The 2-blocking number and the upper chromatic number of $\mathrm{PG}(2, q)$

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September 18, 2012

The problem

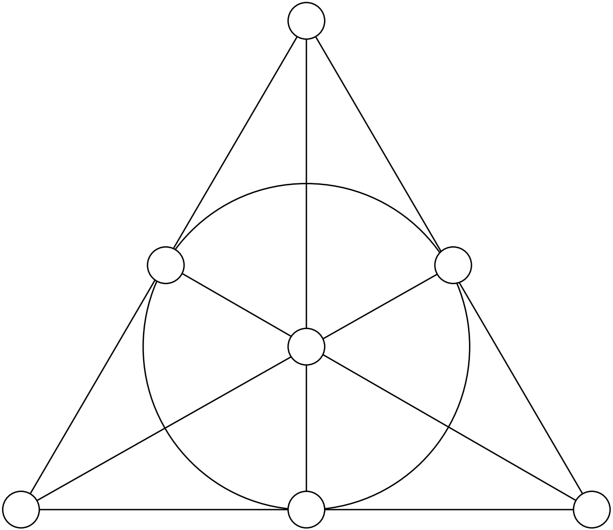
Color the vertices of a hypergraph $\mathcal{H}$.

A hyperedge is *rainbow*, if its vertices have pairwise distinct colors.

The upper chromatic number of $\mathcal{H}$, $\bar{\chi}(\mathcal{H})$: the maximum number of colors that can be used without creating a rainbow hyperedge.

Determining $\bar{\chi}(\Pi_q)$ and $\bar{\chi}(\mathrm{PG}(2,q))$ has been of interest since the mid-1990s.

Example: $\bar{\chi}(\mathrm{PG}(2,2)) = 3$



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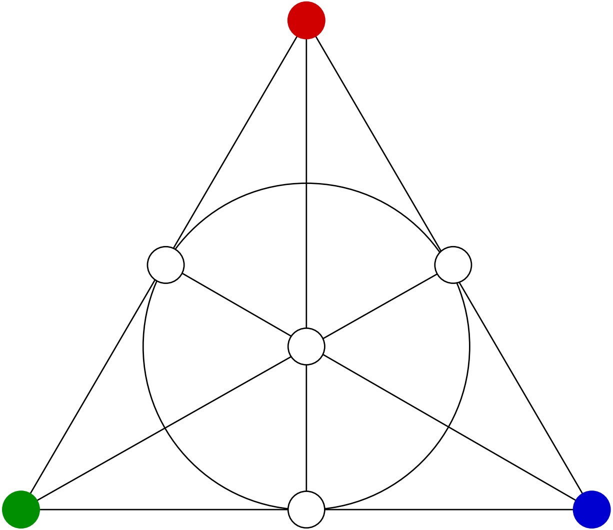

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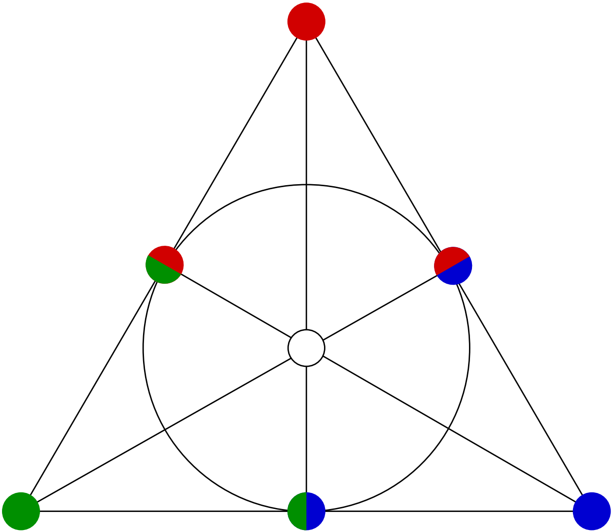

