An application of $\{\delta(q+1), \delta; n+1, q\}$-minihypers on generalized quadrangles

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Abstract

Minihypers in finite projective spaces have been used greatly to study the problem of linear codes meeting the Griesmer bound; thereby showing their importance for coding theory. But they are also important for a great variety of geometrical problems. Using the classification of $\{\delta(q+1), \delta; n+1, q\}$-minihypers we obtain results on spreads of certain finite generalized quadrangles. We discuss both the application and the result.

1 Introduction

In this section we introduce the concept of a generalized quadrangle, or shortly, a GQ.

Definition 1.1 *A (finite)* generalized quadrangle *(GQ) is an incidence structure* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *in which* $\mathcal{P}$ *and* $\mathcal{B}$ *are disjoint non-empty sets of objects called* points *and* lines *(respectively), and for which* $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ *is a symmetric point-line incidence relation satisfying the following axioms:*

(i) Each point is incident with $1+t$ *lines* $(t \geqslant 1)$ *and two distinct points are incident with at most one line.*

(ii) Each line is incident with $1+s$ *points* $(s \geqslant 1)$ *and two distinct lines are incident with at most one point.*

(iii) If x *is a point and* L *is a line not incident with* x*, then there is a unique pair* $(y, M) \in \mathcal{P} \times \mathcal{B}$ *for which* $x \,\mathrm{I}\, M \,\mathrm{I}\, y \,\mathrm{I}\, L$*.*

The integers s *and* t *are the* parameters *of the GQ and* $\mathcal{S}$ *is said to have* order (s, t)*. If* $s = t$*, then* $\mathcal{S}$ *is said to have order* s*.*

Examples: Classical examples are the symplectic space $\mathrm{W}_3(q)$ in $\mathrm{PG}(3, q)$, the hyperbolic quadric $\mathrm{Q}^+(3, q)$ in $\mathrm{PG}(3, q)$, the parabolic quadric $\mathrm{Q}(4, q)$ in $\mathrm{PG}(4, q)$, the elliptic quadric $\mathrm{Q}^-(5, q)$ in $\mathrm{PG}(5, q)$, and the Hermitian varieties $\mathrm{H}(3, q^2)$ and $\mathrm{H}(4, q^2)$ in $\mathrm{PG}(3, q^2)$ and $\mathrm{PG}(4, q^2)$, which have respectively order q, $(q, 1)$, q, (q, q^2), (q^2, q) and (q^2, q^3). The non-classical examples of Tits are given in the following definition:

Definition 1.2 *Let* $n = 2$ *(respectively,* $n = 3$*) and let* $\mathcal{O}$ *be an oval (respectively, an ovoid) of* $\mathrm{PG}(n, q)$*. Furthermore, let* $\mathrm{PG}(n, q)$ *be embedded as a hyperplane in* $\mathrm{PG}(n+1, q)$*.*
Define points as

(i) *the points of* $\mathrm{PG}(n+1,q) \setminus \mathrm{PG}(n,q)$*,*

(ii) *the hyperplanes* X *of* $\mathrm{PG}(n+1,q)$ *for which* $|X \cap \mathcal{O}| = 1$*, and*

(iii) *one new symbol* (∞)*.*

Lines are defined as

(a) *the lines of* $\mathrm{PG}(n+1,q)$ *which are not contained in* $\mathrm{PG}(n,q)$ *and which meet* $\mathcal{O}$ *(necessarily in a unique point), and*

(b) *the points of* $\mathcal{O}$*.*

Incidence is inherited from $\mathrm{PG}(n+1,q)$*, whereas the point* (∞) *is incident with no line of type (a) and with all lines of type (b).*

It is straightforward to show that these incidence structures are GQ's with parameters $s = q$, $t = q^{n-1}$.

Definition 1.3 *A* spread *of a GQ* $\mathcal{S}$ *of order* (s,t) *is a set* S *of lines such that every point of* $\mathcal{S}$ *is incident with exactly one element of* S*. A spread necessarily contains* $1+st$ *lines. A* partial spread *is a set* S *of lines for which every point is incident with at most one line of* S*. A partial spread is called* maximal *if* S *is not contained in a larger partial spread. If the size of a partial spread is* $1+st-\delta$*, then* δ *is said to be the* deficiency *of the partial spread.*

The natural question is whether a partial spread with certain deficiency can be maximal, or, in other words, can a partial spread with small deficiency be extended? Using minihypers we can give answers to this question for partial spreads of the GQ's $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$.

2 The minihypers

Definition 2.1 *An* $\{f,m;N,q\}$-minihyper *is a pair* (F,w)*, where* F *is a subset of the point set of* $\mathrm{PG}(N,q)$ *and where* w *is a weight function* w*:* $\mathrm{PG}(N,q) \to \mathbb{N}$*:* $x \mapsto w(x)$*, satisfying*

1. $w(x) > 0 \iff x \in F$*,*
2. $\sum_{x\in F} w(x) = f$*, and*
3. $\min\{\sum_{x\in H} w(x) \| H \in \mathcal{H}\} = m$*, where* $\mathcal{H}$ *is the set of hyperplanes of* $\mathrm{PG}(N,q)$*.*

Related to certain minihypers are blocking sets of $\mathrm{PG}(2,q)$. The following theorem about blocking sets is used for the final theorem.

