

Maximal partial ovoids of $Q(4, q)$ of size $q^2 - 1$

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Finite Generalized Quadrangles

A finite generalized quadrangle (GQ) is a point-line geometry $\mathcal{S} = \mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ such that

- (i) Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line.
- (ii) Each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point.
- (iii) If x is a point and L is a line not incident with x , then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x I M I y I L$.

- Finite classical GQs: associated to sesquilinear or quadratic forms on a vectorspace over a finite field of Witt index two.
- $Q(4, q)$: set of points of $PG(4, q)$ satisfying

$$X_0^2 + X_1X_2 + X_3X_4 = 0$$

- Complete lines of $PG(4, q)$ are contained in this point set, but no planes . . .
- . . . these points and lines constitute a GQ of order q .

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Ovoids and partial ovoids

Definition

An *ovoid* of a GQ \mathcal{S} is a set \mathcal{O} of points of \mathcal{S} such that every line of \mathcal{S} contains exactly one point of \mathcal{O} .

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A *partial ovoid* of a GQ \mathcal{S} is a set \mathcal{O} of points of \mathcal{S} such that every line of \mathcal{S} contains at most one point of \mathcal{S} . A partial ovoid is *maximal* if it cannot be extended to a larger partial ovoid.

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Existence

- $Q(4, q)$ has always ovoids.
- partial ovoids of size q^2 can always be extended to an ovoid
- We are interested in partial ovoids of size $q^2 - 1 \dots$
- \dots which exist for $q = 3, 5, 7, 11$ and which do not exist for $q = 9$.
- When q is even, maximal partial ovoids of size $q^2 - 1$ do not exist.

Theorem (Payne and Thas)

Let $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a GQ of order (s, t) . Any $(st - \rho)$ -partial ovoid of S with $0 \leq \rho < \frac{t}{s}$ is contained in an uniquely defined ovoid of S .

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The GQ $T_2(\mathcal{C})$

Definition

An oval of $PG(2, q)$ is a set of $q + 1$ points \mathcal{C} , such that no three points of \mathcal{C} are collinear.

Let \mathcal{C} be an oval of $PG(2, q)$ and embed $PG(2, q)$ as a hyperplane in $PG(3, q)$. We denote this hyperplane with π_∞ . Define points as

- (i) the points of $PG(3, q) \setminus PG(2, q)$,
- (ii) the hyperplanes π of $PG(3, q)$ for which $|\pi \cap \mathcal{C}| = 1$, and
- (iii) one new symbol (∞).

Lines are defined as

- (a) the lines of $PG(3, q)$ which are not contained in $PG(2, q)$ and meet \mathcal{C} (necessarily in a unique point), and
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$T_2(\mathcal{C})$ and $Q(4, q)$

Theorem

When \mathcal{C} is a conic of $PG(2, q)$, $T_2(\mathcal{C}) \cong Q(4, q)$.

Theorem

All ovals of $PG(2, q)$ are conics, when q is odd.

Corollary

When q is odd, $T_2(\mathcal{C}) \cong Q(4, q)$.

Suppose now that q is odd and \mathcal{O} is a partial ovoid of $Q(4, q) \cong T_2(\mathcal{C})$. We may assume that $(\infty) \in \mathcal{O}$.

If \mathcal{O} has size k , then $\mathcal{O} = \{(\infty)\} \cup U$, where U is a set of $k - 1$ points of type (i).

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Directions in $AG(3, q)$

- U set of affine points, not determining $q + 1$ points at infinity.
- Suppose that $|U| = q^2 - 2$, can U be extended, such that none of the given directions is determined?
- Denote by D the set of directions determined by U , denote by O the set of points $\pi_\infty \setminus D$.

Classical theorems

Proposition

$q + 1$ points of $AG(2, q)$ determine all directions.

Theorem (Szőnyi)

Suppose that S is a set of points of $AG(2, q)$, $|S| \geq q - \sqrt{q}/2$, determining at most $\frac{q-1}{2}$ directions. Then $|S|$ can be extended to a set of q points determining the same directions

Theorem (Rédei)

A set of p points of $AG(2, p)$, p prime, not on a line, determines at least $\frac{p+3}{2}$ directions.