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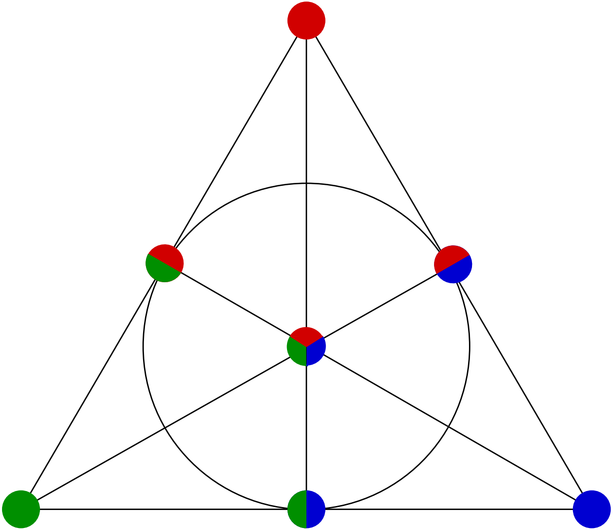

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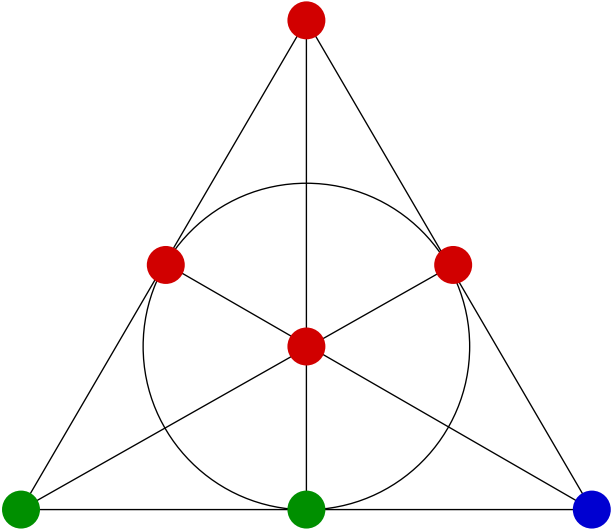
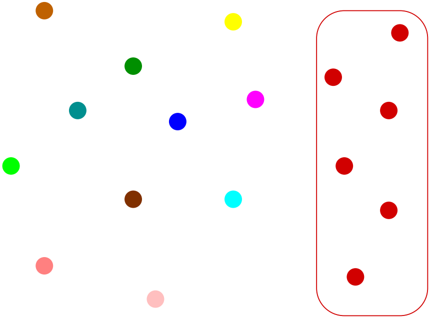


Trivial coloring



$v := q^2 + q + 1$, the number of points in Π_q.

$\tau_2 :=$ the size of the smallest double blocking set in Π_q.

Then $\bar{\chi}(\Pi_q) \geq v - \tau_2 + 1$.

We call this a *trivial coloring*.

Remark: if a coloring contains a monochromatic 2BS, it is not

Theorem (Bacsó, Tuza, 2007)

As $q \to \infty$,

- $\bar{\chi}(\Pi_q) \le v - (2q + \sqrt{q}/2) + o(\sqrt{q})$;
- *for q square,* $\bar{\chi}(\mathrm{PG}(2,q)) \ge v - (2q + 2\sqrt{q} + 1) = v - \tau_2 + 1$;
- $\bar{\chi}(\mathrm{PG}(2,q)) \le v - (2q + \sqrt{q}) + o(\sqrt{q})$;
- *for q non-square,* $\bar{\chi}(\mathrm{PG}(2,q)) \le v - (2q + Cq^{2/3}) + o(\sqrt{q})$.

Theorem

Let $q = p^h$, p prime. Let $\tau_2(\mathrm{PG}(2,q)) = 2(q+1)+c$. Suppose that one of the following two conditions holds:

1. $206 \le c \le c_0 q - 13$, *where* $0 < c_0 < 1/2$, $q \ge q(c_0) = 2(c_0+2)/(2/3 - c_0) - 1$, *and* $p \ge p(c_0) = 50c_0 + 24$.
2. $q > 256$ *is a square.*

Then $\bar{\chi}(\mathrm{PG}(2,q)) = v - \tau_2 + 1$, and equality is reached only by trivial colorings.

Simpler form of the above theorem:

Theorem

Let $q = p^h$, p prime. Suppose that either $q > 256$ is a square, or $h \ge 3$ odd and $p \ge 29$. Then $\bar{\chi}(\mathrm{PG}(2,q)) = v - \tau_2 + 1$, and equality is reached only by trivial colorings.

