New classification results for a certain class of weighted minihypers

J. De Beule and L. Storme

Ghent University
Dept. Pure Mathematics and Computer Algebra, Krijgslaan 281
B 9000 Gent, Belgium
jdebeule@cage.ugent.be
ls@cage.ugent.be

Abstract

Minihypers were introduced for their relation with linear codes meeting the Griesmer bound. Results on minihypers have therefore important applications in coding theory. On the other hand, minihypers are nice geometrical structures since they are a generalization of blocking sets, which have been studied a lot. Therefore also minihypers deserve to be studied from a purely geometrical point of view. At last, results on minihypers have also important applications in finite projective geometry.

1 Introduction

A linear \([N, k, d; q]\)-code is a \(k\)-dimensional subspace of the \(N\)-dimensional vectorspace \(V(N, q)\) over the galois field \(GF(q)\) having minimum Hamming distance \(d\). It is interesting to use linear codes having a minimal length for given \(k, d\) and \(q\). Every linear code satisfies 
\[
N \geq \sum_{i=0}^{k-1} \lceil \frac{d q^i}{q} \rceil,
\]
where \(\lceil x \rceil\) denotes the smallest integer greater than or equal to \(x\). This inequality is known as the Griesmer bound.

Suppose that \(d \geq 1\) and \(k \geq 2\). Then \(d\) can be written in an unique way as 
\[
d = \theta q^{k-1} - \sum_{i=0}^{k-2} \zeta_i q^i,
\]
with \(\theta \geq 1\) and \(0 \leq \zeta_i \leq q-1\), \(i = 0, 1, \ldots, k-2\). Using this expression for \(d\), the Griesmer bound for an \([N, k, d; q]\)-code can be expressed as 
\[
N \geq \theta v_k - \sum_{i=0}^{k-2} \zeta_i v_{i+1},
\]
defining \(v_i := \frac{q^{i-1}}{q-1}\) and \(v_0 := 0\).

We continue with the definition of a minihyper as found in [8].

Definition 1.1 An \(\{f, m; N, q\}\)-minihyper is a pair \((F, w)\), where \(F\) is a subset of the point set of \(PG(N, q)\) and where \(w\) is a weight function \(w: PG(N, q) \rightarrow \mathbb{N}: x \mapsto w(x)\), satisfying

1. \(w(x) > 0 \iff x \in F\),
2. \(\sum_{x \in F} w(x) = f\), and
3. \(\min\{\sum_{x \in H} w(x) \| H \in \mathcal{H}\} = m\), where \(\mathcal{H}\) is the set of hyperplanes of \(PG(N, q)\).
Let $C$ be an $[N, k, d; q]$-code meeting the Griesmer bound and let $G = (g_1^T \cdots g_n^T)$ be a $(k \times N)$-generator matrix of $C$. Then, for each $i \in \{1, 2, \ldots, N\}$, $g_i$ is a nonzero vector in $V(k, q)$; hence it defines a point $p(g_i) \in \text{PG}(k - 1, q)$. Now define a weight function $w' : \text{PG}(k - 1, q) \rightarrow \mathbb{N}$:

$$w'(p) = |\{i \in \{1, 2, \ldots, N\} : p = p(g_i)\}|$$

If $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \zeta_i q^i$, with $\theta$ and $\zeta_i$, $i = 1, 2, \ldots, k - 2$, as above, then $\max \{w'(p) : p \in \text{PG}(k - 1, q)\} = \theta$. Let $w : \text{PG}(k - 1, q) \rightarrow \mathbb{N} : P \mapsto w(p) = \theta - w'(p)$ be a weight function and let $F = \{p \in \text{PG}(k - 1, q) : w(p) > 0\}$. Then $(F, w)$ is a $\{\sum_{i=0}^{k-2} \zeta_i v_{i+1}; \sum_{i=0}^{k-2} \zeta_i v_i; k - 1, q\}$-minihyper. This relation is clearly expressed in the following theorem.