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The Rédei polynomial

Choose $\pi_\infty : X_3 = 0$. Set

$U = \{(a_i, b_i, c_i, 1) : i = 1, \dots, k\} \subset AG(3, q)$, then

$D = \{(a_i - a_j, b_i - b_j, c_i - c_j, 0) : i \neq j\}$

Define

$$R(X, Y, Z, W) = \prod_{i=1}^k (X + a_i Y + b_i Z + c_i W)$$

then

$$R(X, Y, Z, W) = X^k + \sum_{i=1}^k \sigma_i(Y, Z, W) X^{k-i}$$

with $\sigma_i(X, Y, Z)$ the i -th elementary symmetric polynomial of the set $\{a_i Y + b_i Z + c_i W \mid i = 1 \dots k\}$.

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The Rédei polynomial

Lemma

For any $x, y, z, w \in GF(q)$, $(y, z, w) \neq (0, 0, 0)$, the multiplicity of $-x$ in the multi-set $\{ya_i + zb_i + wc_i : i = 1, \dots, k\}$ is the same as the number of common points of U and the plane $yX_0 + zX_1 + wX_2 + xX_3 = 0$.

The Rédei polynomial

We may assume that $\sum a_i = \sum b_i = \sum c_i = 0$, implying $\sigma_1(X, Y, Z) = 0$.

Consider a line L in π_∞ :

$$L : yX_0 + zX_1 + wX_2 = X_3 = 0$$

Suppose that $L \cap O \neq \emptyset$ then

$$R(X, y, z, w)(X^2 - \sigma_2(y, z, w)) = (X^q - X)^q.$$

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Relations for σ

Define

$$S_k(Y, Z, W) = \sum_i (a_i Y + b_i Z + c_i W)^k$$

Lemma

If the line with equation $yX_0 + zX_1 + wX_2 = X_3 = 0$ has at least one common point with O , then $S_k(y, z, w) = 0$ for odd k and $S_k(y, z, w) = -2\sigma_2^{k/2}(y, z, w)$ for even k .

The result for q non prime

Theorem

If $|U| = q^2 - 2$, $q = p^h$ and $|O| \geq p + 2$, then U can be extended by two points to a set of q^2 points determining the same directions.

A property of $(q^2 - 1)$ -partial ovoids

Theorem

Let $S = (\mathcal{P}, \mathcal{B}, \mathbf{I})$ be a GQ of order (s, t) . Let \mathcal{K} be a maximal partial ovoid of size $st - \frac{t}{s}$ of S . Let \mathcal{B}' be the set of lines incident with no point of \mathcal{K} , and let \mathcal{P}' be the set of points on at least one line of \mathcal{B}' and let \mathbf{I}' be the restriction of \mathbf{I} to points of \mathcal{P}' and lines of \mathcal{B}' . Then $S' = (\mathcal{P}', \mathcal{B}', \mathbf{I}')$ is a subquadrangle of order $(s, \rho = \frac{t}{s})$.

Corollary

Suppose that \mathcal{O} is a maximal $(q^2 - 1)$ -partial ovoid of $Q(4, q)$, then the lines of $Q(4, q)$ not meeting \mathcal{O} are the lines of a hyperbolic quadric $Q^+(3, q) \subset Q(4, Q)$.

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Elements of $SL(2, q)$

- $Q(4, q): X_1X_3 - X_2X_4 = X_0^2$.
- $\pi : X_0 = 0$ intersects $Q(4, q)$ in a hyperbolic quadric
- If $P(x_0, x_1, x_2, x_3, x_4) \in \mathcal{O}$, then $x_1x_3 - x_2x_4 = 1$.
- Elements of \mathcal{O} are elements of $SL(2, q)$.
- Question: does the set of elements of \mathcal{O} constitute a subgroup of $SL(2, q)$?

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