Finite linear spaces with many line-axial automorphisms

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Abstract

An automorphism of a linear space is called line-axial if it fixes each point on some line. We call a line-axial automorphism a line-shear if it fixes no second line through any fixed point. For a finite linear space such that each line is fixed pointwise by a suitable group of line-axial automorphisms (in particular, if each line is fixed pointwise by a line-shear of given prime order), we prove that the linear space is one of the following: a desarguesian projective or affine space, a Lüneburg plane, a Witt space, a hermitian unital, a Ree unital, or a certain Hering space.

In particular, among all finite unitals, the hermitian and the Ree unitals are characterized by the property that each line is fixed pointwise by a line-shear of given prime order.

Introduction

For a given linear space, we call an automorphism *line-axial* if it fixes all points on some line. If a line-axial automorphism fixes no second line through any fixed point, we call it a *line-shear*. We study finite linear spaces where each line is fixed pointwise by a suitable set of line-shears; these line-shears will generate a flag-transitive group. The classification of linear spaces with flag-transitive groups can then be used to determine explicitly all linear spaces with sufficiently many line-shears, see 3.6, 3.9 and 3.11 below.

In [12], [13], [14], we have studied a dual phenomenon (in a loose sense), considering translations of unitals (i.e., automorphisms fixing each line through a given point); this leads to characterizations of the hermitian unitals. It has been proved (see [12, 1.3]) that a non-trivial translation fixes no second point. For the line-shears, various examples show that the dual statement is not true, in general.

1 Line-axial automorphisms, and line-shears

1.1 Definitions. A linear space is an incidence structure $S = (P, \mathcal{L})$ where any two points are on a unique line in $\mathcal{L}$, and every line has at least two points (so we may identify each line with the set of points incident with it). A *flag* in a linear space is an incident point-line pair. We will assume that the linear space is not trivial, i.e., some line has at least three points, and there is more than one line. If P is finite and each line has the same number k of points then $(P, \mathcal{L})$ is a 2-$(|P|, k, 1)$ design.

Let $\operatorname{Aut} S$ denote the group of all automorphisms of S. We call $\sigma \in \operatorname{Aut} S$ *line-axial* if some line $A \in \mathcal{L}$ is contained in the set $\operatorname{Fix}(\sigma)$ of fixed points of σ. The set $(\operatorname{Aut} S)_{[A]} := \{\alpha \in \operatorname{Aut} S \mid A \subseteq \operatorname{Fix}(\alpha)\}$ forms a subgroup of $\operatorname{Aut} S$.

Note that a line-axial automorphism may fix more than one line pointwise; in general, the set $\operatorname{Fix}(\sigma)$ will not even be a subspace of S. If an automorphism σ fixes each point on some line A but fixes no second line through any point on A, then σ is called a *line-shear with respect to A*.

Our definition of line-shear does not exclude the existence of further fixed lines, but secures that every fixed point is in A. By definition, an automorphism of a projective or affine space is axial if it fixes each point in some hyperplane, which usually contains many lines. See Section 2 for examples of line-axial automorphisms, and of line-shears.

1.2 Lemma. *Let $S = (P, \mathcal{L})$ be a finite linear space with constant line size k, and let $A \in \mathcal{L}$. Let σ be an automorphism of S whose order is a power of a prime p that does not divide $k-1$. Then σ is a line-shear with respect to A if, and only if, the line A is the set of all fixed points of σ.*

Proof. A line-shear with respect to A fixes no point in $P \setminus A$ since any two points are joined by a unique line. Conversely, assume that $\sigma \in \operatorname{Aut} S$ fixes each point in A and no point in $P \setminus A$. If σ would fix some line $B \neq A$ meeting A, then $B \cap A$ is fixed by σ, and the sizes of the orbits of $\langle\sigma\rangle$ in $B \setminus A$ are multiples of p. Hence p divides $|B \setminus (B \cap A)| = k - 1$, contrary to our assumption. $\square$

2 Examples

2.1 Affine and projective spaces. A projective plane may admit line-axial (i.e., axial) automorphisms, but none of those is a line-shear; it fixes further lines because the existence of an axis implies the existence of a center; see [2, Cor. 2.3], or [27, p. 65].

In any affine plane, each non-trivial line-axial automorphism has a unique axis, and has a unique center at infinity (so there is a unique parallel class fixed linewise). The non-trivial shears as defined in [8, p. 133] are line-axial automorphisms, their center is the point at infinity of their axis. These examples motivate our name “line-shear” for the present context; in general linear spaces, there exist line-shears that would in other contexts be called homologies (e.g., see 2.2 below).

The diagonal matrix $D = \mathrm{diag}(1, -1, 2) \in \mathrm{GL}(3, 5)$ describes a line-shear of the affine space $\mathrm{AG}(3, 5)$, but $D^2 = \mathrm{diag}(1, 1, -1)$ does not. The same holds for the automorphism induced on the projective space $\mathrm{PG}(3, 5)$ by the block diagonal matrix $D = \mathrm{diag}(1, 1, X) \in \mathrm{GL}(4, 5)$ with $X = \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right)$. These examples show that in general the line-shears that fix a given line pointwise do not form a group (not even if we would call the identity a line-shear — but we do not).

