On the smallest minimal blocking sets of $\mathrm{Q}(2n, q)$

Jan De Beule
joint work with Leo Storme

Definitions

An ovoid $\mathcal{O}$ of $\mathrm{Q}(2n, q)$ is a set of points such that every generator meets $\mathcal{O}$ in exactly one point, $|\mathcal{O}| = q^n + 1$

A blocking set $\mathcal{K}$ of $\mathrm{Q}(2n, q)$ is a set of points such that every generator meets $\mathcal{K}$ in at least one point, $|\mathcal{K}| = q^n + 1 + r$.

$\mathcal{K}$ is minimal iff $\mathcal{K} \setminus \{p\}$ is not a blocking set for all $p \in \mathcal{K}$,

Part 1: $\mathrm{Q}(6, q)$, q even

Supose $\mathcal{K}$ is a minimal blocking set of $\mathrm{Q}(6, q)$, q even, of size $q^3 + 1 + r$, $r < q$.

Lemma 1. *If $p \in \mathrm{Q}(6, q)$, $p \in \mathcal{K}$, then $|T_p(\mathrm{Q}(6, q)) \cap \mathcal{K}| \leqslant 1 + r$.*

Lemma 2. *If $p \in \mathrm{Q}(6, q)$, $p \notin \mathcal{K}$, then p projects $\mathcal{K}$ onto a minimal blocking set of $\mathrm{Q}(4, q)$*

Theorem 1. *(Eisfeld, Storme, Szőnyi and Sziklai) A blocking set of $\mathrm{Q}(4, q)$, q even, $q \geqslant 32$, of size $q^2 + 1 + r$, with $0 < r \leqslant \sqrt{q}$, contains an ovoid.*

Corollary 1. *If L is a line of $\mathrm{Q}(6, q)$, then $|L \cap \mathcal{K}| = 0, 1$ or $|L \cap \mathcal{K}| \geqslant 1 + \sqrt{q}$.*

Theorem 2. *Let $\mathcal{K}$ be a minimal blocking set of $\mathrm{Q}(6,q)$, q even, $|\mathcal{K}| \leqslant q^3+q$, $q \geqslant 32$. Then there is a point $p \in \mathrm{Q}(6,q)\backslash\mathcal{K}$ with the following property: $T_p(\mathrm{Q}(6,q)) \cap \mathrm{Q}(6,q) = p\mathrm{Q}(4,q)$ and $\mathcal{K}$ consists of all the points of the lines L on p meeting $\mathrm{Q}(4,q)$ in an ovoid $\mathcal{O}$, minus the point p itself, and $|\mathcal{K}| = q^3+q$.*

Part 2: $\mathrm{Q}(6,5)$

Lemma 3. *If* $\mathcal{B}$ *is a minimal blocking set of* $\mathrm{Q}(4,5)$ *and* $\mathcal{B}$ *is not an ovoid, then* $|\mathcal{B}| > 27$

Theorem 3. *Let* $\mathcal{K}$ *be a minimal blocking set of* $\mathrm{Q}(6, q=5)$, $|\mathcal{K}| \leqslant q^3+q$. *Then there is a point* $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ *with the following property:* $T_p(\mathrm{Q}(6,q)) \cap \mathrm{Q}(6,q) = p\mathrm{Q}(4,q)$ *and* $\mathcal{K}$ *consists of all the points of the lines* L *on* p *meeting* $\mathrm{Q}(4,q)$ *in an ovoid* $\mathcal{O}$, *minus the point* p *itself, and* $|\mathcal{K}| = q^3+q$.

Part 3: $\mathrm{Q}(2n, q)$, q odd

Supose $\mathcal{K}$ is a minimal blocking set of $\mathrm{Q}(2n, q)$, of size $q^n + 1 + r$, $r < q^{n-2}$.

Lemma 4. *If $p \in \mathrm{Q}(2n, q)$, $p \in \mathcal{K}$, then $|T_p(\mathrm{Q}(2n, q)) \cap \mathcal{K}| \leqslant 1 + r$.*

Lemma 5. *If $p \in \mathrm{Q}(2n, q)$, $p \notin \mathcal{K}$, then p projects $\mathcal{K}$ onto a minimal blocking set of $\mathrm{Q}(2n-2, q)$*

Lemma 6. *There exists a point $p \in \mathrm{Q}(2n, q) \setminus \mathcal{K}$ such that $|T_p(\mathrm{Q}(2n, q)) \cap \mathcal{K}| = q^{n-1} + q^{n-3}$.*

Applied on $\mathrm{Q}(8, q)$

Lemma 7. *There exists a line* $L \subset \mathrm{Q}(8, q)$*,* $L \cap \mathcal{K} = \emptyset$*, such that* $|T_L(\mathrm{Q}(8, q)) \cap \mathcal{K}| = q^2 + 1$*, and the set* $T_L(\mathrm{Q}(8, q)) \cap \mathcal{K}$ *is an ovoid of* $\mathrm{Q}(4, q)$

Theorem 4. *Let* $\mathcal{K}$ *be a minimal blocking set of* $\mathrm{Q}(8, q = 5)$*,* $|\mathcal{K}| \leqslant q^4 + q^2$*. Then there is a line* $L \subset \mathrm{Q}(8, q)$*,* $L \cap \mathcal{K} = \emptyset$ *with the following property:* $T_L(\mathrm{Q}(8, q)) \cap \mathrm{Q}(8, q) = L\mathrm{Q}(4, q)$ *and* $\mathcal{K}$ *consists of all the points of the lines on the points of* L *meeting* $\mathrm{Q}(4, q)$ *in an ovoid* $\mathcal{O}$*, minus the points of* L*, and* $|\mathcal{K}| = q^4 + q^2$*.*