# An application of $\{\delta(q+1),\delta;n+1,q\}$-minihypers on generalized quadrangles

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# 1. Introduction

$\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a finite generalized quadrangle if:

(i) Each point is incident with $1+t$ lines $(t \geqslant 1)$ and two distinct points are incident with at most one line.

(ii) Each line is incident with $1+s$ points $(s \geqslant 1)$ and two distinct lines are incident with at most one point.

(iii) If $x$ is a point and $L$ is a line not incident with $x$, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathrm{I} M \mathrm{I} y \mathrm{I} L$.

The integers $s$ and $t$ are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* $(s, t)$. If $s = t$, then $\mathcal{S}$ is said to have order $s$.

# classical examples

1. Quadrics in $\mathrm{PG}(n, q)$.

(a) $n = 3$: $\mathrm{Q}^+(3, q)$: $X_0X_1 + X_2X_3 = 0$, $s = q$, $t = 1$

(b) $n = 4$: $\mathrm{Q}(4, q)$: $X_0^2 + X_1X_2 + X_3X_4 = 0$, $s = t = q$

(c) $n = 5$: $\mathrm{Q}^-(5, q)$: $f(X_0, X_1) + X_2X_3 + X_4X_5 = 0$, $f(X_0, X_1)$ irreducible
$s = q$, $t = q^2$

# classical examples

2. Hermitian varieties in $\mathrm{PG}(n, q)$.
$n = 3, 4$: $\mathrm{H}(n, q^2)$: $X_0^{q+1} + \ldots + X_n^{q+1} = 0$
$n = 3$: $s = q^2$, $= q$
$n = 4$: $s = q^2$, $= q^3$

3. $\mathrm{W}_3(q)$: symplectic polarity of $\mathrm{PG}(3, q)$, $s = t = q$.

# non classical examples of Tits

(i) the points of $\mathrm{PG}(n+1,q) \setminus \mathrm{PG}(n,q)$,

(ii) the hyperplanes $X$ of $\mathrm{PG}(n+1,q)$ for which $|X \cap \mathcal{O}| = 1$, and

(iii) one new symbol $(\infty)$.

(a) the lines of $\mathrm{PG}(n+1,q)$ which are not contained in $\mathrm{PG}(n,q)$ and which meet $\mathcal{O}$ (necessarily in a unique point), and

(b) the points of $\mathcal{O}$.

Incidence is inherited from $\mathrm{PG}(n+1,q)$, whereas the point $(\infty)$ is incident with no line of type (a) and with all lines of type (b).

# spreads

- A *spread* of a GQ is a set of lines which partition the point set
- A *partial spread* of a GQ is a set of lines such that every point of the GQ lies on at most one line.
- A *maximal partial spread* is a partial spread which is not contained in a bigger partial spread.

# 2. Minihypers

A $\{f, m; N, q\}$ minihyper is a pair $(F, w)$, where $F$ is subset of the point set of $\mathrm{PG}(N, q)$ and $w$ is a weight function $w : \mathrm{PG}(N, q) \to \mathbb{N} : x \mapsto w(x)$, satisfying

(i) $w(x) > 0 \iff x \in F$

(ii) $\sum_{x \in F} = f$

(iii) min $\{\sum_{x \in H} w(x) \| H \in \mathcal{H}\} = m$, where $\mathcal{H}$ is the set of hyperplanes of $\mathrm{PG}(N, q)$.

# minihypers: the theorems

**Theorem 1.** *(Govaerts, Storme) Let* $(F, w)$ *be a* $\{\delta(q+1), \delta; N, q\}$*-minihyper,* $q > 2$*, satisfying* $0 \leqslant \delta < \epsilon$*, where* $q + \epsilon$ *is the size of the smallest non trivial blocking set in* $\mathrm{PG}(2, q)$*. Then* $w$ *is the weight function induced on the points of* $\mathrm{PG}(N, q)$ *by a sum of* $\delta$ *lines.*

**Theorem 2.** *(Blokhuis, Storme, Szőnyi) Let* $B$ *be a blocking set in* $\mathrm{PG}(2, q)$*,* $q = p^h$*, prime of size* $q + 1 + c$*. Let* $c_2 = c_3 = 2^{\frac{-1}{3}}$ *and* $c_p = 1$ *for* $p > 3$

*(i) If* $q = p^{2d+1}$ *and* $c < c_p q^{\frac{2}{3}}$ *then* $B$ *contains a line*

*(ii) If* $4 < q$ *and* $q$ *is a square an* $c < c_p q^{\frac{2}{3}}$*, then* $B$ *contains a line or a Baer subplane*

# 3. The Application

**Lemma 1.** *Let $S$ be a partial spread of $T_n(\mathcal{O})$ ($n = 2$ or $3$) which covers $(\infty)$ and which has deficiency $\delta < q$. Then $w_S$ is the weight function of a $\{\delta(q+1), \delta; n+1, q\}$-minihyper $(F, w_S)$.*

**Theorem 3.** *(*M.R. Brown, J. De Beule and L. Storme *) Let $S$ be a partial spread with deficiency $\delta$ of $T_n(\mathcal{O})$ ($n = 2$ or $3$) covering $(\infty)$. If $\delta < \epsilon$, with $q + \epsilon$ the size of the smallest non-trivial blocking set in $\mathrm{PG}(2, q)$, $q > 2$, we can always extend $S$ to a spread.*