Maximal partial ovoids of $\mathrm{Q}(4, q)$ of size $q^2 - 1$

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Finite Generalized Quadrangles

A finite generalized quadrangle (GQ) is a point-line geometry $\mathcal{S} = \mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ such that

(i) Each point is incident with $1 + t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $1 + s$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \mathrm{~I~} M \mathrm{~I~} y \mathrm{~I~} L$.

- Finite classical GQs: associated to sesquilinear or quadratic forms on a vectorspace over a finite field of Witt index two.
- $\mathrm{Q}(4, q)$: set of points of $\mathrm{PG}(4, q)$ satisfying

$$X_0^2 + X_1X_2 + X_3X_4 = 0$$

- Complete lines of $\mathrm{PG}(4, q)$ are contained in this point set, but no planes ...
- ...these points and lines constitute a GQ of order q.

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Ovoids and partial ovoids

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An *ovoid* of a GQ $\mathcal{S}$ is a set $\mathcal{O}$ of points of $\mathcal{S}$ such that every line of $\mathcal{S}$ contains exactly one point of $\mathcal{O}$.

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Existence

- $\mathrm{Q}(4,q)$ has always ovoids.
- partial ovoids of size q^2 can always be extended to an ovoid
- We are interested in partial ovoids of size q^2-1 ...
- ...which exist for $q=3,5,7,11$ and which do not exist for $q=9$.
- When q is even, maximal partial ovoids of size q^2-1 do not exist.

Theorem (Payne and Thas)

Let $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ be a GQ of order (s,t). Any $(st-\rho)$-partial ovoid of $\mathcal{S}$ with $0\leq\rho<\frac{t}{s}$ is contained in an uniquely defined ovoid of $\mathcal{S}$.

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The GQ $T_2(\mathcal{C})$

Definition

An oval of $\mathrm{PG}(2,q)$ is a set of $q+1$ points $\mathcal{C}$, such that no three points of $\mathcal{C}$ are collinear.

Let $\mathcal{C}$ be an oval of $\mathrm{PG}(2,q)$ and embed $\mathrm{PG}(2,q)$ as a hyperplane in $\mathrm{PG}(3,q)$. We denote this hyperplane with π_∞. Define points as

(i) the points of $\mathrm{PG}(3,q) \setminus \mathrm{PG}(2,q)$,
(ii) the hyperplanes π of $\mathrm{PG}(3,q)$ for which $|\pi \cap \mathcal{C}| = 1$, and
(iii) one new symbol (∞).

Lines are defined as

(a) the lines of $\mathrm{PG}(3,q)$ which are not contained in $\mathrm{PG}(2,q)$ and meet $\mathcal{C}$ (necessarily in a unique point), and
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$T_2(\mathcal{C})$ and $\mathrm{Q}(4,q)$

Theorem

When $\mathcal{C}$ is a conic of $\mathrm{PG}(2,q)$, $T_2(\mathcal{C}) \cong \mathrm{Q}(4,q)$.

Theorem

All ovals of $\mathrm{PG}(2,q)$ are conics, when q is odd.

Corollary

When q is odd, $T_2(\mathcal{C}) \cong \mathrm{Q}(4,q)$.

Suppose now that q is odd and $\mathcal{O}$ is a partial ovoid of $\mathrm{Q}(4,q) \cong T_2(\mathcal{C})$. We may assume that $(\infty) \in \mathcal{O}$.
If $\mathcal{O}$ has size k, then $\mathcal{O} = \{(\infty)\} \cup U$, where U is a set of $k-1$ points of type (i).

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Directions in $\mathrm{AG}(3, q)$

- U set of affine points, not determining $q+1$ points at infinity.
- Suppose that $|U| = q^2 - 2$, can U be extended, such that none of the given directions is determined?
- Denote by D the set of directions determined by U, denote by O the set of points $\pi_\infty \setminus D$.

Classical theorems

Proposition

$q+1$ points of $\mathrm{AG}(2,q)$ determine all directions.

