ and $\Delta := \mathsf{E}_{6,1}(\mathbb{K})$. Also, we denote by Γ a graph isomorphic to the skeleton of $\mathsf{D}_{5,5}(1)$.

Lemma 3.1. *Let Γ be isometrically laxly embedded in Δ, and let Δ be naturally embedded in* $\mathsf{PG}(26,\mathbb{K})$. *Then Γ is either contained in a symp, or collinear to a given point. If Γ spans a* 15-*dimensional projective space in* $\mathsf{PG}(26,\mathbb{K})$, *then it is naturally contained in a trace geometry (as the skeleton of an apartment of that trace geometry).*

Proof. We can describe $\Gamma \cong \mathsf{D}_{5,5}(1)$ as the graph with point set $\{\{i,j\} \mid i,j \in \{1,2,3,4,5\}\} \cup \{\infty\}$, where ∞ is adjacent to all pairs $\{i,j\}$, $i \neq j$, $i,j \in \{1,2,3,4,5\}$, the set $\{i,j\}$, whuch can be a singleton (case $i=j$) or a pair (case $i \neq j$) is adjacent to $\{k\}$ if $k \notin \{i,j\}$, $i,j,k \in \{1,2,3,4,5\}$, and the pairs $\{i,j\}$ and $\{i,k\}$ are adjacent if $|\{i,j,k\}| = 3$, $i,j,k \in \{1,2,3,4,5\}$.

The points $\{1\},\{2\},\{3\},\{4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}$ are all contained in the symp $\xi := \xi(\{1\},\{1,5\})$ (and each set of vertices of Γ like this is called a $(4,4)$-*cross-polytope*). Moreover, the singular subspace U generated by $\{1\},\{2\},\{3\}$ and $\{4\}$ has dimension 3 since $\{i,5\}$ is not collinear to $\{i\}$, but collinear to all of $\{\{j\} \mid j \in \{1,2,3,4\} \setminus \{i\}\}$. Similarly the singular subspace W generated by $\{1,5\},\{2,5\},\{3,5\}$ and $\{4,5\}$ has dimension 3.

Suppose for a contradiction that U and W are not opposite in ξ. Then there is a point $u \in U$ collinear to W. Hence it is contained in each plane $\langle \{i\},\{j\},\{k\} \rangle$, $|\{i,j,k\}| = 3$, $i,j,k \in \{1,2,3,4\}$. But that intersection is clearly empty, as $\{\{1\},\{2\},\{3\},\{4\}\}$ is a basis for U. Hence U and W are opposite in ξ.

Next, suppose that $\{5\} \in \xi$. Since $\{5\} \perp \{1,2\} \perp \{1,5\}$, the point $\{1,2\}$ is contained in ξ. Similarly the points $\{1,3\},\{1,4\},\{2,3\},\{2,4\}$ and $\{3,4\}$ belong to ξ, and also ∞, since the latter is collinear to the noncollinear points $\{1,2\}$ and $\{3,4\}$. Hence Γ is entirely contained in ξ. In this case it cannot span a 15-dimensional subspace of $\mathsf{PG}(26,\mathbb{K})$ as ξ only spans a 9-dimension subspace.

So we may from now on assume that $\{5\} \notin \xi$. Similarly, $\infty \notin \xi$.

Hence $\{5\}$ is contained in the unique 5-space U^* containing U, cf. Section 2.3.1(8), but not in ξ. It follows that $\{1\},\{2\},\{3\},\{4\}$ and $\{5\}$ generate a $4'$-space, and in similar way, the same thing holds for every 5-clique of Γ. Let z be the intersection of the $4'$-spaces of ξ containing U and W, respectively. Then $z \in U^* \cap W^*$, where W^* is the unique 5-space containing W, and which contains also ∞. Hence $\{5\} \perp z \perp \infty$.

Assume for a contradiction that z is not collinear to $\{1,2\}$. Then the points $\{1,5\}$, $\{2,5\}$, $\{3\}$, $\{4\}$, $\{5\}$ and ∞ are contained in $\xi(z,\{1,2\})$. Since the latter thus contains the noncollinear points ∞ and $\{3\}$, it also contains the point $\{4,5\}$, collinear with both. But then it contains the noncollinear points $\{4,5\}$ and $\{4\}$, which belong to ξ. Consequently $\xi = \xi(z,\{1,2\})$, which contains ∞. But we just argued above that we may assume $\infty \notin \xi$, a contradiction. Hence $\{1,2\} \perp z$, and similarly every other point of Γ is collinear to z.