The Lüneburg planes are (non-desarguesian) affine translation planes of order q^2, where $q \ge 8$ is a power of 2 with odd exponent; compare [23, Sect. 23]. The non-trivial line-axial automorphisms of such a plane are the involutory automorphisms fixing at least one point, they are shears, and there are $q - 1$ shears with given axis, see [8, 5.2.17, p. 235]. The full automorphism group of a Lüneburg plane has been determined in [21, 6.9]. The stabilizer of a point is isomorphic to $(\mathbb{F}_q^\times \times \mathrm{Sz}(q)) \rtimes \mathrm{Aut}\,\mathbb{F}_q$, where $\mathrm{Sz}(q)$ is the Suzuki group (i.e., a group of twisted Lie type ${}^2\mathrm{B}_2$, see [31], [32], [35], [23, Ch. IV]), acting 2-transitively on the pencil (of size $q^2 + 1$). That stabilizer is not 3-transitive on the pencil.

Apart from the Lüneburg planes, the only further example of a non-desarguesian affine plane with flag-transitive automorphism group is the nearfield plane S_9 of order 9, see [23, Sect. 8]. That plane admits line-axial automorphisms, but no line-shears; indeed, for every flag (x, A) of S_9 the point stabilizer $(\mathrm{Aut}\,S_9)_x$ acts imprimitively on the pencil of lines through x (and on the line at infinity of S_9), with blocks of imprimitivity of size 2; see [1, § 6] or [23, p. 36]. This implies that $(\mathrm{Aut}\,S_9)_{x,A}$ fixes a second line through x.

There are affine planes where some, but not all lines are axes of shears; e.g., take a plane over a proper semifield (where the projective closure has Lenz type V).

2.2 Hermitian unitals. Let $\mathbb{E}$ be the finite field of order q^2, where q is a prime power, and consider the non-degenerate hermitian form

$$h\colon \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{E}\colon \big((x_0, x_1, x_2), (y_0, y_1, y_2)\big) \mapsto x_0 y_2^q + x_1 y_1^q + x_2 y_0^q\,.$$

The *hermitian unital* HU_q has the point set $U_q := \left\{\mathbb{E}x \;\middle|\; 0 \ne x \in \mathbb{E}^3, h(x, x) = 0\right\}$; the intersections of U_q with secants (i.e., lines of $\mathrm{PG}(2, q^2)$ that contain at least two points of U_q) are the lines of HU_q.

For $s \in \mathbb{E}$, we have $s^{q+1} = 1$ if, and only if, the map $\delta_s\colon \mathbb{E}(x_0, x_1, x_2) \mapsto \mathbb{E}(x_0, sx_1, x_2)$ leaves U_q invariant, and then induces an automorphism of HU_q. If $s^{q+1} = 1 \ne s$ then $A := \left\{\mathbb{E}(1, 0, z) \;\middle|\; z \in \mathbb{E}, z^q = -z\right\} \cup \{\mathbb{E}(0, 0, 1)\}$ is the set of fixed points of δ_s inside U_q.

That set consists of all points on the line $\mathbb{E}(1,0,0) + \mathbb{E}(0,0,1)$ of $\mathrm{PG}(2,q^2)$ that lie in U_q, and forms a line of HU_q. The point $\mathbb{E}(0,1,0)$ does not belong to U_q. Each line of $\mathrm{PG}(2,q^2)$ through $\mathbb{E}(0,1,0)$ intersects U_q either in a single point (which then belongs to A), or in a line of HU_q that is fixed by δ_s, but contains no element of A. So δ_s is a line-shear of HU_q, as defined in 1.1. Note that δ_s acts as a homology on the projective plane $\mathrm{PG}(2,q^2)$.

If q is odd then every non-trivial line-axial automorphism of HU_q is one of the line-shears above, and $(\mathrm{Aut}\,\mathrm{HU}_q)_{[A]}$ is a cyclic group of order $q+1$.

If q is even then δ_{-1} is trivial, and HU_q does not admit any line-shears of order 2 (but there are line-shears of order $q+1$). The Baer involution $\beta\colon \mathbb{E}(x_0,x_1,x_2) \mapsto \mathbb{E}(x_0^q,x_1^q,x_2^q)$ then fixes each point of HU_q that lies on the line $\mathbb{E}(1,0,1)+\mathbb{E}(0,1,0)$. So β is a line-axial automorphism, but not a line-shear of HU_q. We obtain that $(\mathrm{Aut}\,\mathrm{HU}_q)_{[A]}$ is a non-abelian group of order $2(q+1)$; the line-shears with respect to A form a cyclic subgroup of order $q+1$.

2.3 Ree unitals. Consider $q := 3^{2n+1}$ with a non-negative integer n. The Ree group $\mathrm{Ree}(q)$ is a finite group of twisted Lie type ${}^2\mathrm{G}_2$, see [28], [36].

We construct a unital of order q, as follows (see [22]). The points are the Sylow 3-subgroups of $\mathrm{Ree}(q)$, and the lines are the involutions in $\mathrm{Ree}(q)$. A point S and a line j are incident if, and only if, the involution j normalizes the group S. This unital is called the Ree-Tits unital of order q, we denote it by RU_q.

Via conjugation, the Ree group $\mathrm{Ree}(q)$ acts by automorphisms on RU_q. That action is transitive on the set of flags. Every involution j fixes each point incident with the line j, but no other points. From 1.2 it follows that j is a line-shear.

The stabilizer $\mathrm{Ree}(q)_{S,T}$ of two points S, T of RU_q is the centralizer of the involution j, where j is the line joining S and T in RU_q. The group induced on the set of points incident with the line j is isomorphic to $\mathrm{PSL}(2,q)$, and j is the only non-trivial line-axial automorphism with respect to j, see [22, p. 257].

If $q > 3$ then $\mathrm{Ree}(q)$ is simple, and generated by its involutions, i.e., by the line-shears of RU_q.