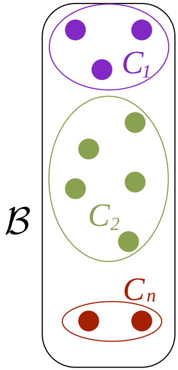
Remark: if $\tau_2(\mathrm{PG}(2,q)) < 8q/3$, $q > q(\tau_2)$, then $\bar{\chi} \lesssim v - \tau_2 + 10$.

$C_1, \ldots, C_n$: color classes of size at least two (only these are useful)

C_i colors the line ℓ iff $|\ell \cap C_i| \geq 2$.

All lines have to be colored, so $\mathcal{B} = \bigcup_{i=1}^{n} C_i$ is a double blocking set.

We use $v - |\mathcal{B}| + n$ colors.



To reach the trivial coloring, we must have $v - |\mathcal{B}| + n \geq v - \tau_2 + 1$, thus we need

$$n \geq |\mathcal{B}| - \tau_2 + 1$$

colors in $\mathcal{B}$. Also $n \leq |B|/2$, so $|B| \leq 2\tau_2 \leq 6q$.

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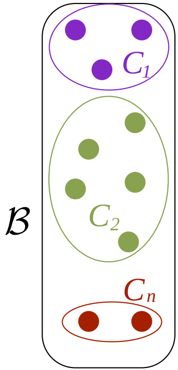

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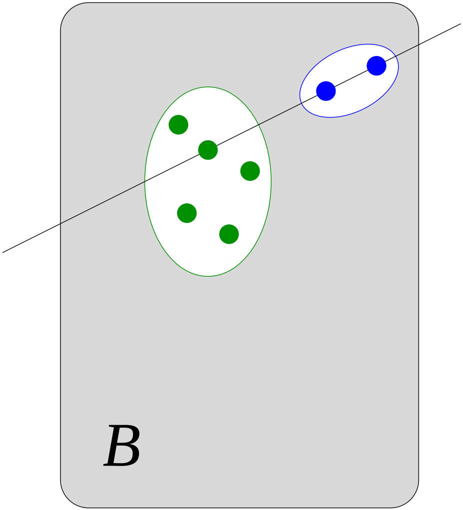
Eliminating color classes of size two



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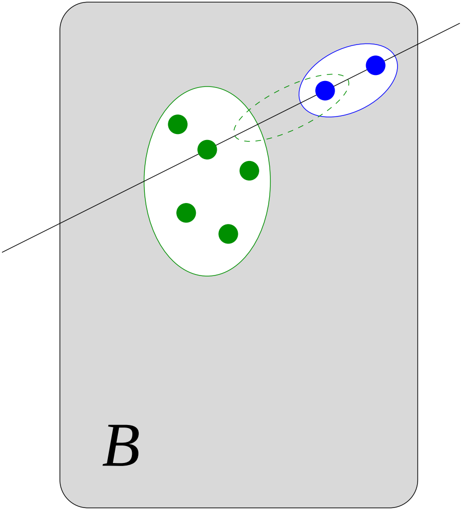

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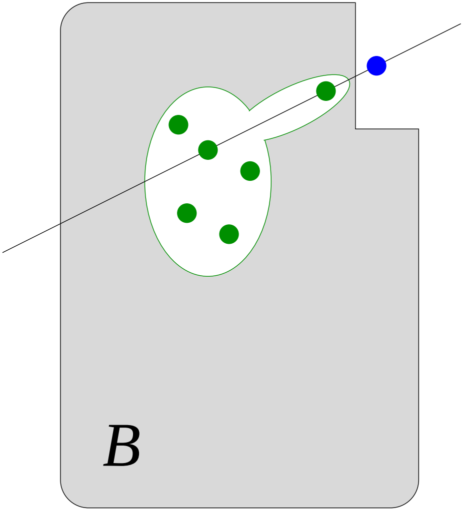