In order to complete the proof of the proposition, we may assume that Γ spans a 15-dimensional subspace V in $\mathsf{PG}(26,\mathbb{K})$. We claim that $z \notin V$.

Suppose for a contradiction $z \in V$. From the construction above follows that the subspace spanned by $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,5\}$, $\{2,5\}$, $\{3,5\}$ and $\{4,5\}$ is a hyperbolic quadric Q_1 not containing z. Likewise, $\{5\}$, ∞, $\{i,j\}$, $i,j \in \{1,2,3,4\}$, $i \neq j$ span a quadric Q_2 not containing z. Hence the subspaces $\langle z, Q_1\rangle$ and $\langle z, Q_2\rangle$ share exactly a line L, contradicting Section 7 of [9]. The claim is proved.

Hence we can consider the cone with vertex z over $V(\Gamma)$; this defines, by Theorem A of [5], an apartment in the residue of z.

Now, using Section 2.3.1(6), we can consider the set Ξ of symps defined by the $4'$-spaces generated by the 5-cliques of Γ. They form an isometrically embedded graph $\Gamma' \cong \mathsf{D}_{5,4}(1)$, which is also isomorphic to $\mathsf{D}_{5,5}(1)$, in the dual $\Delta^* \cong \mathsf{E}_{6,6}(\mathbb{K})$ of Δ. Hence, by the foregoing, each member of Ξ is adjacent to some fixed symp ζ.

Assume for a contradiction that z is incident with ζ. Then Corollary 1.3 of [1] implies that we can find a $(4,4)$-cross-polytope P in Γ defining a symp of Δ through z locally opposite ζ. Consider any 5-clique C of Γ containing a 4-clique of P. Every symp containing C and a 4-space of ζ obviously contains z, which is a contradiction since that symp would then contain a 5-space.

Assume, again for a contradiction, that z is close to ζ. Then, again by Corollary 1.3 of [1], we find a vertex v of Γ such that the line zv is locally opposite the 5-space through z intersecting ζ in a $4'$-space. Then v is far from ζ and can hence never be contained in a symp intersecting ζ in a 4-space.

Hence ζ and z are opposite and the assertion now follows. □

Proposition 3.2. *Let $\Omega \cong \mathsf{D}_{5,5}(\mathbb{K})$ be fully embedded in $\Delta \cong \mathsf{E}_{6,1}(\mathbb{K})$. Then Ω coincides with a trace geometry.*

Proof. Consider a symp ξ of Ω. If ξ is not isometrically embedded, then by Lemma 2.1, it is embedded in a singular subspace W of Δ. But ξ contains disjoint solids, contradicting $\dim W \leq 5$. Since each pair of points of Ω is contained in a symp of Ω, we conclude that Ω is isometrically embedded in Δ.

Select an apartment with skeleton graph Γ in Ω and note that, by Proposition 2.1 of [7], the latter spans a subspace of dimension 15 of $\mathsf{PG}(26,\mathbb{K})$. Hence also the former does, by Theorem A of [5]. Also, the previous paragraph implies that Γ is isometrically embedded in Δ. By Lemma 3.1 the graph Γ is naturally embedded in a trace geometry. Since Γ generates Ω, and a trace geometry is a subspace, it follows that Ω is contained in a trace geometry, say $\Omega \subseteq z^{\perp} \cap \zeta^{\not\equiv}$, for a point z and an opposite symp ζ.

We now claim that $X(\Omega) = z^{\perp} \cap \zeta^{\not\equiv}$, which will conclude the proof of the proposition.

Indeed, with the notation of the proof of Lemma 3.1, the quadrics Q_1 and Q_2 together span a 15-dimensional subspace U of $\langle z^{\perp}\rangle$ (generation in $\mathsf{PG}(26,\mathbb{K})$). Both are also symps of the trace geometry $z^{\perp} \cap \zeta^{\not\equiv}$ (and note that this trace geometry is isomorphic to $\mathsf{D}_{5,5}(\mathbb{K})$ by Section 2.3.1(9)). Now the construction of $\mathsf{D}_{5,5}(\mathbb{K})$ out of two opposite symps explained in Section 5.1 of [11] shows the wanted equality. □

4. THE UNIQUENESS OF $\mathsf{E}_{6,1}(\mathbb{K})$ IN $\mathsf{E}_{7,7}(\mathbb{K})$

In this section, we set $\Omega \cong \mathsf{E}_{6,1}(\mathbb{K})$ and $\Delta \cong \mathsf{E}_{7,7}(\mathbb{K})$. We assume that Ω is fully embedded in Δ. Here is the main result of this section.