The Ree unital RU_3 is isomorphic to the Witt(-Bose-Shrikhande) space $W(2^3)$, see 2.4 below. In particular, the involutions (viz., the line-shears) in $\mathrm{Ree}(3)$ generate a proper subgroup $\Lambda < \mathrm{Ree}(3)$ that acts flag-transitively on RU_3. In fact, one has $\mathrm{Ree}(3) \cong \mathrm{Aut}\,\mathrm{SL}(2,8) \cong \mathrm{P\Gamma L}(2,8) \cong \mathrm{Aut}\,W(2^3)$, and $\Lambda \cong \mathrm{SL}(2,8) \cong \mathrm{PGL}(2,8)$.

2.4 Witt spaces. We use the description of [5, p. 279] here: start with a hyperoval C in $\mathrm{PG}(2,2^n)$; namely, a non-degenerate conic together with its nucleus N. The set P of points consists of those lines of $\mathrm{PG}(2,2^n)$ that contain no point of C, the set of lines consists of all points of $\mathrm{PG}(2,2^n)$ not in C. Incidence is the obvious one. The resulting linear space $W(2^n)$ is known as a Witt(-Bose-Shrikhande) space; see [5, p. 279] for other descriptions and historical remarks. One should not confuse it with a Witt design (as related to Mathieu groups).

Clearly, every element of the stabilizer of C in the full automorphism group of $\mathrm{PG}(2,2^n)$ induces an automorphism of $W(2^n)$. If $n>2$, we obtain all automorphisms of $W(2^n)$ in this way. That stabilizer is isomorphic to $\mathrm{P\Gamma L}(2,2^n)$. We have $2^{n-1}(2^n-1)$ points and $2^{2n}-1$ lines in total, each point is on 2^n+1 lines, and each line has 2^{n-1} points. So the linear space $W(2^n)$ is not trivial if $n>2$, we only consider this case from now on.

Let γ be an automorphism of $W(2^n)$ that fixes some line A of $W(2^n)$ pointwise. Then γ is induced by a collineation $\hat\gamma$ of $\mathrm{PG}(2,2^n)$ that fixes the point A of $\mathrm{PG}(2,2^n)$ and at least $2^{n-1}\ge 4$ lines of $\mathrm{PG}(2,2^n)$ through A. With respect to a suitable coordinatization of the line pencil, the set of fixed lines through A apart from $A+N$ therefore forms a subfield of $\mathbb{F}_{2^n}$. Now $n>2$ yields $2^{n-1}>\sqrt{2^n}$, and that subfield is all of $\mathbb{F}_{2^n}$. So $\hat\gamma\in\mathrm{PGL}(3,2^n)$, and $\hat\gamma$ fixes each line through A, and $\hat\gamma^2$ fixes each point of C. Then $\hat\gamma^2$ is trivial, and $\hat\gamma$ is the unique involution with center A and axis $A+N$ that leaves C invariant. In particular, the points of $W(2^n)$ incident with A are the only fixed points of $\hat\gamma$ in $W(2^n)$. As each line contains an even number of points, the line-axial automorphism γ will not fix any line meeting A, and γ is a line-shear (see also 1.2).

If an automorphism δ of $W(2^n)$ fixes a point of $W(2^n)$ linewise then that point is a line L of $\mathrm{PG}(2,2^n)$ containing no point of C, and δ fixes each point on L and leaves C invariant. Then the nucleus of C is fixed linewise, and δ fixes each point of C. This implies that δ is trivial.

The commutator subgroup of $\mathrm{Aut}\,W(2^n)\cong\mathrm{P\Gamma L}(2,2^n)$ is isomorphic to $\mathrm{SL}(2,2^n)$, and acts flag-transitively on $W(2^n)$. The Witt space $W(2^n)$ can thus also be described in group theoretical terms (see [5, 2.6], [18, 6.3]); points of $W(2^n)$ are the dihedral subgroups of order $2(2^n+1)$ in $\mathrm{PSL}(2,2^n)\cong\mathrm{SL}(2,2^n)$, lines are the involutions in $\mathrm{PSL}(2,2^n)$, and a point D is incident with a line i if $i\in D$.

Note that the Witt space $W(2^3)$ is isomorphic to the smallest Ree unital RU_3, see 2.3.

2.5 Hering's examples. Let V be the vector space $\mathbb{F}_3^6$. Hering [15] describes a subgroup $G<\mathrm{GL}(6,3)$ with $G\cong\mathrm{SL}(2,13)$, in the form $G=\langle h,r,s\rangle$ with explicitly given matrices $h,r,s\in\mathrm{GL}(6,3)$. The group G is transitive on $V\smallsetminus\{0\}$, hence the semidirect product TG with the translation group T of V is doubly transitive on V.

This gives rise to three flag-transitive linear spaces with the point set V, by taking as the line set the TG-orbit of a suitable subspace of V. The *Hering plane* S_0 arising from a 3-dimensional subspace U_0 of V is an affine translation plane of order 27; see [15]. The *Hering spaces* S_1 and S_2 arise from two 2-dimensional subspaces U_1 and U_2 of V; see [16] or [17]. (There are two more geometries related to G, see [4] or [26]: one has $G<\mathrm{Sp}(6,3)$, and the symplectic polarity transforms S_1 and S_2 into two doubly transitive 2-designs which are not linear spaces: any two vectors are joined by 10 blocks.)