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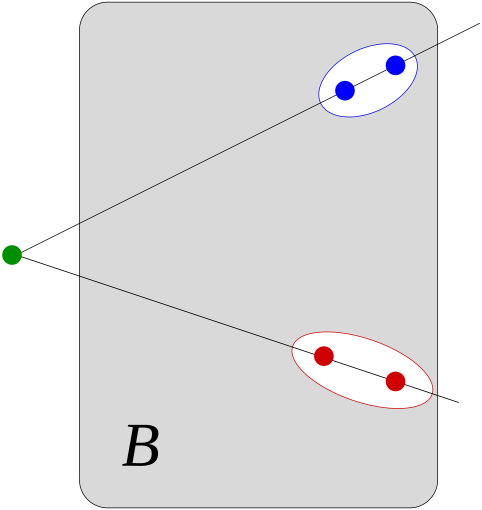

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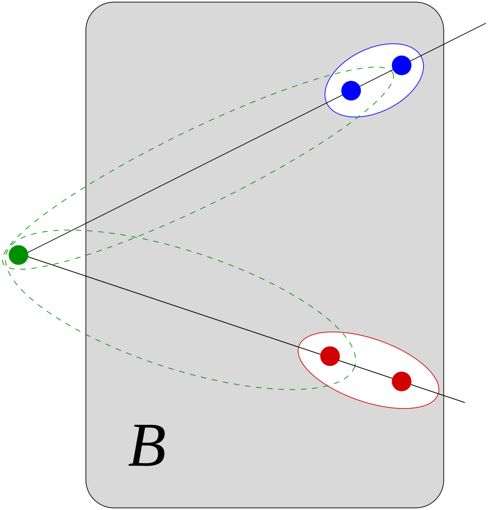

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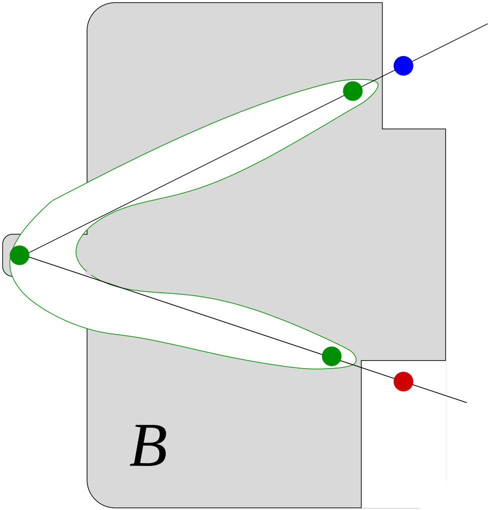

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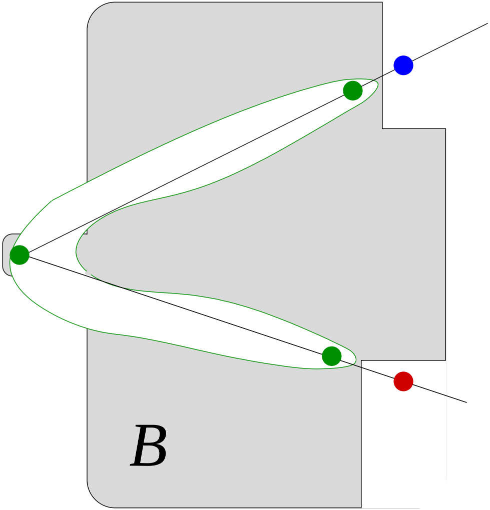

So there is at most one color class of size two.

$|\mathcal{B}| \gtrsim 3q - \varepsilon$

Recall that $\tau_2 \lesssim 2.5q$.

$L(C_i) :=$ the number of lines colored by C_i. Then $L(C_i) \leq \binom{|C_i|}{2}$.

By convexity, to satisfy

$$q^2 + q + 1 \leq \sum L(C_i) \leq \sum \binom{|C_i|}{2},$$

the best is to have one giant, and many dwarf color classes. But as

$$|\mathcal{B}| - \tau_2 + 1 \leq n \leq 1 + \frac{|\mathcal{B}| - \left|C_{\text{giant}}\right|}{3},$$

$|C_g iant| \leq 3\tau_2 - 2|B|$, too small.

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However, if $\left|C_{\text{giant}}\right| \ge q + 2$, we use $L(C_i) \le \frac{(q+1)}{2}|C_i|$.