Proposition 4.1. *If Ω is fully embedded in Δ, then it is isometrically embedded and it coincides with a trace geometry.*

Proof. We break up the proof in a few parts.

Part 1: The embedding is isometric. Let ξ be a symp of Ω. Then $\xi \cong \mathsf{D}_{5,1}(\mathbb{K})$ is not embedded in a singular subspace of Δ, as the maximum dimension of such a subspace is 6 (and ξ contains disjoint singular 4-spaces). Hence, by Lemma 2.1, ξ is isometrically embedded in a symp of Δ. Since every pair of points of Ω is contained in a symp, the distance between those points in Ω is 2 if and only if the distance between those points in Δ is 2. Hence the embedding is isometric.

Note that this implies that Ω is a subspace of Δ.

Part 2: Ω is contained in $p^\perp$, for some point $p \in X(\Delta)$. Select $x \in X(\Omega)$. Then the geometry $\Omega_x \cong \mathsf{D}_{5,5}(\mathbb{K})$ is fully and isometrically embedded in Δ_x (using Part 1) and hence, by Proposition 3.2, there exists a line $L \in \mathscr{L}(\Delta)$ through x, not belonging to Ω_x, collinear to Ω_x.

Now let $y \in X(\Omega)$ be collinear to x. Then, similarly, there exists a line $M \in \mathscr{L}(\Delta)$ through y collinear to Ω_y. Now note first that the intersection $\Omega_x \cap \Omega_y$ is not contained in a symp (since it is a geometry isomorphic to a cone with vertex the line xy and base a subspace S isomorphic to $\mathsf{A}_{4,2}(\mathbb{K})$, which contains a point and a line violating the one-or-all axiom). Hence it immediately follows that the lines L and M are collinear. Suppose for a contradiction that they generate a solid Σ. Then select noncollinear points $u, v \in S$. Since singular subspaces of Δ inside a symp have dimension at most 5, the subspace Σ intersects each plane of $P := u^\perp \cap v^\perp$ (the perp is taken inside S) in at least a point. By Lemma 2.2, the whole of Σ is generated by these intersections, and since the embedding is isometric, Σ is a subspace of P, which is ridiculous since P does not contain 3-spaces. Hence L and M generate a plane and therefore intersect in a point p.

Now let $z \in X(\Omega)$ be collinear to x, but not to y. Letting z play the role of y, the previous paragraph yields a point $q \in L \setminus \{x\}$ such that $\Omega_z \subseteq q^\perp$. Assume for a contradiction that $p \neq q$. Let A be the intersection of $z^\perp$ and $y^\perp$, where both perps are taken in Ω. Then A contains points not collinear to x, whereas A is collinear to both p and q, and hence to L, including x, the sought contradiction.

Now it is easy to see that for every pair of points of $\mathsf{D}_{5,5}(\mathbb{K})$, there exists a point at distance 2 from both. This implies by the previous paragraphs and the arbitrariness of z that $\Omega_t \subseteq p^\perp$, for every $t \in X(\Omega)$ with $t \perp x$. This, however, covers all points of Ω and Part 2 is proved.

Part 3: Every line of Δ through p contains a unique point of Ω. Clearly, if some line L of Δ through p contained at least two points of Ω, then, since Ω is a subspace, also p would belong to Ω, contradicting Part 1 (as no point in Ω is collinear to all other points of Ω).

Again by Part 1, the lines through p containing some point of Ω constitute the point set of a fully and isometrically embedded subgeometry Ω' in Δ_p isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$. Let $x \in X(\Omega')$ be arbitrary. Select an arbitrary trace geometry Γ in $x^\perp$ (the perp is in Ω'). Then $\Gamma \cong \mathsf{D}_{5,5}(\mathbb{K})$ and so, by Proposition 3.2, it coincided with a trace geometry in Δ_p. Hence every line of Δ_p through x is also a line of Ω'. It now follows from Lemma 2.3 that Δ_p and Ω' coincide, which concludes the proof of Part 3.

Define, for each symp ξ of Ω, the point p_ξ as the unique point of the symp ξ^* of Δ containing ξ collinear to all points of ξ and distinct from p.