For each i, we determine the non-trivial line-axial automorphisms of S_i. Let γ be such an automorphism. The group TG is the full automorphism group of S_i, see [24] and [16], and $|TG| = 3^6 \cdot (3^6 - 1) \cdot 3$. The stabilizer $(TG)_{0,v} = G_v$ has order 3 for $0 \neq v \in V$, hence it is a Sylow 3-subgroup of $G \cong \mathrm{SL}(2, 13)$. It follows that γ has order 3, and generates the kernel of the action of the stabilizer G_{U_i} on U_i.

The matrix $h \in G$ has order 12, see [15, equation (1), page 204], thus h^4 has order 3. The Jordan normal form of $h^{\pm 4}$ consists of two unipotent Jordan blocks of size 3×3, see [26, p. 251] or check that $h^4 - 1$ has rank 4. Hence h^4 (and also h^{-4}) fixes precisely 9 vectors in V.

We conclude that the Hering plane S_0 does not admit such a γ, so it admits no line-axial automorphisms except for the identity.

For $i \in \{1, 2\}$, the flag-stabilizers $X_i = (TG)_{0,U_i} = G_{U_i}$ both have order 24, but they are different: As $X_1/\langle -1\rangle$ is the dihedral group of order 12, the group X_1 has a normal subgroup of order 3, while $X_2 \cong \mathrm{SL}(2, 3)$ has not (compare [4, Figure 1]). Since the kernel $\langle\gamma\rangle$ of the action on U_i is normal in the flag-stabilizer, the Hering space S_2 does not admit such a γ, but S_1 does. Every such $\gamma \in \mathrm{Aut}\, S_1$ is conjugate to h^4 (since $r^{-1}hr = h^{-1}$, see [15, p. 206]). By 1.2, the line-axial automorphism γ is indeed a line-shear of S_1.

3 Transitivity conditions and line-axial automorphisms

If F is a field and F_0 is a subfield of F then F is a vector space over F_0, and the point set F with the set $\left\{a + bF_0 \,\middle|\, a, b \in F, b \neq 0\right\}$ of all one-dimensional affine F_0-subspaces as lines is an affine space which we denote by $\mathrm{AG}(F, F_0)$.

3.1 Proposition. *Let F be a finite field, and let $S = (F, \mathcal{L})$ be a non-trivial linear space such that $\mathrm{Aut}\, S \leq \mathrm{A\Gamma L}(1, F)$ is flag-transitive. Assume that $\alpha \in \mathrm{Aut}\, S$ is a line-axial automorphism of S. Then F has a subfield F_0 with $[F : F_0] = |\langle\alpha\rangle|$ such that the lines of the affine space $\mathrm{AG}(F, F_0)$ are subspaces of S (i.e., S is obtained from $\mathrm{AG}(F, F_0)$ via "refinement" of the lines). If α fixes precisely the points of some line of S then $S = \mathrm{AG}(F, F_0)$.*

Proof. For distinct $x, y \in F$ let $x \vee y \in \mathcal{L}$ be the line joining x and y. Without loss of generality, assume that α fixes each point on $0 \vee 1$. We have $\alpha \in (\mathrm{Aut}\, S)_{0,1} \leq \mathrm{A\Gamma L}(1, F)_{0,1} = \mathrm{Aut}\, F$, and $F_0 := \mathrm{Fix}(\alpha)$ is a subfield of F with $[F : F_0] = |\langle\alpha\rangle|$.

Each set $x \vee y \in \mathcal{L}$ is contained in a unique line $\overline{x \vee y}$ of $\mathrm{AG}(F, F_0)$; indeed, $0 \vee 1 \subseteq F_0$ and $\mathrm{Aut}\, S$ is transitive on $\mathcal{L}$ and F_0 is invariant under $\mathrm{Aut}\, F$. Every line of $\mathrm{AG}(F, F_0)$

is of the form $\overline{x \vee y}$, and for distinct $u, v \in \overline{x \vee y}$ we have $u \vee v \subseteq \overline{u \vee v} = \overline{x \vee y}$. This means that $\overline{x \vee y}$ is a subspace of S.

If $0 \vee 1 = \mathrm{Fix}(\alpha)$ then $0 \vee 1 = \overline{0 \vee 1} = F_0$ and $S = \mathrm{AG}(F, F_0)$. □

There are many constructions of linear spaces with subgroups of $\mathrm{A\Gamma L}(1, q)$ acting flag-transitively, using refinements of $\mathrm{AG}(d, q)$, see [19] or [25].

A group G is called *almost simple* if it has a simple non-abelian normal subgroup N such that the centralizer of N in G is trivial. In other words, the group G is a subgroup of $\mathrm{Aut}\, N$ containing all inner automorphisms.

3.2 Theorem. *Let $S = (P, \mathcal{L})$ be a non-trivial finite linear space such that for each point $x \in P$ the stabilizer $(\mathrm{Aut}\, S)_x$ is transitive on the line pencil $\mathcal{L}_x$. Then the following hold.*

(a) *The group $\mathrm{Aut}\, S$ is flag-transitive.*

(b) (Saxl [29]) *If $\mathrm{Aut}\, S$ is almost simple, then S is a desarguesian projective space $\mathrm{PG}(d, q)$ with $d \geq 2$ (and $q \geq 4$ if $d = 2$), or a hermitian unital with order $q > 2$, or a Ree unital, or a Witt space.*

Each one of these linear spaces admits non-trivial line-axial automorphisms, and admits line-shears unless S is a projective plane.