**Theorem 1.2 (Hamada [?])** Let $q$ be a prime power and let $k, \theta$ and $\zeta_i$, $i = 0, 1, \ldots, k - 2$, be integers satisfying $k \geq 3$, $\theta \geq 1$, $0 \leq \zeta_i \leq q - 1$ and $(\zeta_0, \zeta_1, \ldots, \zeta_{k-2}) \neq (0, 0, \ldots, 0)$. Let $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \zeta_i q^i$. Then there is a one-to-one correspondence between the set of all nonequivalent $[N, k, d; q]$-codes meeting the Griesmer bound and the set of all

$$\left\{\sum_{i=0}^{k-2} \zeta_i v_{i+1}; \sum_{i=0}^{k-2} \zeta_i v_i; k - 1, q\right\} - \text{minihypers } (F, w)$$

satisfying $w(p) \leq \theta$ for each point $p \in \text{PG}(k - 1, q)$.

Both the example and the theorem show clearly that there is an interesting link between minihypers and linear codes meeting the Griesmer bound. From now on, we will consider minihypers as purely geometrical objects.

Minihypers can easily be constructed in $\text{PG}(k - 1, q)$ as a sum of certain geometrical objects. Consider for example an arbitrary set $\mathcal{L}$ of lines of $\text{PG}(k - 1, q)$. The sum of lines of $\mathcal{L}$ is defined as the set of all points contained in a line of $\mathcal{L}$, and the weight of a point $P \in \text{PG}(k - 1, q)$ is the number of lines of $\mathcal{L}$ on $P$. If all lines are two by two skew, then all weights of the points in the induced set $F$ are one. Hence the weight function itself is not necessary to determine the minihyper. We call a minihyper $(F, w)$ a non-weighted minihyper if $w(p) = 1$ for all points $p \in F$. If this condition on the weight function is not satisfied, then $(F, w)$ is called a weighted minihyper and the minihyper is not completely determined by the set $F$. We can construct minihypers using different subspaces of $\text{PG}(k - 1, q)$, e.g., planes, 3-dimensional subspaces \ldots, even subgeometries of $\text{PG}(k - 1, q)$ (if they exist) can be used.

## 2 An application of a known classification result

In this section we describe a known classification result of a certain class of minihypers. We also mention an interesting application of this result in finite geometry. We need the following definition.

**Definition 2.1** A blocking set of $\text{PG}(2, q)$ is a set $B$ of points such that every line of $\text{PG}(2, q)$ meets $B$ in at least one point. A non-trivial blocking set $B$ in $\text{PG}(2, q)$ is a blocking set containing no line of $\text{PG}(2, q)$.

On minihypers with a non-trivial weight function, the following result is known:
Theorem 2.2 (Govaerts and Storme [6]) Let \((F, w)\) be a weighted \(\{\delta v_{i+1}, \delta w_i; k - 1, q\}\)-minihyper, where \(\delta \leq \epsilon\) with \(q + \epsilon + 1\) the size of the smallest non-trivial blocking sets in \(\text{PG}(2, q)\), then \((F, w)\) is a sum of \(\delta\) \(\mu\)-dimensional subspaces of \(\text{PG}(k - 1, q)\).

A lot of other results on weighted and non-weighted minihypers are known. We refer to [5, 6, 2, 3, 4]

Consider now the quadric \(Q(4, q), q\) odd. This quadric contains points and lines but no planes of \(\text{PG}(4, q)\). A spread of \(Q(4, q)\) is a set \(S\) of lines of \(Q(4, q)\) partitioning the pointset. A partial spread is a set \(P\) of two by two skew lines of \(Q(4, q)\). A partial spread is maximal if and only if it cannot be extended to a larger partial spread. Using the result of theorem 2.2, the following result on partial spreads of \(Q(4, q)\) is proved:

Theorem 2.3 Let \(P\) be a partial spread of size \(q^2 + 1 - \delta\). If \(\delta < \epsilon\), with \(q + \epsilon\) the size of the smallest non-trivial blocking sets in \(\text{PG}(2, q), q > 2\). Then \(P\) can be extended to a spread.

To prove this theorem, we prove that there is a link between the set of points of \(Q(4, q)\) that do not lie on a line of the partial spread \(S\) and a \(\{\delta v_{i+1}, \delta w_i; k - 1, q\}\)-minihyper. Then the characterisation of such a minihyper from Theorem 2.2 is used to find lines extending the partial spread \(P\). This result shows clearly that characterisation results on minihypers can have nice applications in projective geometry. For more applications of these results we refer to [1, 7]

The following result concerns non-weighted minihypers.