Theorem 2.2 *(*A. Blokhuis, L. Storme, and T. Szőnyi [3]*) Let B be a blocking set in* $\mathrm{PG}(2,q)$*, $q=p^h$, p prime, of size $q+1+c$. Let $c_2=c_3=2^{-1/3}$ and $c_p=1$ for $p>3$.*

1. *If $q=p^{2d+1}$ and $c<c_pq^{2/3}$, then B contains a line.*
2. *If $4<q$ and q is a square and $c<c_pq^{2/3}$, then B contains a line or a Baer subplane.*

To establish the connection between blocking sets and certain minihypers we need one more definition.

Definition 2.3 *Let $\mathcal{A}$ be the set of all lines of* $\mathrm{PG}(N,q)$*. A* sum of lines *is a weight function w: $\mathcal{A}\to\mathbb{N}$: $L\mapsto w(L)$. A sum of lines induces a weight function on the points of* $\mathrm{PG}(N,q)$*, which is given by $w(x)=\sum_{L\in\mathcal{A},x\in L}w(L)$. In other words, the weight of a point is the sum of the weights of the lines passing through that point. A sum of lines is said to be a* sum of n lines *if the sum of all the weights of the lines is n.*

The connection is finally expressed in the following theorem, which will be of direct use for our application

Theorem 2.4 *(*Govaerts and Storme [1]*) Let (F,w) be a $\{\delta(q+1),\delta;N,q\}$-minihyper, $q>2$, satisfying $0\leqslant\delta<\epsilon$, where $q+\epsilon$ is the size of the smallest non-trivial blocking set in* $\mathrm{PG}(2,q)$*. Then w is a weight function induced on the points of* $\mathrm{PG}(N,q)$ *by a sum of δ lines.*

3 The application

Considering an arbitrary partial spread of $T_n(\mathcal{O})$, we will define an $\{\delta(q+1),\delta;n+1,q\}$-minihyper.

Definition 3.1 *Let S be a partial spread of a GQ. A* hole *with respect to S is a point of the GQ which is not incident with any line of S.*

Consider a partial spread S of $T_n(\mathcal{O})$, $n=2$ or $n=3$, of size $q^n+1-\delta$. Referring to the definition of the GQ $T_n(\mathcal{O})$, let $\pi_0=\mathrm{PG}(n,q)$ which contains $\mathcal{O}$ and which is embedded in $\mathrm{PG}(n+1,q)$ as a hyperplane. We remark that a partial spread contains at most one line of type (b) of the GQ, because all lines of type (b) intersect in (∞).

Definition 3.2 *Let S be a partial spread of $T_n(\mathcal{O})$ ($n=2$ or $n=3$). Define w_S:* $\mathrm{PG}(n+1,q)\to\mathbb{N}$ *as follows:*

(i) if $x\in\mathrm{PG}(n+1,q)\backslash\pi_0$ and x is a hole with respect to S, then $w_S(x)=1$, otherwise $w_S(x)=0$,

(ii) *suppose* $x \in \mathcal{O}$*, define* $w_S(x) = \delta_x$*, with* $q - \delta_x$ *the number of lines of* $\mathcal{S}$ *through* x*.*

(iii) $w_S(x) = 0, \forall x \in \pi_0 \setminus \mathcal{O}$*.*

This weight function determines a set F *of points of* $\mathrm{PG}(n+1,q)$*. We will denote the defined minihyper by* (F, w_S)*.*

We can now prove

Lemma 3.3 *Let* S *be a partial spread of* $T_n(\mathcal{O})$ *(*$n = 2$ *or* 3*) which covers* (∞) *and which has deficiency* $\delta < q$*. Then* w_S *is the weight function of a* $\{\delta(q+1), \delta; n+1, q\}$*-minihyper* (F, w_S)*.*

This lemma leads immediately to

Theorem 3.4 *(*M.R. Brown, J. De Beule and L. Storme [2]*) Let* S *be a partial spread with deficiency* δ *of* $T_n(\mathcal{O})$ *(*$n = 2$ *or* 3*) covering* (∞)*. If* $\delta < \epsilon$*, with* $q + \epsilon$ *the size of the smallest non-trivial blocking set in* $\mathrm{PG}(2,q)$*,* $q > 2$*, we can always extend* S *to a spread.*

References

[1] P. Govaerts and L. Storme. On a particular class of minihypers and its applications. I. The result for general q. *Des. Codes Cryptogr.,* accepted.

[2] M.R. Brown, J. De Beule and L. Storme. Partial spreads of $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$. *European J. Combin.,* accepted.

[3] A. Blokhuis, L. Storme and T. Szőnyi. Lacunary polynomials, multiple blocking sets and Baer subplanes. *J. London Math. Soc. (2)*, 60(2):321–332, 1999.

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