(c) (Liebeck [20]) *If $\mathrm{Aut}\, S$ is not almost simple then the point set P can be identified with a vector space $\mathbb{F}_p^n$ for some prime p such that $\mathrm{Aut}\, S \leq \mathrm{AGL}(n, p)$. Moreover, the linear space S is an affine space $\mathrm{AG}(d, q)$ with $d \geq 2$, or a Lüneburg plane, or the affine nearfield plane of order 9, or one of Hering's examples S_0, S_1, S_2 as in 2.5, or one has $\mathrm{Aut}\, S \leq \mathrm{A\Gamma L}(1, p^n)$.*

(d) *If $\mathrm{Aut}\, S$ is not almost simple and contains some non-trivial line-axial automorphism then S is a desarguesian affine space $\mathrm{AG}(d, q)$ with $d \geq 2$, or a Lüneburg plane, or the affine nearfield plane of order 9, or the Hering space S_1, or S is a refinement of some affine space $\mathrm{AG}(d, q)$ and $\mathrm{Aut}\, S \leq \mathrm{A\Gamma L}(1, q^d)$.*

Proof. (a) Any two pencils share some line, and $\bigcup_{x \in P} \mathcal{L}_x = \mathcal{L}$. Hence $\mathrm{Aut}\, S$ is transitive on $\mathcal{L}$, and then also on P (see [3, Th. 2], cp. [8, 2.3.1, p. 78]). Thus $\mathrm{Aut}\, S$ is flag-transitive.

All finite linear spaces with flag-transitive automorphism groups are known explicitly, except for those where $\mathrm{Aut}\, S$ is a subgroup of $\mathrm{A\Gamma L}(1, E)$ for some finite field E. The pertinent classification has been announced in [6] (see also [5]), proofs appeared in [20] (using unpublished results from the dissertation [7, Chapter 3]) and [29].

(b) If $\operatorname{Aut} S$ is almost simple then the second corollary of Saxl's Main Theorem in [29, p. 324] says that S is $\mathrm{PG}(d,q)$ with $d \geq 2$, or a hermitian unital of order $q > 2$, or a Ree unital, or a Witt space. That corollary relies on the classification of finite simple groups.

(c) Assume that $\operatorname{Aut} S$ is not almost simple. By [5, Section 5] (or [7, Chapter 3], [37]), the group $\operatorname{Aut} S$ is of affine type; this entails that the point set of S can be identified with the vector space $\mathbb{F}_q^d$ such that $\operatorname{Aut} S \leq \mathrm{AGL}(d,q)$. Liebeck's Main Theorem in [20] relies on the classification of finite simple groups and yields assertion (c).

(d) The Hering space S_1 admits non-trivial line-axial automorphisms, but the Hering plane S_0 and the Hering space S_2 do not, see 2.5.

If $\operatorname{Aut} S \leq \mathrm{A\Gamma L}(1,p^n)$ then S is obtained by refining an affine space by 3.1. □

Note that the hermitian unital of order 2 is isomorphic to the affine plane $\mathrm{AG}(2,3)$ (e.g., see [34, 10.16]), and that the Ree unital of order 3 is isomorphic to the Witt space $W(2^3)$, see 2.4.

3.3 Remark. If the flag-transitive group $\operatorname{Aut} S$ in 3.2 contains an automorphism fixing precisely the points on some line of S (in particular, if it contains a line-shear) then each refinement occurring in 3.2(d) is an affine space by 3.1.

3.4 Lemma. *Let F be a finite field and assume that some subgroup G of $\Gamma\mathrm{L}(1,F)$ has a doubly transitive permutation representation of degree $r \geq 3$. If $|F| < r(r-1)$ then either $|F| = 4$ and $r = 3$, or $|F| = 16$ and $r = 5$.*

Proof. The groups $\Gamma\mathrm{L}(1,F)$ and G are metacyclic, hence the doubly transitive permutation group $\overline{G}$ induced by G has a normal subgroup N of prime order. Then N is (sharply) transitive, hence $|N| = r$ is a prime, and $\overline{G}$ is the permutation group $N \rtimes \operatorname{Aut} N$. The commutator group $\overline{G}' = N$ is induced by $G' \leq \mathrm{GL}(1,F)$. Let $|F| = p^n$ where p is the characteristic of F. Some field automorphism of F acts on N as an element of order $r-1$, hence $r-1$ divides n; thus $n > 1$. We infer that $2^n \leq p^n = |F| < r(r-1) \leq n(n+1)$, which entails that $n \leq 4$. Then $n(n+1) \leq 20$, and $p^n \in \{2^4, 3^2, 2^3, 2^2\}$. If $p^n = 3^2$ then $r = 3$, which is a contradiction to $p^n < r(r-1)$. If $p^n = 2^3$ then $r = 4$, which is not a prime. Thus $|F| = p^n \in \{2^4, 2^2\}$. □

3.5 Theorem. *Let $S = (P, \mathcal{L})$ be a non-trivial finite linear space.*

(a) *If $(\operatorname{Aut} S)_x$ is 2-transitive on $\mathcal{L}_x$ for each point $x \in P$ then S is a desarguesian affine or projective space, or a Lüneburg plane.*

(b) *If $(\operatorname{Aut} S)_x$ is 3-transitive on $\mathcal{L}_x$ for each point $x \in P$ then S is a desarguesian affine or projective plane.*

Proof. By 3.2(a), the group $\mathrm{Aut}\, S$ is flag-transitive. Inspection of transitivity properties of the automorphism groups of the examples obtained in 3.2(b) and 3.2(c) yields that either S is a desarguesian affine or projective space, or S is a Lüneburg plane, or $\mathrm{Aut}\, S \le \mathrm{A\Gamma L}(1, F)$, for some finite field F.