Theorem 2.4

The aim of this research is to find an analogues characterisation for weighted minihypers

3 New results

In this section we describe a characterisation of

\[
\left\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k - 1, q\right\} - \text{minihypers (} F, w)\right.,
\]

where \(k \geq 3\) and \(\sum_{i=0}^{k-2} \epsilon_i \leq \sqrt{q}\).

We will use several inductive arguments. Therefore, we start with the case \(k = 3\). We prove the following theorem.

Theorem 3.1 Let \((F, w)\) be a \(\{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; k - 1, q\}\)-minihyper with \(\epsilon_0 + \epsilon_1 \leq \sqrt{q} + 1\). Then \((F, w)\) is the sum of \(\epsilon_0\) points and \(\epsilon_1\) lines of \(\text{PG}(k - 1, q)\).

We start with a minihyper in \(\text{PG}(2, q)\), i.e. \(k = 3\). Such a minihyper \((F, w)\) is a weighted \(\epsilon_1\)-fold blocking set of \(\text{PG}(2, q)\), i.e., every line contains at least \(\epsilon_1\) points of \((F, w)\), were a point is counted according to its weight. Since we supposed that \(\epsilon_1 + \epsilon_0 \leq \sqrt{q}\), a result of S. Ball [?] implies that \(F\) contains a complete line \(L\) of \(\text{PG}(2, q)\), i.e. \(w(p) > 0\) for all points \(p \in L\). Any line \(M \neq L\) of \(\text{PG}(2, q)\) intersects \(L\) in exactly one point. Define \(w'(p) = w(p)\) for all points of \(\text{PG}(2, q) \setminus L\) and \(w'(p) := w(p) - 1\) for all points \(p \in L\). The weight function \(w'\) induces a new set of points \(F'\). It is clear that every line \(M \neq L\) contains at least \(\epsilon_1 - 1\) points of \(F'\), but it is unsure if \(L\) meets \(F'\) in at least \(\epsilon_1 - 1\) points.

Using so-called lacunary polynomials, we prove the following lemma:
Lemma 3.2 The set \((F', \omega')\) is a \(\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1-1; k-1, q\}\)-minihyper with 
\[\epsilon_0 + \epsilon_1 - 1 \leq \sqrt{q}\]

This lemma implies that “removing” the line \(L\) from the minihyper \((F, \omega)\), i.e. subtracting 1 from the weight of every point of \(L\), yields a minihyper \((F', \omega')\) containing \(q+1\) points less and satisfying the same conditions of the minihyper \((F, \omega)\). It is clear that the process of removing a line can be repeated, we may conclude the following theorem:

Theorem 3.3 Let \((F, \omega)\) be a \(\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 2, q\}\)-minihyper with \(\epsilon_0 + \epsilon_1 \leq \sqrt{q} + 1\). Then \((F, \omega)\) is the sum of \(\epsilon_0\) points and \(\epsilon_1\) lines of \(\text{PG}(2, q)\).

We consider now a \(\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 3, q\}\)-minihyper \((F, \omega)\). If we project this minihyper from an arbitrary point \(p \notin F\), then a \(\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; 2, q\}\)-minihyper \((F', \omega')\) is obtained. We know that \((F', \omega')\) is a sum of \(\epsilon_1\) lines and \(\epsilon_0\) points of \(\text{PG}(2, q)\). Suppose that \(L'\) is a line of \((F', \omega')\). The plane \((p, L')\) intersects \((F, \omega)\) in a \(\{\mu_1(q+1) + \mu_0, \epsilon_1; 2, q\}\)-minihyper \((F'', \omega'')\), with \(\mu_1 + \mu_0 \leq \sqrt{q}\). This minihyper is again a sum of \(\mu_1\) lines and \(\mu_0\) points. The \(\mu_1\) lines are projected on the line \(L'\), hence, \(L'\) has weight \(\mu_1\) in \((F', \omega')\). We can consider any line in \((F', \omega')\), and we get a total weight of \(\epsilon_1\). Hence \((F, \omega)\) contains \(\epsilon_1\) lines, and, necessary, \((F, \omega)\) is the sum of \(\epsilon_1\) lines and \(\epsilon_0\) points of \(\text{PG}(3, q)\).

We can now repeat this argument for any \(\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; k-1, q\}\)-minihyper with \(\epsilon_0 + \epsilon_1 \leq \sqrt{q} + 1\), using an induction on \(k\), and we conclude Theorem 3.1.