$\tau_2 + \varepsilon' \lesssim |\mathcal{B}| \lesssim 3q - \varepsilon$

Lemma

Let $\mathcal{B}$ be t-fold blocking set in $\mathrm{PG}(2,q)$, $|\mathcal{B}| = t(q+1)+k$, and $P \in \mathcal{B}$ be an essential point of $\mathcal{B}$. Then there are at least $(q+1-k-t)$ t-secants of $\mathcal{B}$ through P.

Corollary

Let $\mathcal{B}$ be a t-fold blocking set with $|\mathcal{B}| \le (t+1)q$ points. Then there is exactly one minimal t-fold blocking set in $\mathcal{B}$, namely the set of essential points.

Remark

Harrach has a recent result on the unique reducibility of weighted t-fold $(n-k)$-blocking sets in the projective space $\mathrm{PG}(n,q)$.

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Proof of the lemma

Let $P \in B$ essential, and suppose to the contrary that there are more than $k+t$ long secants through P. Let ℓ be t-secant, $P \notin \ell$.

$$R(M,B) = \prod_{i=1}^{t-1}(M-m_i)\prod_{i=1}^{tq+k}(Mx_i+B-y_i) = g(M)\prod_{i=1}^{tq+k}(Mx_i+B-y_i).$$

As B is a t-fold blocking set,

$$R(M,B) \quad = \quad \sum_{j=0}^{t}(M^q-M)^j(B^q -$$

$(B^q-B)^t g(M)F_0^*(M,B)+(M^q-M)h(M,b),$

where $\deg(F_0^*) \leq k$.
Then for any $m \notin \{m_1,\dots,m_{t-1}\}$,

$$|\{Y=mX+B\} \cap B| > t \iff (B^q-B)^{t+1} \mid R(m,B) \iff (B-b) \mid F_0^*(r$$

Proof of the lemma

Let $P = (x_1, y_1)$. More than $k + t$ long sec's on $P \Rightarrow$ more than k long sec's with $m \notin \{m_1, \ldots, m_{t-1}\}$.

So $Mx_1 + B - y_1 = 0$ and $F_0^*(M, B) = 0$ have more than k common points. Thus $(Mx_1 + B - y_1)$ is a factor of $F_0^*(M, B)$, thus the lines through P and a point of $\ell \setminus B$ are long secants. This gives a contradiction if P is essential.

Clear: if ℓ is a 2-secant to $\mathcal{B}$, then $\ell \cap \mathcal{B}$ is monochromatic.

Let $|\mathcal{B}| = 2(q+1) + k$. Then

Proposition

Every color class containing an essential point of $\mathcal{B}$ has at least $(q-k) \approx 3q - |\mathcal{B}|$ points.

$\mathcal{B} = \mathcal{B}^* \cup \mathcal{B}'$, where $\mathcal{B}^*$ is the set of essential points, $|\mathcal{B}^*| \geq \tau_2$.

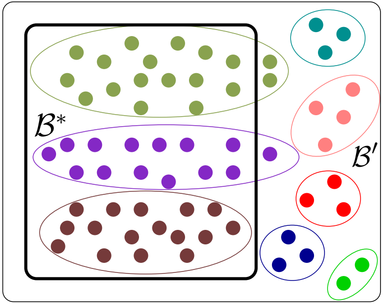
We have

$$|\mathcal{B}| - \tau_2 + 1 \leq n \leq \frac{|\mathcal{B}| - |\mathcal{B}^*|}{3} + \frac{|\mathcal{B}^*|}{q-k},$$

so

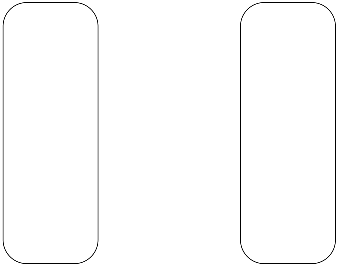
$$\frac{2}{3}(|\mathcal{B}| - \tau_2)(q-k) \leq \tau_2.$$

$\mathcal{B}$



$|\mathcal{B}| \leq \tau_2 + \varepsilon$, $q > 256$ square (so $\tau_2 = 2(q + \sqrt{q} + 1)$)

Blokhuis, Storme, Szőnyi: $\mathcal{B}$ contains two disjoint Baer subplanes, $\mathcal{B}_1$ and $\mathcal{B}_2$. $\mathcal{B}^* = \mathcal{B}_1 \cup \mathcal{B}_2$ can not be monochromatic.