For the proof of assertion (a), it remains to treat the case where $\mathrm{Aut}\, S \le \mathrm{A\Gamma L}(1, F)$; then $(\mathrm{Aut}\, S)_x \le \Gamma\mathrm{L}(1, F)$. We have $|F| = 1 + r(k-1)$ where $r \ge 3$ is the number of lines through x, and the number k of points per line satisfies $2 < k \le r$ since S is a non-trivial linear space. Thus 3.4 yields that $|F| = 16$ and $k = 4$. This means that S is a 2-$(16, 4, 1)$ design, i.e. the unique affine plane of order 4, and thus desarguesian. This completes the proof of assertion (a).

In an affine or projective space S of rank higher than 2, the stabilizer $(\mathrm{Aut}\, S)_x$ of a point x cannot move coplanar triples of lines through that point to triples that are not coplanar. The stabilizer of a point in a Lüneburg plane is not 3-transitive on the pencil, see 2.1. This proves assertion (b). □

The point stabilizers in the automorphism groups of the linear spaces in 3.5 indeed have transitivity properties, as stated.

Linear spaces admitting many line-axial automorphisms

An action of a group on a set is called *semi-regular* if each point of the set has trivial stabilizer in the group.

3.6 Theorem. *Let $S = (P, \mathcal{L})$ be a non-trivial finite linear space, and let $m > 1$ be an integer. Assume that for each flag (a, A) of S there exists some subgroup $\Sigma_{a,A} \le (\mathrm{Aut}\, S)_{[A]}$ of order m acting semi-regularly on $\mathcal{L}_a \setminus \{A\}$. Then $\mathrm{Aut}\, S$ is flag-transitive, and the linear space S is a desarguesian affine or projective space, or a hermitian unital with order $q > 2$, or a Ree unital, or a Witt space, or a Lüneburg plane, or the Hering space S_1. Explicitly, we have the following.*

- **(a)** *If $m > 3$ then S is a desarguesian affine or projective space (and m divides the order of that space), or a hermitian unital HU_q (and m divides $q + 1$).*
- **(b)** *For $m = 3$ there is, apart from the examples in* (a)*, the additional possibility that S is the Hering space S_1.*
- **(c)** *For $m = 2$ there are, apart from the examples in* (a)*, the additional possibilities that S is a Ree unital, or a Witt space, or a Lüneburg plane.*

Proof. Let p be a prime dividing m, and consider a point $a \in P$. For each flag (a, A), there is an automorphism $\gamma_a \in (\operatorname{Aut} S)_{[A]}$ of order p fixing no other line through a. Thus γ_a induces a permutation of order p on the pencil $\mathcal{L}_a$ of lines through a. Now Gleason's Lemma [10, 1.7] (see [8, 4.3.14, 4.3.15, p. 190 f] or [12, 4.1]) applies, and yields that the stabilizer $(\operatorname{Aut} S)_a$ is transitive on the pencil $\mathcal{L}_a$. By 3.2(a), we obtain that $\operatorname{Aut} S$ is flag-transitive, and S is one of the spaces listed in 3.2(b) and 3.2(d).

The group $\Sigma_{a,A} \leq (\operatorname{Aut} S)_{[A]}$ acts semi-regularly on $P \setminus A$. So 3.3 yields that each refinement of an affine space occurring under the present hypotheses is an affine space. Inspection of the orders of line-axial automorphisms (cp. 2.1, 2.2, 2.3, 2.4, 2.5) yields the explicit results, as stated. □

The flag stabilizers in the automorphism groups of the linear spaces in 3.6 indeed contain semi-regular subgroups, as stated; see 2.1, 2.2, 2.3, 2.4, 2.5. In particular, groups as $\Sigma_{a,A}$ in 3.6 exist for each desarguesian projective space $\mathrm{PG}(d, q)$ with $d \geq 2$. Note, however, that projective planes do not admit line-shears in the sense of our definition 1.1.

3.7 Corollary. *Let $S = (P, \mathcal{L})$ be a non-trivial linear space, and let p be a prime number. If each line is is fixed pointwise by a line-shear of order p, then S is a desarguesian projective space $\mathrm{PG}(d, q)$ with $d \geq 3$, a desarguesian affine space, a hermitian unital with order $q > 2$, a Ree unital, a Witt space, a Lüneburg plane, or the Hering space S_1.*

3.8 Remark. In the special case of finite affine planes, Lüneburg [23, 39.2] assumes, for each line L, the existence of a shear with axis L. Without any restriction on the orders of those shears, he then proves that the plane is either desarguesian or a Lüneburg plane. He also explicitly determines the group generated by the automorphisms in question.

3.9 Theorem. *Let $S = (P, \mathcal{L})$ be a non-trivial finite linear space, and assume that for each flag (a, A) in S there exists a subgroup $\Sigma_{a,A} \leq (\operatorname{Aut} S)_{[A]}$ that acts sharply transitively on $\mathcal{L}_a \setminus \{A\}$. Then S is a desarguesian affine or projective space.*

Conversely, every desarguesian affine or projective space admits such groups.

Proof. As S is not trivial, each line pencil $\mathcal{L}_a$ has at least three elements. Our transitivity assumption then yields that $(\operatorname{Aut} S)_a$ is doubly transitive on $\mathcal{L}_a$, and $\operatorname{Aut} S$ is transitive on the set of flags by 3.2(a). Now 3.6 applies, with $m = |\mathcal{L}_a| - 1$.