Part 4*: The set* $X := \{p_\xi \in X(\Delta) \mid \xi \in \Xi(\Omega)\}$ *is the point set of an isometrically fully embedded geometry* Ω^* *isomorphic to* $\mathsf{E}_{6,1}(\mathbb{K})$. By Section 2.3.1(7), X carries in a natural way the structure of $\mathsf{E}_{6,6}(\mathbb{K})$, since every point of X corresponds to a unique symp of Ω, and no two symps of Ω define the same point (which is obvious). Hence, due to Part 1, it suffices to show that each line in this natural structure coincides with a line of Δ. Now, a line in X consists of the points corresponding to the symps of Δ containing a given 5-space U through p. The corresponding points in X are, by definition, collinear to the same hyperplane $H \not\ni p$ of U. Let p^* be such a point and consider an arbitrary symp ξ^* containing U, but not p^*. Then, using Section 2.3.1(14), p^*, being collinear to H, is collinear to a $5'$-space U^* of ξ^*, and obviously U^* contains a member q^* of X. So $p^* \perp q^*$. One now deduces that all points of X corresponding to U are contained in the 6-space generated by U^* and p^*. Let p_1^*, p_2^* and p_3^* be three such points and assume for a contradiction that they are not contained in a common line. Then, inside the 6-space $\langle U^*, p^*\rangle$, the line $p_1^*p_2^*$ intersects the $5'$-space $\langle H, p_3^*\rangle$ in some point $r_3 \notin H$ distinct from p_3^*. Note that r_3 is not collinear to p.

Set $\xi_i = p^\perp \cap p_i^{*\perp} \in \Xi(\Omega)$. Since $r_3 \notin \{p, p_3^*\}$, there exists a point s_3 in ξ_3 not collinear to r_3. Since s_3 is collinear to at least a 3-space of ξ_2, it is, by (5) and (8) of Section 2.3.1, contained in a 5-space of Ω intersecting both ξ_1 and ξ_2 in 4-spaces. It follows that any line L through s_3, contained in that 5-space and disjoint from the solid $s_3^\perp \cap H$ intersects ξ_1 and ξ_2 in distinct points s_1 and s_2, respectively. The symp of Δ through s_1 and p_2^* contains L, hence s_3, and $p_1^*p_2^*$, hence r_3. Hence r_3 is collinear to some point t_3 of L, which belongs to $\xi(p, r_3)$. Hence $t_3 \in L \cap \xi(p, r_3) = \{s_3\}$, contradicting the choice of s_3 not being collinear to r_3.

Hence X is a full embedding of $\mathsf{E}_{6,6}(\mathbb{K})$ and, applying Part 1 to that embedding, Part 4 is proved.

Part 5*: There exists a unique point* q *opposite* p *and not opposite each point of* Ω*, that is,* Ω *coincides with the trace geometry* $p^\perp \cap q^{\not\equiv}$. By Part 2, there is a unique point q collinear to all points of X. Then $X(\Omega) \subseteq q^{\not\equiv}$. It is obvious that $p \neq q$. Also, if $q \perp p$, then it lies in each symp through p, a contradiction. If q is at distance 2 from p, then, by Section 2.3.1(14), the unique point of Ω on any line through p locally opposite $\xi(p,q)$ is opposite q, a contradiction. This proves existence. Let ξ_p be any symp through p. Then, since q is opposite p, it follows from Section 2.3.1(14) that q is collinear to a unique point of ξ_p, which automatically belongs to X. Let q' now be any point opposite p distinct from q. Then by the uniqueness of q as point collinear to all points of X, there exists a symp ξ^* containing a symp ξ of Ω such that the unique point r of ξ^* collinear to q' (where we again use Section 2.3.1(14)) is not equal to p_ξ. Then r is collinear to some point s on a line px, with $x \in X(\Omega)$ and $s \neq x$. Consequently q' is not opposite s, and since it is opposite p, it is also opposite x. Hence q' is opposite some point of Ω and we conclude that q is unique. □

5. The uniqueness of $\mathsf{E}_{7,7}(\mathbb{K})$ in $\mathsf{E}_{8,8}(\mathbb{K})$

In this section, we set $\Omega \cong \mathsf{E}_{7,7}(\mathbb{K})$ and $\Delta \cong \mathsf{E}_{8,8}(\mathbb{K})$. We assume that Ω is fully embedded in Δ. Here is the main result of this section.

Proposition 5.1. *If* Ω *is fully embedded in* Δ*, then it is isometrically embedded and it coincides with a trace geometry.*

Proof. We again break up the proof in several steps. Although one will discover great similarity with the structure of the proof of Proposition 4.1, some of the arguments are a little different and less direct because of the growing complexity that comes with the rank and the fact that we move from strong parapolar spaces of diameter 2, over strong ones with diameter 3, to non-strong ones with diameter 3. Nevertheless, arguments very similar or the same as in the proof of Proposition 4.1 will not be repeated.

