On stability theorems in finite geometry

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Stability in mathematics

- structure with parameters (e.g. size)
- bound on the parameter(s)
- example(s) meeting the bound
- Stability: what is known if an example is “close” to an extremal case?
- Spectrum: second, third, etc. smallest/largest example

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An example in set theory

- $A := \{1, 2, \ldots, n\}$,
- $\mathcal{F} \subseteq 2^A$,
- $F \in \mathcal{F} \Rightarrow |F| = k$; k fixed, $2k < n$,
- $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \neq \emptyset$

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Let $\mathcal{K}$ be the set of all $k-1$ subsets of A not containing a given element of A, say 1. Define

$$\mathcal{F} = \{\{1\} \cup K \| K \in \mathcal{K}\}$$

Then $\mathcal{F}$ is an extremal example.

Theorem (Hilton-Milner)

- *The above example is the unique extremal example.*
- $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-1}{n-k-1} + 1$ *when* $\bigcap \mathcal{F} = \emptyset$.
- *example: (recall: $2k < n$)*
  $\mathcal{F}' := \mathcal{F} \setminus \{F \| F \cap \{2,3,\ldots,k+1\} = \emptyset\} \cup \{2,3,\ldots,k+1\}$

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Arcs and Segre

- Suppose S is a set of vectors in $V(k,q)$, $q = p^h$ with the property that every subset of size k is a basis

Theorem (Bose)

If $p \geq k = 3$, then $|S| \leq q + 1$

Theorem (Segre)

If $p \geq k = 3$, and $|S| = q + 1$, then S is a normal rational curve

- Going the the projective space $\mathrm{PG}(k-1, q)$, we talk about *arcs*.
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Arcs and Segre

Theorem (Segre)

If $\mathcal{K}$ is arc in $\mathrm{PG}(2,q)$ with $|\mathcal{K}| \geq q - \sqrt{q} + 1$ when q is even and $|\mathcal{K}| \geq q - \sqrt{q}/4 + 7/4$ when q is odd, then $\mathcal{K}$ is contained in an arc of maximum size (that is, in an oval or hyperoval).

MDS-conjecture

Conjecture

$|S|$ has size at most $q+1$ when q is odd, unless q is even, $k=3$ or $k=q-1$, then $|S|$ has size at most $q+2$.

Theorem (Ball)

$|S|$ has size at most $q+k+1-\min(k,p)$, where $k \leq q$.

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An example in finite geometry

- Consider $V(2, \mathrm{GF}(q)) = \mathrm{AG}(2, q)$.
- Suppose $v_1, v_2 \in \mathrm{AG}(2, q)$, denote $v_1 = (x_1, y_1), v_2 = (x_2, y_2)$. Define $d := \langle x_1 - x_2, y_1 - y_2 \rangle$.
- There are $q + 1$ *directions*: $\{(0, 1)\} \cup \{(1, x) \| x \in \mathrm{GF}(q)\}$.
- Any pointset $A \subseteq \mathrm{AG}(2, q)$ of size at least $q + 1$ determines **all** directions.

Theorem (Szőnyi)

A set of $q - k > q - \sqrt{q}/2$ points of $\mathrm{AG}(2, q)$ which does not determine a set $\mathcal{D}$, of more than $(q + 1)/2$ directions, can be extended to a set of q points not determining the set of directions $\mathcal{D}$.

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partitioning $V(2t+2, \mathrm{GF}(q))$

- Consider the vector space $V(2t+2, q)$
- Partition the set of non-zero vectors by $t+1$-dimensional sub vector spaces?

- $V(2t+2, q) \setminus \{0\}, \cdot = \mathrm{GF}(q^{2t+2}) \setminus \{0\}, \cdot =: L$, $K := \mathrm{GF}(q^2) \setminus \{0\}$,
- $\mathcal{S} := \{tK \| t \in \mathrm{GF}(q^{2t+2})\}$, i.e. the cosets of $K \subset L$,
- Alle elements of $\mathcal{S}$ are $\mathrm{GF}(q)$ vector spaces, sharing no element of $V(2t+2, q) \setminus \{0\}$
- This is the standard example of a partition, clearly $|\mathcal{S}| = \frac{q^{2t+2}-1}{q^2-1}$.