Among the spaces obtained in 3.6, the only ones with $|\mathcal{L}_a| - 1 \leq 3$ are $\mathrm{AG}(2, 3)$, $\mathrm{PG}(2, 3)$, and $\mathrm{PG}(2, 2)$. In all other cases, we have $m > 3$, and by 3.6(a) the linear space S is either a desarguesian affine or projective space, or a hermitian unital. If S is a hermitian unital of order q then the group $(\operatorname{Aut} S)_{[A]}$ has order $q + 1$ if q is odd, and has order $2(q + 1)$ if q is even (see 2.2). So transitivity of $\Sigma_{a,A}$ on $\mathcal{L}_a \setminus \{A\}$ implies $q = 2$, and $S \cong \mathrm{HU}_2 \cong \mathrm{AG}(2, 3)$ is an affine plane. □

The special case of unitals

Recall that a unital of order q is a linear space with q^3+1 points and $q+1$ points per line.

The following result is a very special case of Catalan's conjecture (which has been proved by Mihăilescu; compare [30]). We provide an elementary and short proof for this special case.

3.10 Lemma. *If q is a positive integer such that q^3+1 is a power of some prime then $q \le 2$.*

Proof. We have $q^3+1=(q+1)(q^2-q+1)$, and these two factors of q^3+1 are powers of the same prime. If $q \ge 2$ then $q+1 \le q^2-q+1=(q+1)^2-3q$, hence $q+1$ divides $(q+1)^2-3q$ and then also $3q$ and 3. This gives $q=2$. □

3.11 Theorem. *Let $S=(P,\mathcal{L})$ be a unital of order q, and let $m>1$ be an integer. Assume that for each flag (a,A) in S there is a group $\Sigma_{a,A} \le (\operatorname{Aut} S)_{[A]}$ of order m acting semi-regularly on $\mathcal{L}_a \smallsetminus \{A\}$.*

Then either S is a hermitian unital (and m divides $q+1$), or S is a Ree unital (and $m=2$).

Proof. For the orders of line-shears in the hermitian and Ree unitals, see 2.2 and 2.3.

The group $\operatorname{Aut} S$ is flag-transitive by 3.6. By the Main Theorem in [5, Section 5] (or [7, Chapter 3], [37]), the group $\operatorname{Aut} S$ is either almost simple, or of affine type.

Assume first that $\operatorname{Aut} S$ is almost simple. According to 3.2(b), we have to exclude the cases where S is a projective space, or a Witt space $W(2^n)$ with $n>3$.

In a projective space $\mathrm{PG}(r,s)$, there are $(s^{r+1}-1)/(s-1)$ points, and $s+1$ points per line. If these are the parameters of a unital of order q then $s+1=q+1$ and $s^{r+1}-1=(q^3+1)(s-1)$ imply $s=q$ and then $s^{r+1}-1=s^4-s^3+s-1$. This is impossible.

In the Witt space $W(2^n)$ (see 2.4), each point is on 2^n+1 lines, and each line has 2^{n-1} points. If $W(2^n)$ is a unital of order q then $2^{n-1}=q+1$ leads to $2^n+1=q^2=(2^{n-1}-1)^2$ and then to $2n-2=n+1$. So $n=3$, and $S \cong W(2^3) \cong \mathrm{RU}_3$.

If $\operatorname{Aut} S$ is of affine type then $\operatorname{Aut} S$ has a (normal) elementary abelian subgroup acting transitively on P. So $|P|$ is a power of a prime, and $|P|=q^3+1$ yields $q \le 2$ by 3.10. Then $S \cong \mathrm{HU}_2 \cong \mathrm{AG}(2,3)$. (For unitals, we have thus avoided Liebeck's involved classification [20] for the affine case required in the proof of 3.2(c).) □

3.12 Corollary. *Let $S=(P,\mathcal{L})$ be a unital of order q, and let p be a prime. Assume that every line $A \in \mathcal{L}$ is fixed pointwise by a line-shear of order p. Then either S is a hermitian unital (and p divides $q+1$), or S is a Ree unital (and $p=2$).*

3.13 Remark. In [9], the hermitian and Ree unitals are characterized by the existence of a line-transitive group G of automorphisms such that the socle of G is a group of Lie type of characteristic p, where p is a "significant prime" (i.e., it divides the number of lines as well as $v-1$, where v is the number of points).

We note that a significant prime occurs as the order of a line-shear for the unitals (of course) but not for affine spaces, and not for Witt spaces, for instance.

Some flag-transitive groups generated by line-shears

Assume that S is a non-trivial linear space, and let X be a set of line-shears such that every line of S is fixed pointwise by some line-shear in X. We also assume that each one of those line-shears generates a group that acts semi-regularly, as required in 3.6. The latter condition is satisfied if the line-shears in question have prime order. Then the group $\langle X\rangle$ is flag-transitive. For any given $\sigma \in X$, every line of S is fixed pointwise by a line-shear in the conjugacy class C of σ in $\langle X\rangle$. We concentrate, therefore, on the case where X forms a conjugacy class in $\langle X\rangle$.

Desarguesian affine or projective spaces: We refrain from a discussion of the higher rank cases. In a finite desarguesian affine plane $\mathrm{AG}(2,q)$, the line-shears with axes through 0 are transvections, they belong to $\Delta_0 := \mathrm{SL}(2,q)$, and their order is the unique prime p dividing q. The stabilizer $\langle X\rangle_0$ is a subgroup of Δ_0 that acts transitively on the set of one-dimensional vector subspaces, and p divides the order of $\langle X\rangle_0$. From the list of finite subgroups of $\mathrm{SL}(2,F)$ over the algebraic closure of $\mathbb{F}_q$ (see [33, 6.17, Case II]) one sees that either $\langle X\rangle_0$ contains $\mathrm{SL}(2,q)$, or q is even and $\langle X\rangle_0$ is a dihedral group of order $2(q+1)$, or $q=9$ and $\langle X\rangle_0 \cong \mathrm{SL}(2,5)$.