Part 1*: The embedding is isometric.* As in the proof of Proposition 4.1, one deduces that symplectic points of Ω are also symplectic points of Δ. Now let $\{x,y\}$ be an opposite pair of points of Ω. Then, by Section 2.3.2(2), it is not an opposite pair in Δ since x is collinear to a symplectic point to y. Let ξ be any symp of Ω containing x. Then Section 2.3.1(14) asserts that there is a unique point z collinear to y and contained in ξ. Considering Δ_z, we see that, by the fact that Ω_z is isometrically embedded in Δ_z and Section 2.3.2(4), exactly a line of the symp of Δ containing ξ is collinear to y. This means, by Section 2.3.2(4) again, that y is special to x (as z is symplectic to x). Hence the embedding is isometric.

Part 2*: Ω is contained in $p^\perp$, for some point $p \in X(\Delta)$.* This is entirely similar to Part 2 of the proof of Proposition 4.1, except that we have to push it one step further and repeat the main argument for points collinear to y (with the notation of the proof of Proposition 4.1).

Part 3*: Every line of Δ through p contains a unique point of Ω.* Also this step is completely similar to the corresponding part in the proof of Proposition 4.1.

We again define, for each symp ξ of Ω, the point p_ξ as the unique point of the symp ξ^* of Δ containing ξ, collinear to all points of ξ and distinct from p.

Part 4*: The set $X := \{p_\xi \in X(\Delta) \mid \xi \in \Xi(\Omega)\}$ is an equator geometry with p one of its poles.* Completely similar to the proof of Part 4 in the proof of Proposition 4.1 one shows that X is the point set of an isometrically fully embedded geometry Ω^* isomorphic to $\mathsf{E}_{7,1}(\mathbb{K})$. By Proposition 4.9 of [8], the assertion follows. Note that the set of poles of X is an imaginary line $\mathscr{C}$.

Part 4*: There exists a unique imaginary line $\mathscr{C}$ containing p each point q of which distinct from and hence opposite p is not opposite each point of Ω, that is, Ω is a trace geometry.* Part 4 yields already existence of $\mathscr{C}$. Uniqueness follows with the same arguments as in Part 5 of the proof of Proposition 4.1. □

6. The uniqueness of $\mathsf{A}_{5,3}(\mathbb{K})$ in $\mathsf{E}_{6,2}(\mathbb{K})$

Proposition 6.1. *If a Lie incidence geometry $\Omega \cong \mathsf{A}_{5,3}(\mathbb{K})$ is fully embedded in another Lie incidence geometry $\Delta \cong \mathsf{E}_{6,2}(\mathbb{K})$, then it is isometrically embedded and it coincides with a trace geometry.*

The proof of this proposition is completely the same as the proof of Proposition 5.1, as soon as we prove the analogue of Proposition 3.2 for the Lie incidence geometries $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ not only fully, but also assumed to be isometrically embedded in $\mathsf{A}_{5,3}(\mathbb{K})$. However, this is just an extended exercise in projective geometry, which we shall not carry out in detail. We just hint at the fact that an efficient proof uses the fact that a pair of planes of $\mathsf{A}_{5,3}(\mathbb{K})$ with the property that each point of each plane is collinear to a unique point of the other plane always arises, up to duality in $\mathsf{PG}(5,\mathbb{K})$, from the set of planes of $\mathsf{PG}(5,\mathbb{K})$ through fixed points x_1 and x_2, and contained in given 3-spaces Σ_1 and Σ_2, respectively, where $x_i \in \Sigma_j$ if and only if $i = j$, and $\Sigma_1 \cap \Sigma_2$ is a plane.

7. The uniqueness of $\mathsf{D}_{6,6}(\mathbb{K})$ in $\mathsf{E}_{7,1}(\mathbb{K})$

Proposition 7.1. *If a Lie incidence geometry $\Omega \cong \mathsf{D}_{6,6}(\mathbb{K})$ is fully embedded in another Lie incidence geometry $\Delta \cong \mathsf{E}_{7,1}(\mathbb{K})$, then it is isometrically embedded and it coincides with a trace geometry.*

Proof. As in the previous section, the proof of this proposition is completely the same as the proof of Proposition 5.1, as soon as we prove the analogue of Proposition 4.1 for the Lie incidence geometries $\mathsf{A}_{5,2}(\mathbb{K})$ and $\mathsf{D}_{6,6}(\mathbb{K})$, assuming we have an isometric embedding. That one, on its turn, is completely similar to Proposition 4.1 once we show the analogue of Proposition 3.2 for the Lie incidence geometries $\mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$ and $\mathsf{A}_{5,2}(\mathbb{K})$, assuming we have an isometric embedding. That is what we will now do.

