Characterising point sets in $\mathrm{AG}(3, q)$ from intersection numbers

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Definitions

Definition

Consider a set U of points of $\mathrm{AG}(n, q)$. A direction is called *determined by U* if and only if it is the slope of the line determined by two points of U. Denote by U_D the set of directions determined by U.

Corollary

If $|U| > q^{n-1}$, then all directions are determined by U.

a stability question

Consider a point set U in $\mathrm{AG}(3, q)$, $|U| = q^2 - \epsilon$, not determining a set N of directions. Can we extend U such that N remains unaffected?

partial ovoids of $Q(4,q)$

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An *ovoid* of $Q(4,q)$ is a set $\mathcal{O}$ of points of $Q(4,q)$ such that every line of $Q(4,q)$ contains exactly one point of $\mathcal{O}$.

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A *partial ovoid* of $Q(4,q)$ is a set $\mathcal{O}$ of points of $Q(4,q)$ such that every line of $Q(4,q)$ contains at most one point of $\mathcal{S}$. A partial ovoid is *maximal* if it cannot be extended to a larger partial ovoid.

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another representation

Theorem

$Q(4,q) \cong T_2(\mathcal{O}) \iff \mathcal{O}$ *is a conic.*

A (partial) ovoid $\rightarrow$ becomes a set of points not determining the points of a conic at infinity.

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a stability question in $\mathrm{AG}(3, q)$

Suppose that U is a pointset of size $q^2 - 2$ in $\mathrm{AG}(3, q)$, not determining the points of a conic at infinity. Can U be extended with two points?

(i) Yes, when $q = p^h$, $h > 1$: p odd: DB and Gács (2008) (p even: Brown, DB and Storme (2003)).

(ii) Maximal examples exist when $p \in \{3, 5, 7, 11\}$.

An (alternative) description of the known examples

Theorem (K. Coolsaet, DB, A. Siciliano)

A maximal partial ovoid of size $q^2 - 2$ *of* $\mathrm{Q}(4, q)$*,* q *odd, is equivalent with a* sharply transitive subset *of size* $q^2 - 1$ *of* $\mathrm{SL}(2, q)$*.*

computation using Rédei polynomial

(i) $U = \{(a_i, b_i, c_i, 1) \| i = 1 \dots q^2 - 2\}$

(ii) $R(X, Y, Z, W) = \prod_{i=1}^{q^2-2}(X + a_i Y + b_i Z + c_i W)$

$$R(X, y, z, w) \mid (X^q - X)^q$$

if $yX_1 + zX_2 + wX_2 = X_3 = 0$ is a line containing a non-determined direction.

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(iii) $\sigma_1(Y, Z, W) = 0$ (by affine translation)

(iv) a conic is not determined implies $\sigma_{2k}(Y, Z, W) = \sigma_2(Y, Z, W)^k$, $k = 1 \ldots \frac{q-1}{2}$, $\sigma_{2j+1}(Y, Z, W) = 0$, $j = 1 \ldots \frac{q-1}{2}$

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computation using Rédei polynomial

$$P(X,Y,Z,W) := \sum_{i=1}^{q^2-2} (X + a_i Y + b_i Z + c_i W)^{q-1} \quad (1)$$

$$= -2\frac{X^{q+1} - (\sigma_2(Y,Z,W))^{\frac{q+1}{2}}}{X^2 - \sigma_2(Y,Z,W)} \quad (2)$$

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hypothesis on intersection numbers

Suppose that $P(X, Y, Z, W) = 0$.

Conjecture

Suppose that U is a set of q^2 points in $\mathrm{AG}(3, q)$, q prime, such that every plane intersects U in $0 \mod q$ points. Then U is a cylinder, i.e. the set of q^2 points on q distinct lines in one parallel class.

A general equality

Lemma

Suppose that $R(X_1, \ldots, X_n) = \prod_{i=1}^{d}(a_i^1 X_1 + \ldots + a_i^n X_n)$, $a_i^j \in \mathbb{F}_q$, $\in \mathbb{N}$, *and consider* $P(X_1, \ldots, X_n) = \sum_{i=1}^{d}(a_i^1 X_1 + \ldots + a_i^n X_n)^{q-1}$. *Then*

$$P \cdot R = X_1^q \frac{\partial R}{\partial X_1} + \ldots + X_n^q \frac{\partial R}{\partial X_n}$$

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If we also suppose that U does not determine $q+1$ directions, assuming $P(X,Y,Z,W)=0$ implies

$$\sigma_k(Y,Z,W) \equiv 0\,, k = lq+1 \dots (l+1)q - l,$$
$$l = 0 \dots q-1$$

$$(-j+1)\sigma_{j+q-1}(Y,Z,W) + (Y^q\frac{\partial\sigma_j}{\partial Y} + Z^q\frac{\partial\sigma_j}{\partial Z} + W^q\frac{\partial\sigma_j}{\partial W}) \equiv 0\,,$$
$$j = q+1 \dots q^2 - q$$

$$Y^q\frac{\partial\sigma_j}{\partial Y} + Z^q\frac{\partial\sigma_j}{\partial Z} + W^q\frac{\partial\sigma_j}{\partial W} \equiv 0\,,$$
$$j = q^2-q+1 \dots q^2$$

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Intersections with lines

Substitution $Y := sZ + tW$ enables to use $R(X, Y, Z, W)$ to investigate intersections with the q^2 lines through $(0, 1, -s, -t)$.

$$\sigma_k^{s,t}(Z, W) \equiv 0\,, k = lq + 1 \ldots (l+1)q - l, \quad l = 0 \ldots q-1$$

$$(-j+1)\sigma_{j+q-1}^{s,t}(Z, W) + (Z^q \frac{\partial \sigma_j^{s,t}}{\partial Z} + W^q \frac{\partial \sigma_j^{s,t}}{\partial W}) \equiv 0\,, \quad j = q+1 \ldots q^2 - q$$

$$Z^q \frac{\partial \sigma_j^{s,t}}{\partial Z} + W^q \frac{\partial \sigma_j^{s,t}}{\partial W} \equiv 0\,, \quad j = q^2 - q + 1 \ldots q^2$$

Suppose that U is a pointset of size $q^2 + 1$ in $\mathrm{AG}(3, q)$, such that $q + 1$ points of a given conic at infinity have the property that each line on such a point meets U in at least one point. Can one point of U be removed?

- it is easy to find examples where a point can be removed
- for q an odd prime, the answer is yes (DB and Metsch 2005).
- the case $q = p^h$, p odd prime, $h > 1$ is open.
- this problem seems to be related to the cylinder conjecture