Every flag-transitive group on $\mathrm{AG}(2,q)$ contains the group T of all translations (see [8, 4.4.22, p. 214]). Thus either we have $\langle X\rangle = T\Delta_0 \cong \mathbb{F}_q^2 \rtimes \mathrm{SL}(2,q)$, or q is even and $\langle X\rangle \cong \mathbb{F}_q^2 \rtimes (\mathrm{C}_{q+1} \rtimes \mathrm{C}_2)$, or $q=9$ and $\langle X\rangle \cong \mathbb{F}_9^2 \rtimes \mathrm{SL}(2,5)$.

Lüneburg planes: See 2.1. Again, the group $\langle X\rangle$ contains the group T of all translations. In the Lüneburg plane of order $q=2^{2k+1}$ with a positive integer k, the line-shears fixing 0 are involutions in $\Delta_0 := \mathrm{Sz}(q)$, and form a single conjugacy class in Δ_0. Every line is the axis of a unique involution, so X consists of the involutions fixing at least two points, and $\langle X\rangle = T\Delta_0 \cong \mathbb{F}_q^4 \rtimes \mathrm{Sz}(q)$; cp. [32, Theorem 9].

Hermitian unitals: We assume $q>2$ because HU_2 is the affine plane of order 3. The line-shears of HU_q are of the form δ_s with $s \in \mathbb{F}_{q^2}$ and $s^{q+1} = 1 \neq s$ (see 2.2). Note that δ_s is represented by an element of $\mathrm{GL}(3,q^2)$ with determinant s, and δ_s belongs to the simple group $\mathrm{PSU}(3,\mathbb{F}_{q^2}|\mathbb{F}_q)$ if s is a cube in $\mathbb{F}_{q^2}$. If 3 does not divide $q+1$ then each line-shear of HU_q belongs to $\mathrm{PSU}(3,\mathbb{F}_{q^2}|\mathbb{F}_q)$, and X generates $\mathrm{PSU}(3,\mathbb{F}_{q^2}|\mathbb{F}_q)$. If 3 divides $q+1$ then there exist line-shears outside $\mathrm{PSU}(3,\mathbb{F}_{q^2}|\mathbb{F}_q)$. The conjugacy class X of any such line-shear generates $\mathrm{PU}(3,\mathbb{F}_{q^2}|\mathbb{F}_q)$, and $\mathrm{PSU}(3,\mathbb{F}_{q^2}|\mathbb{F}_q)$ has index 3 in the latter (in any case, that index is $\gcd(3,q+1)$, see [34, p. 118]).

Ree unitals: Here $p=2$, see 2.3, and the line-shears *are* the lines of the linear space. If $q>3$ then $\langle X\rangle = \mathrm{Ree}(q)$. The situation is different for $q=3$; the smallest Ree unital RU_3 is isomorphic to the Witt space $W(2^3)$, and $\langle X\rangle = \mathrm{Ree}(3)' \cong \mathrm{SL}(2,8)$ is a subgroup of index 3 in $\mathrm{Ree}(3)$.

Witt spaces: See 2.4. The line-shears are the involutions in $\mathrm{SL}(2,2^n)$, and every line is fixed pointwise by a unique involution, so $\langle X\rangle = \mathrm{SL}(2,2^n)$.

Hering spaces: See 2.5. Only S_1 admits line-shears; for any given line, there are exactly two line-shears fixing that line pointwise, and they are inverse to each other. The line-shears form a single conjugacy class of elements of order 3 in $G=\mathrm{SL}(2,13)$. So $\langle X\rangle = TG \cong \mathbb{F}_3^6 \rtimes \mathrm{SL}(2,13)$.

4 Automorphisms fixing each line through a given point

We give a companion of 3.7 for the dual situation.

4.1 Theorem. *Let S be a non-trivial finite linear space such that $\operatorname{Aut} S$ is flag-transitive and contains a non-trivial automorphism fixing each line through some point of S.*

(a) *If $\operatorname{Aut} S$ is almost simple, then S is a desarguesian projective space $\mathrm{PG}(d,q)$ with $d\ge 2$, or a hermitian unital with order $q>2$.*

(b) *If $\operatorname{Aut} S$ is not almost simple, then S is a desarguesian affine space $\mathrm{AG}(d,q)$ with $d\ge 2$, a Lüneburg plane, the affine nearfield plane of order 9, the Hering plane S_0, or one of the Hering spaces S_1, S_2 (see 2.5), or S is a refinement of some affine space $\mathrm{AG}(d,q)$ with $\operatorname{Aut} S \le \mathrm{A\Gamma L}(1,q^d)$.*

Proof. As in the proof of 3.6, we have to consider the candidates for linear spaces with flag-transitive groups.

For unitals, non-trivial automorphisms fixing each line through a point were called "translations" in [12]. The hermitian unitals admit such automorphisms, but the Ree unitals and the Witt spaces do not, see [11, 1.8] and 2.4. □

For the affine translation planes in 4.1(b), there is a non-trivial element in the kernel of a corresponding quasifield, leading to an automorphism as required in 4.1. In the Hering spaces S_1 and S_2, the central involution of $\mathrm{SL}(2,13)$ fixes each line through the point 0.

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