Let $\Omega \cong \mathsf{A}_{1,1}(\mathbb{K}) \times \mathsf{A}_{3,1}(\mathbb{K})$, which is just the Cartesian product of a projective line over $\mathbb{K}$ with a projective space of dimension 3 over $\mathbb{K}$, be isometrically and fully embedded in $\Delta \cong \mathsf{A}_{5,2}(\mathbb{K})$. We argue in the corresponding projective space $\mathsf{PG}(5,\mathbb{K})$. Pick a maximal singular subspace Σ_1 of Ω of dimension 3. This corresponds to the set of lines of $\mathsf{PG}(5,\mathbb{K})$ through some point x_1 inside some hyperplane H_1. A point of Δ is collinear to exactly one point of Σ if and only if it corresponds to a line L of $\mathsf{PG}(5,\mathbb{K})$ not through x_1 and not in H_1. Hence a second maximal singular subspace Σ_2 of Ω corresponds to a point $x_2 \notin H_1$ and a hyperplane $H_2 \not\ni x_1$. It follows that the points of Ω correspond to the lines of $\mathsf{PG}(5,\mathbb{K})$ intersecting both $H_1 \cap H_2$ and x_1x_2, that is, it coincides with the trace geometry $p^\perp \cap \xi^{\not\equiv}$, where p is the point corresponding to the line x_1x_2 and ξ is the symp corresponding to the solid $H_1 \cap H_2$ (through the Klein correspondence). $\square$

8. A general consequence

Before we go to the more tricky case of $\mathsf{E}_{6,1}(\mathbb{K})$ in $\mathsf{E}_{8,8}(\mathbb{K})$, we mention a global consequence of all previous results. Note that the standard embedding of a long root subgroup geometry in projective space is the one arising from the adjoint module.

Corollary 8.1. *Let Δ be one of the Lie incidence geometries $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,7}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$ and $\mathsf{E}_{8,8}(\mathbb{K})$ with standard embedding in $\mathsf{PG}(d,\mathbb{K})$ (and $d = 26$, 77, 55, 132 and 247, respectively). Let p be any point of Δ en let H be a hyperplane in the subspace of $\mathsf{PG}(d,\mathbb{K})$ spanned by all points of Δ collinear to p, not containing p. Then $H \cap X(\Delta)$ is a trace geometry. In particular, there exists a point $q \in X(\Delta)$ not opposite each point of $H \cap X(\Delta)$, unique if Δ is not a long root subgroup geometry, otherwise the imaginary line containing p and q is unique.*

Proof. The set $H \cap X(\Delta)$ is an embedded geometry isomorphic to a point residual. The assertion now follows from Propositions 3.2, 6.1, 4.1, 7.1 and 5.1. $\square$

9. The uniqueness of $\mathsf{E}_{6,1}(\mathbb{K})$ in $\mathsf{E}_{8,8}(\mathbb{K})$

In this section let Ω be isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$ and Δ to $\mathsf{E}_{8,8}(\mathbb{K})$. We first aim to show that Ω is a trace geometry with respect to two opposite lines.

But before that, we need to study the full embeddings of $\Omega' \cong \mathsf{D}_{5,5}(\mathbb{K})$ in $\Delta' \cong \mathsf{E}_{7,7}(\mathbb{K})$. We head off with a partial analogue to Lemma 3.1.

Denote again by Γ a graph isomorphic to the skeleton of $\mathsf{D}_{5,5}(1)$.

Lemma 9.1. *Let Γ be isometrically laxly embedded in Δ', and let Δ' be naturally embedded in $\mathsf{PG}(55,\mathbb{K})$. Then Γ is either contained in a symp, or collinear to a given line.*

Proof. We will follow the strategy of the proof of Lemma 3.1. Since now symps have larger Witt index, some arguments need to be revised.

We take the same notation for the vertices of Γ as in the proof of Lemma 3.1. The arguments of the first few paragraphs of that proof can then be copied, so that we have the following situation:

The points $\{1\},\{2\},\{3\},\{4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}$ are all contained in a common symp ξ. Moreover, the singular subspaces U generated by $\{1\},\{2\},\{3\}$ and $\{4\}$, and W generated by $\{1,5\},\{2,5\},\{3,5\}$ and $\{4,5\}$, are opposite and 3-dimensional. If $\{5\}\in\xi$ then all vertices of Γ are contained in ξ. So we may assume that both $\{5\}$ and ∞ are not contained in ξ.

