The Hermitian variety $H(5,4)$ has no ovoid

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Introduction

The Hermitian variety $\mathrm{H}(d, q^2)$ is the set of points of $\mathrm{PG}(d, q^2)$ satisfying the equation

$$X_0^{q+1} + X_1^{q+1} + \ldots X_d^{q+1} = 0$$

When $d = 2n + 1$, $2n$ respectively, $\mathrm{H}(d, q^2)$ contains points, lines, . . . , n-dimensional subspaces of $\mathrm{PG}(d, q^2)$, $(n - 1)$-dimensional subspaces of $\mathrm{PG}(d, q^2)$ respectively.

The Hermitian variety $\mathrm{H}(d, q^2)$ is a example of a so-called classical polar space. The subspaces of maximal dimension are also called generators.

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Ovoids

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If $\mathrm{H}(d-2, q^2)$ has no ovoids, then $\mathrm{H}(d, q^2)$ has no ovoids.

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Known results

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$$p^{2n+1} > \binom{2n+p}{2n+1}^2 - \binom{2n+p-1}{2n+1}^2$$

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A. Klein: $\mathrm{H}(2n+1, q^2)$ has no ovoids if $n > q^3$.

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Ovoids of $\mathrm{H}(3,4)$

Suppose that $\mathcal{O}$ is an ovoid of $\mathrm{H}(3,4)$. There exists a plane π, $\pi \cap \mathrm{H}(3,4) = \mathrm{H}(2,4)$, such that either

- $\pi \cap \mathrm{H}(3,4) = \mathrm{H}(2,4) = \mathcal{O}$, or
- $\mathcal{O} = (\mathrm{H}(2,4) \setminus L) \cup (L^{\perp} \cap \mathrm{H}(3,4))$, L a line of π, $L \cap \mathrm{H}(3,4) = \mathrm{H}(1,4)$.

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If π is a plane, $|\pi \cap \mathcal{O}| = 3$, then the points of $\pi \cap \mathcal{O}$ are collinear.

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Ovoids of $\mathrm{H}(5,4)$

Suppose that $\mathcal{O}$ is an ovoid of $\mathrm{H}(5,4)$. Let p be a point of $\mathrm{H}(5,4) \setminus \mathcal{O}$. Then $|p^\perp \cap \mathcal{O}| = 9$. If π is a plane in $p^\perp$, $\pi \cap H(5,4) = \mathrm{H}(2,4)$, then $|\langle p, \pi \rangle| \in \{0, 1, 2, 3, 6, 9\}$

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Suppose that $\mathcal{O}$ is an ovoid of $H(5,q^2)$. Consider a plane π that meets the variety in $\mathrm{H}(2,q^2)$ and put $m := |\pi \cap \mathcal{O}|$. Suppose furthermore that $1 \leq m < q^3+1$. Let A, resp. B, be the set consisting of all points $x \in \mathcal{O} \setminus \pi$ such that $\langle \pi, x \rangle$ meets $\mathrm{H}(5,q^2)$ in a cone $s\mathrm{H}(2,q^2)$, resp. an $\mathrm{H}(3,q^2)$.

- We have $|A| = (q^2-1)(q^2-1+m)$ and $|B| = q^2(q^3-q^2+2-m)$.
- If $q=2$ and x is a point of $(\pi \cap \mathrm{H}(5,4)) \setminus \mathcal{O}$, then $|x^\perp \cap B| \in \{0,3,6,7,8,9\}$.

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The last steps

Suppose that $\mathcal{O}$ is an ovoid of $\mathrm{H}(5,4)$. Then $|\pi \cap \mathcal{O}| \leq 3$ for every plane π, $\pi \cap \mathrm{H}(5,4) = \mathrm{H}(2,4)$ and $|\alpha \cap \mathcal{O}| < 6$ for every 3-dimensional space α, $\alpha \cap \mathrm{H}(5,4) = \mathrm{H}(3,4)$.

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