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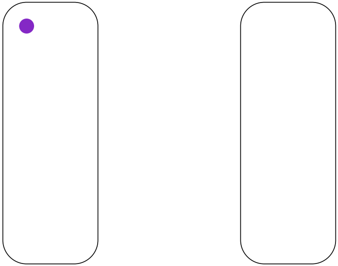

Let $P \in \mathcal{B}_1$ be purple.

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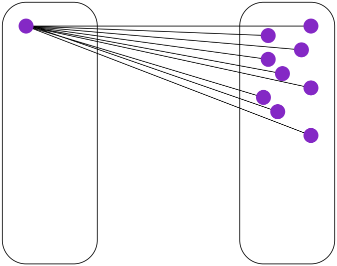

Let $P \in \mathcal{B}_1$ be purple. There are at least $(q - \sqrt{q} - \varepsilon - 1)$ 2-secants on P, so there are a lot of purple points in $\mathcal{B}_2$.

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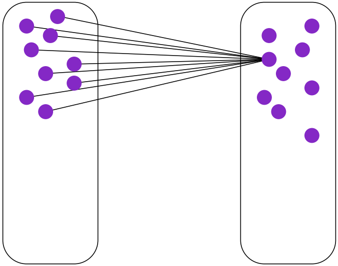

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The same from $\mathcal{B}_2$: we have at least $2(q - \sqrt{q} - \varepsilon - 1)$ purple points.

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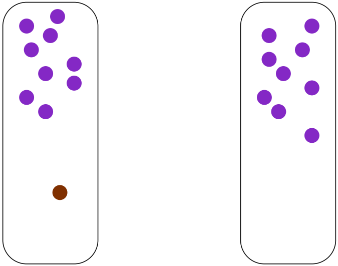

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If we have brown points as well:

$|\mathcal{B}| \leq \tau_2 + \varepsilon$, $q > 256$ square (so $\tau_2 = 2(q + \sqrt{q} + 1)$)

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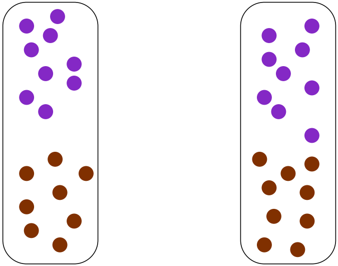

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If we have brown points as well: $|\mathcal{B}| \geq 4(q - \sqrt{q} - \varepsilon - 1)$ $\lightning$

By melting color classes, we may assume $n = 2$, $\mathcal{B}^* = \mathcal{B}^r \cup \mathcal{B}^g$, and let $|\mathcal{B}| = |\mathcal{B}^*| = 2(q+1) + k < 2.5q$.

Theorem (Blokhuis, Lovász, Storme, Szőnyi)

Let B be a minimal t-fold blocking set in $\mathrm{PG}(2, q)$, $q = p^h$, $h \geq 1$, $|B| < tq + (q+3)/2$. Then every line intersects B in $t \pmod{p}$ points.

For a line ℓ, let

$$\begin{aligned} n_\ell^r &= |\mathcal{B}^r \cap \ell|, \\ n_\ell^g &= |\mathcal{B}^g \cap \ell|, \\ n_\ell &= n_\ell^r + n_\ell^g = |\mathcal{B} \cap \ell|. \end{aligned}$$

Define the set of red, green and balanced lines as

$$\begin{aligned} \mathcal{L}^r &= \{\ell \in \mathcal{L} \colon n_\ell^r > n_\ell^g\}, \\ \mathcal{L}^g &= \{\ell \in \mathcal{L} \colon n_\ell^g > n_\ell^r\}, \\ \mathcal{L}^= &= \{\ell \in \mathcal{L} \colon n_\ell^r = n_\ell^g\}. \end{aligned}$$

Using double counting, we get

$$\sum_{\ell \in \mathcal{L}} n_\ell = |\mathcal{B}^*|(q+1), \text{ hence}$$

$$\sum_{\ell \in \mathcal{L} \colon n_\ell > 2} n_\ell \geq \sum_{\ell \in \mathcal{L}} (n_\ell - 2) = |\mathcal{B}^*|(q+1) - 2(q^2+q+1) \gtrsim kq.$$