We now go on with the proof, slightly diverging from the proof of Lemma 3.1. By the above, $\{5\}$ is contained in a unique 6-space U^* containing U and intersecting ξ in a $5'$-space U'. Likewise, ∞ is contained in a unique 6-space W^* containing W and intersecting ξ in a $5'$-space W'. Suppose that $U'\cap W'=\emptyset$. Then, by Section 2.3.1(14), the points $\{5\}$ and ∞ are opposite in Δ', a contradiction. Hence U' and W' share exactly a line Z.

Now the arguments in the proof of Lemma 3.1 can be repeated to prove that all vertices of Γ are collinear to Z. □

We can now show:

Proposition 9.2. *Let $\Omega'\cong\mathsf{D}_{5,5}(\mathbb{K})$ be fully embedded in $\Delta'\cong\mathsf{E}_{7,7}(\mathbb{K})$. Then Ω' coincides with a trace geometry with respect to lines.*

Proof. Since the symps of Ω' do not admit any full embedding in $\mathsf{PG}(6,\mathbb{K})$, the embedding is isometric. Since Ω' is generated by Γ (see [5]), it is, by Lemma 9.1, contained in $Z^\perp$, for some line Z. We select two (distinct) points p,z on Z. In Δ'_p, the cone with vertex p and base Ω' induces a full embedding, which, by Proposition 3.2, is a trace geometry of Δ'_p. Hence there is a symplecton ζ of Δ' through p such that each point of Ω' is collinear to a $5'$-space of ζ through p. We select arbitrarily a point p' in ζ not collinear to p, and we set $\zeta' := p^\perp\cap p'^\perp$. Note that p' is not opposite (in Δ') any point of Ω'.

Now we consider the (universal) embedding of Δ' in $\mathsf{PG}(55,\mathbb{K})$. By construction, the subspace of $\mathsf{PG}(55,\mathbb{K})$ generated by $p,z,X(\Omega')$ and ζ' coincides with $\langle p^\perp\rangle$ (generation in $\mathsf{PG}(55,\mathbb{K})$) and is hence 27-dimensional. On the other hand, the subspace U generated by $z,X(\Omega')$ and ζ' has dimension at most $(((0+15)+1)+9)+1=26$. It follows that it has dimension precisely 26 and that is does not contain p. Hence $U\cap p^\perp$ (perp in Δ) is an embedded geometry Ω^* isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$. Then we know from Proposition 4.1 that there is a point $z'\in X(\Delta')$ not opposite each point of $\{z\}\cup X(\Omega')\cup\zeta'$. The proof of Proposition 4.1 also directly implies that z' is collinear to p'. Since no point of Ω' is now opposite either p' or z', no point of Ω' is opposite any point of the line $p'z'$. Also, since pz is locally opposite ζ by construction, the points p' and z are opposite in Δ. Hence pz is opposite $p'z'$ and $X(\Omega')\subseteq(pz)^\perp\cap(p'z')^{\not\equiv}$. (Here, the notation $(p'z')^{\not\equiv}$ means the set of points not opposite any point of the line $p'z'$.)

We now claim $X(\Omega')=(pz)^\perp\cap(p'z')^{\not\equiv}$. If suffices to prove that every point $u\in(pz)^\perp\cap(p'z')^{\not\equiv}$ is contained in $X(\Omega')$. Let u be such a point. Since u is not opposite z', it is contained in $U\cap(pz)^\perp$, which coincides with $z^\perp\cap\Omega^*$. Hence u is collinear to z (and obviously distinct from it). Let x be the unique point of uz contained in $X(\Omega')$. Since p' is opposite z (see above) and not opposite x, it is opposite each member of $(uz)^* \setminus \{x\}$. Hence $u = x$ and the claim is proved.

This completes the proof of the proposition. □

Proposition 9.3. *Let $\Omega \cong \mathsf{E}_{6,1}(\mathbb{K})$ be fully embedded in $\Delta \cong \mathsf{E}_{8,8}(\mathbb{K})$. Then Ω coincides with a trace geometry with respect to lines.*

Proof. We begin with following the strategy of the proof of Proposition 4.1. Part 1 is completely similar and so the embedding is isometric.

For Part 2, the dimensions are different in the current case. For $x \in X(\Omega)$, using Proposition 9.2, there now exists a plane α through x, not belonging to Ω_x (in fact only intersecting it in x), collinear to Ω_x. For $y \in X(\Omega)$ collinear to x, we find another plane β through y collinear to Ω_y. As in the proof of Proposition 4.1, these planes are contained in a common singular subspace Σ. If $\dim \Sigma \in \{4,5\}$, then, with the notation of the proof of Proposition 4.1, Σ intersects every singular plane of P, and by Lemma 2.2, these intersections generate at least a 3-space, leading to the same contradiction as in the proof of Proposition 4.1. Hence Σ is a solid and $\alpha \cap \beta$ is a line L.

