Characterization of certain weighted t-fold blocking sets of $PG(n, q)$, and an application

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Combinatorics 2008

Blocking sets

Definition

A blocking set of $\mathrm{PG}(2, q)$ is a set B of points such that every line of $\mathrm{PG}(2, q)$ contains at least one point of B. A blocking set is called minimal if no point of B can be removed.

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A blocking set is called non-trivial when it contains no line of $\mathrm{PG}(2, q)$.

Theorem

(A. Bruen) *If B is a non-trivial blocking set of $\mathrm{PG}(2, q)$ then $|B| \geq q + \sqrt{q} + 1$.*

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t-fold blocking sets

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Theorem (A. Bruen)

If B is a t-fold blocking set, $t > 1$, and B contains a line, then $|B| \geq tq + q - t + 2$.

Theorem (S. Ball)

If B is a t-fold blocking set, $t > 1$, and B contains no line, then $|B| \geq tq + \sqrt{tq} + 1$.

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Weighted t-fold blocking sets in $\mathrm{PG}(2, q)$

Definition

A weighted t-fold blocking set (B, w) of $\mathrm{PG}(2, q)$ is a set B of points and a weight function w, such that every line of $\mathrm{PG}(2, q)$ contains at least t points of B, each point counted according to its weight.

Theorem (JDB, K. Metsch, L. Storme)

If (B, w) is a weighted t-fold blocking of $\mathrm{PG}(2, q)$, $1 \leq t < q - 1$, $q \geq 3$, then B contains a line or $|(B, w)| \geq tq + \sqrt{tq} + 1$ (with $|(B, w)| = \sum_{x \in B} w(x)$).

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Weighted t-fold blocking sets in $\mathrm{PG}(2, q)$

Theorem (JDB, K. Metsch, L. Storme)

Suppose that (B, w) is an ϵ_1-fold blocking set (B, w) of $\mathrm{PG}(2, q)$, $q \geq 4$, with

$$|(B, w)| = \epsilon_1(q + 1) + \epsilon_0,$$

$$\epsilon_0 + \epsilon_1 < \sqrt{q} + 1,$$

then (B, w) is a sum of ϵ_1 lines and ϵ_0 points.

Weighted t-fold blocking sets in $\mathrm{PG}(n, q)$

Definition

A weighted $\{f, t\}$-blocking set (B, w) of $\mathrm{PG}(n, q)$ is a set B of points and a weight function w, such that $|(B, w)| = f$, and every hyperplane of $\mathrm{PG}(n, q)$ contains at least t points of B, each point counted according to its weight.

Weighted t-fold blocking sets in $\mathrm{PG}(n,q)$

Theorem (JDB, K. Metsch, L. Storme)

Suppose that (B,w) is a weighted $\{f,\epsilon_1\}$-blocking set of $\mathrm{PG}(n,q)$, $q \geq 4$, with

$$f = \epsilon_1(q+1) + \epsilon_0,$$

$$\epsilon_0 + \epsilon_1 < \sqrt{q} + 1,$$

then (B,w) is a sum of ϵ_1 lines and ϵ_0 points.

Weighted t-fold blocking sets in $\mathrm{PG}(n, q)$

Theorem (JDB, K. Metsch, L. Storme)

Suppose that (B, w) *is a weighted* $\{f, t\}$*-blocking set of* $\mathrm{PG}(n, q)$*,* $q \geq 4$*, with*

$$v_{i+1} = \frac{q^{i+1} - 1}{q - 1} = |\mathrm{PG}(i, q)|, i = 1 \ldots n, v_0 := 0,$$

$$f = \sum_{i=0}^{t} \epsilon_i v_{i+1}, t = \sum_{i=0}^{t} \epsilon_i v_i, \sum_{i=0}^{t} \epsilon_t < \sqrt{q} + 1,$$

then (B, w) *is a* sum of ϵ_t t-dimensional subspaces, …, ϵ_1 lines and ϵ_0 points.

Linear codes and the Griesmer bound

Suppose that C is a linear $[n, k, d]$-code, with generator matrix $G = (g_1, \dots, g_n)$.

The *Griesmer bound* states that

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil = g_q(k, d),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

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Linear codes meeting the Griesmer bound

Suppose that $d \leq q^{k-1}$, then $d = q^{k-1} - \sum_{i=1}^{h} q^{\lambda_i}$,

- $0 \leq \lambda_1 \leq \ldots \leq \lambda_k < k - 1$
- at most $q - 1$ of the λ_i are equal to a given value.

It is shown that $g_i \neq \rho g_j$ when $i \neq j$ and that the projective points of $\mathrm{PG}(k-1, q) \setminus \{g_1, \ldots, g_n\}$ determine a non-weighted $\{f, t\}$-blocking set of $\mathrm{PG}(k-1, q)$, with

$$f = \sum_{i=1}^{h} v_{\lambda_i+1} = \sum_{i=0}^{k-2} \epsilon_i v_{i+1}$$

$$t = \sum_{i=1}^{h} v_{\lambda_i} = \sum_{i=0}^{k-2} \epsilon_i v_i$$

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$$f = \sum_{i=1}^{h} v_{\lambda_i + 1} = \sum_{i=0}^{k-2} \epsilon_i v_{i+1}$$

$$t = \sum_{i=1}^{h} v_{\lambda_i} = \sum_{i=0}^{k-2} \epsilon_i v_i$$

Linear codes meeting the Griesmer bound

Suppose that $d > q^{k-1}$.

- Denote the pointset of $\mathrm{PG}(k-1, q) = \{s_1, \ldots, s_{v_k}\}$.
- Define $m_i(G)$ as the number of columns of G definining s_i, and $\theta := \max\{m_i(G) \| i = 1, \ldots, v_k\}$.
- Define $w(s_i) := \theta - m_i(G)$. Then this weight function determines a weighted $\{f, t\}$-blocking set of $\mathrm{PG}(k-1, q)$, with

$$f = \sum_{i=0}^{k-2} \epsilon_i v_{i+1}, t = \sum_{i=0}^{k-2} \epsilon_i v_i$$

where $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \epsilon_i q^i$.