On the other hand, $\displaystyle\sum_{\ell\in\mathcal{L}:\ n_\ell>2} n_\ell =$

$$\sum_{\ell\in\mathcal{L}^r:\ n_\ell>2} (n_\ell^r + n_\ell^g) + \sum_{\ell\in\mathcal{L}^g:\ n_\ell>2} (n_\ell^r + n_\ell^g) + \sum_{\ell\in\mathcal{L}^=:\ n_\ell>2} (n_\ell^r + n_\ell^g) \leq$$

$$\sum_{\ell\in\mathcal{L}^r:\ n_\ell>2} 2n_\ell^r + \sum_{\ell\in\mathcal{L}^g:\ n_\ell>2} 2n_\ell^g + \sum_{\ell\in\mathcal{L}^=:\ n_\ell>2} 2n_\ell^r \leq 4\cdot \sum_{\ell\in\mathcal{L}^r\cup\mathcal{L}^=:\ n_\ell>2} n_\ell^r.$$

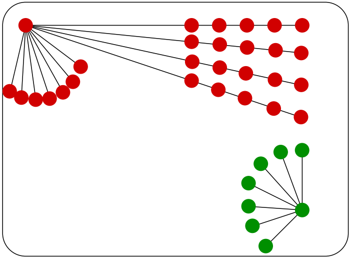
Recall $|\mathcal{B}_r| \leq |\mathcal{B}| - |\mathcal{B}^g| \leq 2q + k - (q-k) = q + 2k < 2q.$

Thus for the average number of long red secants through a red point,

$$\frac{\sum_{\ell\in\mathcal{L}^r\cup\mathcal{L}^=:\ n_\ell>2} n_\ell^r}{\mathcal{B}^r} \geq \frac{kq}{4|\mathcal{B}^r|} \geq \frac{k}{8}.$$

So we see:

$\frac{kp}{16}$ red points on the red long secants through P,
$q-k$ red points on the red two-secants through P,
and $q-k$ green points.



Thus $2q + k \gtrsim |\mathcal{B}| \geq 2q - 2k + \frac{kp}{16}$ ↯

Two disjoint blocking sets

Let $q = p^h$, $h \geq 3$ odd, p not necessarily prime, p odd. Let $m = (q-1)/(p-1) = p^{h-1} + p^{h-2} + \ldots + 1$. Note that m is odd.

Let $f(x) = a(x^p + x)$, $a \in \mathrm{GF}(q)^*$. Then f is $\mathrm{GF}(p)$-linear, and determines the directions $\left\{ \frac{f(x)-f(y)}{(x-y)} : x \neq y \right\}$ $=\{f(x)/x : x \neq 0\}=$ $\{(1 : f(x)/x : 0) : x \neq 0\}=$ $\{(x : f(x) : 0) : x \neq 0\}$. Thus

$$B_1 = \underbrace{\{(x : f(x) : 1)\}}_{A_1} \cup \underbrace{\{(x : f(x) : 0)\}_{x \neq 0}}_{l_1}$$

is a blocking set of Rédei type. Similarly, for $g(x) = x^p$,

$$B_2 = \underbrace{\{(y : 1 : g(y))\}}_{A_2} \cup \underbrace{\{(y : 0 : g(y))\}_{y \neq 0}}_{l_2}$$

is also a blocking set.

Two disjoint blocking sets

$$B_1 = \underbrace{\{(x : f(x) : 1)\}}_{A_1} \cup \underbrace{\{(x : f(x) : 0)\}_{x \neq 0}}_{l_1}$$

$$B_2 = \underbrace{\{(y : 1 : g(y))\}}_{A_2} \cup \underbrace{\{(y : 0 : g(y))\}_{y \neq 0}}_{l_2}$$

$f(x) = 0$ iff $x^p + x = x(x^{p-1} + 1) = 0$. As

$$-1 = (-1)^m \neq x^{(p-1)m} = x^{q-1} = 1,$$

$f(x) = 0$ iff $x = 0$. Also $g(x) = 0$ iff $x = 0$.