The rest of the arguments of the proof of Part 2 in the proof of Proposition 4.1 are also valid here and we conclude that Ω is contained in $L^{\perp}$.

Select $p \in L$ arbitrarily. We apply Proposition 4.1 to Δ_p. Then we find a line pz' and a cone with vertex p and base a geometry Ω'' isomorphic to $\mathsf{E}_{6,6}(\mathbb{K})$ such that pz' is locally opposite L, collinear to $X(\Omega'')$ and not locally opposite each line through p and a point of $X(\Omega)$.

Consider the natural embedding of Δ in $\mathsf{PG}(247,\mathbb{K})$. We claim that L is disjoint from the subspace of $\mathsf{PG}(247,\mathbb{K})$ generated by $X(\Omega)$. Indeed, suppose not, and assume some point $x \in L$ is contained in $\langle X(\Omega)\rangle$. We may assume with loss that $x = p$. Then Δ_x is contained in the subspace of $\mathsf{PG}(247,\mathbb{K})$ generated by $X(\Omega), X(\Omega''), pz'$ and a point on L distinct from p. This is at most a 55-dimensional space, which is a contradiction. The claim is proved.

Hence we can select a point $p \in L$ and a hyperplane H in $\langle p^{\perp}\rangle$ not containing p, but containing $X(\Omega)$. Corollary 8.1 implies that there exists a point q opposite p such that $H \cap X(\Delta)$ is contained in $q^{\not\equiv}$. We may assume $z' \in H$; then $\{z', q\}$ is a special pair. Let w be the unique point of Δ collinear to both q and z'. Since z' is symplectic to each point of Ω, the point w is not opposite any point of Ω. Hence, as before, Ω is contained in $L^{\perp} \cap (qx)^{\not\equiv}$. Similarly as in the last paragraph of the proof of Proposition 9.2 one shows now that $X(\Omega) = L^{\perp} \cap (qx)^{\not\equiv}$. □

Since the automorphism group of Δ acts transitively on opposite pairs of lines (by the so-called *BN*-property, or strongly transitivity), the embedding of $\mathsf{E}_{6,1}(\mathbb{K})$ in $\mathsf{E}_{8,8}(\mathbb{K})$ is unique. Since, by Section 2.3.2, a geometry isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$ is also contained as a full subgeometry in an arbitrary equator geometry, every such embedding also arises in this way. We now make this more concrete.

Connection with equator geometries. Let us go back to the last paragraph of the proof of Proposition 9.3. We proved that z' is symplectic to all points of Ω. Let z be the point in $L \cap H$. Then $\{z, q\}$ is special. Let u be the unique point collinear to both z and q. Since z is collinear to each point of Ω, the point u is at distance a most 2 from each point of Ω. It is not collinear to any point of Ω as such point is also collinear to p and, by Section 2.3.2(2), u is special to p with z the unique point collinear to both u and p. If u were special to a point t of Ω, then again Section 2.3.2(2) would imply that q is opposite t, a contradiction. Hence each point of Ω is symplectic to u and we conclude $X(\Omega) \subseteq E(u,z')$.

In general, Let p and q be two opposite points in Δ. A para in $\mathsf{E}_{7,1}(\mathbb{K})$ corresponds to a vertex of type 7 in the Coxeter diagram, hence to a point of $\mathsf{E}_{7,7}(\mathbb{K})$. It follows that a para of $E(p,q)$ corresponds to a line L through p. More exactly, each symp through L contains a point of $E(p,q)$ (see Section 2.3.2(6)) and the set of these points forms a para $\Pi \cong \mathsf{E}_{6,1}(\mathbb{K})$. The same reference implies that each point of Π is collinear to the unique point $u \in L$ special to q. Similarly, there exists a point w special to p and collinear to u such that $X(\Pi) \subseteq w^{\perp}$, and so $X(\Pi) \subseteq (uw)^{\perp}$. Now let M be a line through p locally opposite L and let R be the line through the point x of M special to q, and containing a point y collinear to q. Since x is collinear to p and p is symplectic to each point of Π, we deduce from Section 2.3.2(2) that no point of Π is opposite x. Likewise, no point of Π is opposite y. Hence no point of R is opposite any point of Π. Then the proof of Proposition 9.3 implies that $X(\Pi) = (uw)^{\perp} \cap R^{\not\equiv}$. This explains the freedom we have in choosing the line R. Note that we do not obtain additional lines like R by replacing q by another point of the imaginary line through p and q.

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