Going from $V(2t+2, q)$ to $\mathrm{PG}(2t+1, q)$, we call $\mathcal{S}$ a spread of $\mathrm{PG}(2t+1, q)$.

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partitioning a symplectic space

- We stick to $t = 1$, $V = V(4, q)$.
- Consider an non-degenerate alternating form $f : V(4, q) \to \mathrm{GF}(q)$, i.e. $f(x, x) = 0$ for any vector x, and $\mathrm{Rad}(f) = \{0\}$.
- e.g. $f(x, y) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$
- Can we partition $V(4, q) \setminus \{0\}$ now using vector planes that are *totally isotropic* with relation to f.

Going from $V(4, q)$ to $\mathrm{PG}(3, q)$, we denote (V, f) as $\mathrm{W}(3, q)$, and call it the symplectic polar space of rank 2.

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stability for spreads of $\mathrm{PG}(2t+1,q)$

- $\mathcal{S}$ is a partial t spread if it consists of mutually skew t-dimensional subspaces of $\mathrm{PG}(2t+1,q)$, $|\mathcal{S}| = \frac{q^{2t+2}-1}{q^2-1} - \delta$
- $\mathcal{S}$ is *maximal* if no t-dimensional subspace of $\mathrm{PG}(2t+1,q)$ is skew to all elements of $\mathcal{S}$.

Theorem (Metsch)

A maximal partial t-spread in $\mathrm{PG}(2t+1,q)$, q non square, with deficiency $\delta > 0$ satisfies $8\delta^3 - 18\delta^2 + 8\delta + 4 \geq 3q^2$

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stability for spreads of $W(3, q)$

Theorem (Brown, DB, Storme)

Suppose that $\mathcal{S}$ is a maximal partial spread of $W(3, q)$, q even, with deficiency $\delta > 0$. Then $\delta \geq q - 1$. This bound is sharp, i.e., examples of size $q^2 - q$ exist.

Theorem (Govaerts, Storme, Van Maldeghem)

Suppose that $\mathcal{S}$ is a spread of $W(3, q)$ with deficiency $0 < \delta < \sqrt{q}$. Then δ must be even.

Corollary

A partial spread of $W(3, q)$ of size q^2 can always be extended to a spread.

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stability for spreads of $\mathrm{W}(3, q)$

- First open case: do maximal partial spreads of size $q^2 - 1$ of $\mathrm{W}(3, q)$, q odd, exist?
- This is a huge difference with the $\mathrm{PG}(3, q)$ case.

A direction problem in $\mathrm{AG}(3, q)$

We define a graph $\Gamma = (V, E)$

- Set V of vertices := points of $\mathrm{AG}(3, q)$
- Choose a fixed set of directions D
- Define $x, y \in V$ adjacent if and only if $\langle x - y \rangle \notin D$.

Lemma

A maximal partial spread of $\mathrm{W}(3, q)$ of size $q^2 - 1$ is equivalent to a maximal clique of size $q^2 - 2$ in Γ if D is a conic.

A direction problem in $\mathrm{AG}(3, q)$

Theorem

A maximal partial spread of $\mathrm{W}(3, q)$, $q = p^h$, p odd prime, does not exist if $h > 1$

- Open case: $h = 1$, known examples for $p \in \{3, 5, 7, 11\}$, but not for larger values.
- Known examples can be constructed from a subgroup of size $q^2 - 1$ of $\mathrm{PSL}(2, q)$.

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Jaeger's conjecture

Conjecture

for all matrices $X \in \mathrm{GL}(n, q)$, there exists a vector $y \in \mathrm{GF}(q)^n$ with the property that y and Xy have no zero coordinate.

true for q a non-prime