$l_2 \cap B_1$ is empty, as $(0 : 0 : 1) \notin l_2$.

If $(x : f(x) : 0) \equiv (y : 1 : g(y)) \in l_1 \cap A_2$, then $g(y) = 0$, hence $y = 0$ and $x = 0$, a contradiction. So $l_1 \cap A_2 = \emptyset$.

Two disjoint blocking sets

$$B_1 = \underbrace{\{(x : f(x) : 1)\}}_{A_1} \cup \underbrace{\{(x : f(x) : 0)\}_{x \neq 0}}_{I_1}$$

$$B_2 = \underbrace{\{(y : 1 : g(y))\}}_{A_2} \cup \underbrace{\{(y : 0 : g(y))\}_{y \neq 0}}_{I_2}$$

Now we need $A_1 \cap A_2 = \emptyset$.

$(y : 1 : g(y)) \equiv (x : f(x) : 1)$ $(x \neq 0)$ iff

$(y; 1; g(y)) = (x/f(x); 1; 1/f(x))$, in which case

$1/f(x) = g(x/f(x)) = g(x)/g(f(x))$.

Thus we need that $g(x) = g(f(x))/f(x) = f(x)^{p-1}$ that is, $x^p = (a(x^p + x))^{p-1} = a^{p-1}x^{p-1}(x^{p-1} + 1)^{p-1}$ has no solution in $\mathrm{GF}(q)^*$.

Two disjoint blocking sets

Equivalent form:

$$\frac{1}{a^{p-1}} = \frac{(x^{p-1}+1)^{p-1}}{x} = (x^{p-1}+1)^{p-1}x^{q-2} =: h(x)$$

should have no solutions $x \in \mathrm{GF}(q)^*$.

Let $D = \{x^m : x \in \mathrm{GF}(q)^*\} = \{x^{(p-1)} : x \in \mathrm{GF}(q)^*\}$. Then $1/a^{p-1} \in D$.

Note that $h(x) \in D \iff x \in D$.

So to find an element a such that $1/a^{(p-1)}$ is not in the range of h, we need that $h|_D : D \to D$ does not permute D.

Permutation polynomials

Theorem (Hermite-Dickson)

Let $f \in \mathrm{GF}(q)[X]$, $q = p^h$, p prime. Then f permutes $\mathrm{GF}(q)$ iff the following conditions hold:

- *f has exactly one root in $\mathrm{GF}(q)$;*
- *for each integer t, $1 \leq t \leq q-2$ and $p \not| t$, $f(X)^t \pmod{X^q - X}$ has degree at most $q-2$.*

A variation for multiplicative subgroups of $\mathrm{GF}(q)^*$:

Theorem

Suppose $d \mid q-1$, and let $D = \{x^d : x \in \mathrm{GF}(q)^*\}$ be the set of nonzero d^{th} powers, $m = |D| = (q-1)/d$. Assume that $g \in \mathrm{GF}(q)[X]$ maps D into D. Then $g|_D$ is a permutation of D if and only if the constant term of $g(x)^t \pmod{x^m - 1}$ is zero for all $1 \leq t \leq m-1$, $p \not| t$.

Two disjoint blocking sets

Recall that $h(X) = (X^{p-1}+1)^{p-1}X^{q-2}$. Let $t = p-1$, that is, consider

$$h^{p-1}(X) = \sum_{k=0}^{(p-1)^2} \binom{(p-1)^2}{k} X^{k(p-1)+(p-1)(q-2)} \pmod{X^m - 1}.$$

Since $k(p-1)+(p-1)(q-2) \equiv (k-1)(p-1) \pmod{m}$, the exponents reduced to zero have $k = 1 + \ell\frac{m}{(m,p-1)}$. Let r be the characteristic of the field $\mathrm{GF}(q)$. As $\binom{(p-1)^2}{1} \equiv 1 \pmod{r}$, it is enough to show that $\binom{(p-1)^2}{k} \equiv 0 \pmod{r}$ for the other possible values of k.

Suppose $h \geq 5$. Then $m/(m,p-1) > m/p > p^{h-2} > p^2$, thus by $k \leq (p-1)^2$, $\ell \geq 1$ does not occur at all. The case $h = 3$ can also be done.

Thank you for your attention!