Theorem (Szőnyi)

Suppose that $\mathcal{S}$ is a set of points of $\mathrm{AG}(2,q)$, $|\mathcal{S}| \geq q - \sqrt{q}/2$, determining at most $\frac{q-1}{2}$ directions. Then $|\mathcal{S}|$ can be extended to a set of q points determining the same directions

Theorem (Rédei)

A set of p points of $\mathrm{AG}(2,p)$, p prime, not on a line, determines at least $\frac{p+3}{2}$ directions.

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The Rédei polynomial

Choose $\pi_\infty : X_3 = 0$. Set
$U = \{(a_i, b_i, c_i, 1) : i = 1, \ldots, k\} \subset \mathrm{AG}(3,q)$, then
$D = \{(a_i - a_j, b_i - b_j, c_i - c_j, 0) : i \neq j\}$
Define

$$R(X,Y,Z,W) = \prod_{i=1}^{k}(X + a_iY + b_iZ + c_iW)$$

then

$$R(X,Y,Z,W) = X^k + \sum_{i=1}^{k} \sigma_i(Y,Z,W)X^{k-i}$$

with $\sigma_i(X,Y,Z)$ the i-th elementary symmetric polynomial of the set $\{a_iY + b_iZ + c_iW | i = 1 \ldots k\}$.

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The Rédei polynomial

Lemma

For any $x, y, z, w \in GF(q)$, $(y, z, w) \neq (0, 0, 0)$, the multiplicity of $-x$ in the multi-set $\{ya_i + zb_i + wc_i : i = 1, \ldots, k\}$ is the same as the number of common points of U and the plane $yX_0 + zX_1 + wX_2 + xX_3 = 0$.

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We may assume that $\sum a_i = \sum b_i = \sum c_i = 0$, implying $\sigma_1(X, Y, Z) = 0$.

Consider a line L in π_∞:

$$L : yX_0 + zX_1 + wX_2 = X_3 = 0$$

Suppose that $L \cap O \neq \emptyset$ then $R(X, y, z, w)(X^2 - \sigma_2(y, z, w)) = (X^q - X)^q$.

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Relations for σ

Define

$$S_k(Y,Z,W) = \sum_i (a_i Y + b_i Z + c_i W)^k$$

Lemma

If the line with equation $yX_0 + zX_1 + wX_2 = X_3 = 0$ has at least one common point with O, then $S_k(y,z,w) = 0$ for odd k and $S_k(y,z,w) = -2\sigma_2^{k/2}(y,z,w)$ for even k.

The result for q non prime

Theorem

If $|U| = q^2 - 2$, $q = p^h$ and $|O| \geq p + 2$, then U can be extended by two points to a set of q^2 points determining the same directions.

A property of $(q^2 - 1)$-partial ovoids

Theorem

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, t). Let $\mathcal{K}$ be a maximal partial ovoid of size $st - \frac{t}{s}$ of $\mathcal{S}$. Let $\mathcal{B}'$ be the set of lines incident with no point of $\mathcal{K}$, and let $\mathcal{P}'$ be the set of points on at least one line of $\mathcal{B}'$ and let I' be the restriction of I to points of $\mathcal{P}'$ and lines of $\mathcal{B}'$. Then $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is a subquadrangle of order $(s, \rho = \frac{t}{s})$.

Corollary

Suppose that $\mathcal{O}$ is a maximal $(q^2 - 1)$-partial ovoid of $\mathrm{Q}(4, q)$, then the lines of $\mathrm{Q}(4, q)$ not meeting $\mathcal{O}$ are the lines of a hyperbolic quadric $\mathrm{Q}^+(3, q) \subset \mathrm{Q}(4, Q)$.

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Elements of $\mathrm{SL}(2, q)$

- $\mathrm{Q}(4, q)$: $X_1X_3 - X_2X_4 = X_0^2$.
- $\pi : X_0 = O$ intersects $\mathrm{Q}(4, q)$ in a hyperbolic quadric
- If $P(x_0, x_1, x_2, x_3, x_4) \in \mathcal{O}$, then $x_1x_3 - x_2x_4 = 1$.
- Elements of $\mathcal{O}$ are elements of $\mathrm{SL}(2, q)$.
- Question: does the set of elements of $\mathcal{O}$ constitute a subgroup of $\mathrm{SL}(2, q)$?

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