In the next step, we consider a \(\{\epsilon_2(q^2 + q + 1) + \epsilon_1(q+1) + \epsilon_0, \epsilon_1; k-1, q\}\)-minihyper \((F', \omega')\), \(k \geq 4\), \(\epsilon_2 + \epsilon_1 + \epsilon_0 \leq \sqrt{q}\).

We start with a lemma that reveals partially some structure of \((F, \omega)\).

Lemma 3.4 The minihyper \((F, \omega)\) contains a sum of \(\epsilon_2\) planes.

To prove this rather technical lemma, we use the following lemma ([5, Lemma 1.1]).

Lemma 3.5 Consider the minihyper \((F, \omega)\) and suppose that \(H\) is a hyperplane of \(\text{PG}(k-1, q)\) containing \(\epsilon_2(q+1) + \epsilon_1\) points of \((F, \omega)\). Then \((F, \omega)\) induces a \(\{\epsilon_2(q+1) + \epsilon_1, \epsilon_1; k-2, q\}\)-minihyper \((F', \omega')\) contained in \(H\).

By Theorem 3.3 we know that \((F', \omega')\) is the sum of \(\epsilon_2\) lines and \(\epsilon_1\) points. This enables us, using Theorem 3.3 inductively, and, using another technical lemma from [5], to prove the lemma in several steps. We end this part with a new lemma, which finally characterizes the minihyper \((F, \omega)\).

Lemma 3.6 If \((F, \omega)\) is a \(\{\epsilon_2(q^2 + q + 1) + \epsilon_1(q+1) + \epsilon_0, \epsilon_1; k-1, q\}\)-minihyper, \(k \geq 4\), where \(\epsilon_2 + \epsilon_1 + \epsilon_0 \leq \sqrt{q}\), containing a sum of \(\epsilon_2\) planes, then \((F, \omega)\) is a sum of \(\epsilon_2\) planes, \(\epsilon_1\) lines and \(\epsilon_0\) points.

At this point, we have a theorem for arbitrary \(k \geq 3\), but with the restriction that \(\epsilon_j = 0\), for all \(j > 2\). As induction hypothesis we assume now that any

\[
\left\{\sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=0}^{t-1} \epsilon_i v_i; k-1, q\right\} \text{ is a minihyper,}
\]

where \(t-1 < k-2\), \(\sum_{i=0}^{t-1} \epsilon_i \leq \sqrt{q}\) is a sum of \(\epsilon_{t-1}\) \((t-1)\)-dimensional subspaces, \(\epsilon_{t-2}\) \((t-2)\)-dimensional subspaces, \(\ldots\), \(\epsilon_1\) \((1)\)-dimensional subspaces (lines) and \(\epsilon_0\) \((0)\)-dimensional subspaces (points). We suppose that \((F, \omega)\) is a

\[
\left\{\sum_{i=0}^{t} \epsilon_i v_{i+1}, \sum_{i=0}^{t} \epsilon_i v_i; k-1, q\right\} \text{ is a minihyper,}
\]
where \( t \leq k - 2, \sum_{i=0}^{t} \epsilon_i \leq \sqrt{q} \). The induction hypothesis is crucial to prove the next lemma, but quite some technical steps are also needed.

**Lemma 3.7** The minihyper \((F, w)\) contains a sum of \( \epsilon_t \) \( t \)-dimensional subspaces of \( \text{PG}(k-1, q) \).

Having this lemma, which is very similar to Lemma 3.4, we can now prove the final theorem

**Theorem 3.8** If \((F, w)\) is a

\[
\left\{ \sum_{i=0}^{t} \epsilon_{i}v_{i+1}, \sum_{i=0}^{t} \epsilon_{i}v_{i}; k - 1, q \right\} - \text{minihyper},
\]

where \( k \geq 3, t \leq k - 2 \) and \( \sum_{i=0}^{t} \epsilon_{i} \leq \sqrt{q} \), then \((F, w)\) is a sum of \( \epsilon_t \) \((t)\)-dimensional subspaces, \( \epsilon_{t-1} \) \((t-1)\)-dimensional subspaces, \( \ldots \), \( \epsilon_1 \) \((1)\)-dimensional subspaces (lines) and \( \epsilon_0 \) \((0)\)-dimensional subspaces (points) of \( \text{PG}(k-1, q) \).

**References**


