The smallest minimal blocking sets of $Q^+(2n+1, q)$, $q = 2, 3, 4$

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August 15, 2005 / Geometric and Algebraic Combinatorics 3

Polar spaces

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Ovoids of polar spaces

Definition

An *ovoid* is a set $\mathcal{O}$ of points such that every generator contains exactly one point of $\mathcal{O}$.

- Ovoids of $Q(6,q)$ are known for $q = 3^h$ (Kantor; Thas).
- Ovoids of $Q(6,q)$ do not exist when $q = 2^h$ (Thas) and when $q > 3$ is an odd prime (Govaerts, Storme and Ball).
- $Q^+(7,2)$, $Q^+(7,3)$ and $Q^+(7,4)$ each have a unique ovoid (Kantor; Patterson; Gunawardena resp.).
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(Non)-existence of ovoids in high dimensions?

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$$p^n > \binom{2n+p}{2n+1} - \binom{2n+p-2}{2n+1}$$

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Some facts

- $Q^+(9,3)$ has no ovoid
- All ovoids of $Q^+(7,3)$ span a hyperplane π of $PG(6,3)$ and are ovoids of $Q(6,3) = \pi \cap Q^+(7,3)$.
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Look in tangent cones of points

Suppose that $\mathcal{B}$ is a minimal blocking set of $Q^+(9,3)$, $|\mathcal{B}| \leq q^4 + q$

- for all $p \in \mathcal{B}$, $|p^\perp \cap \mathcal{B}| \leq q$
- If $p \notin \mathcal{B}$ and $|p^\perp \cap \mathcal{B}| = q^3 + 1$ then points of $p^\perp \cap \mathcal{B}$ are *projected* from p onto an ovoid of $Q^+(7,3)$, which is an ovoid of $Q(6,3)$ and lies in 6 dimensions.
- most difficult problem: show that the set $p^\perp \cap \mathcal{B}$ *is* an ovoid of $Q^+(7,3)$.

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Special properties of the ovoid of $Q^+(7,3)$

- We consider parts of the projection: intersections with $Q^+(5,3)$ that constitute ovoids of $Q^+(5,3)$.
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use the ovoids

- Show that the points of $\mathcal{B}$ lie in a 7-dimensional space.

Theorem

$\mathcal{B}$ is a truncated cone $r^*\mathcal{O}$, $\mathcal{O}$ an ovoid of $Q(6,3)$.

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Induction hypothesis

- Suppose that $\mathcal{B}$ is a minimal blocking set of $Q^+(2n+1,3)$, $|\mathcal{B}| \leq q^n + q^{n-3}$.
- Suppose that the smallest minimal blocking sets of $Q^+(2n_0+1,3)$, $4 \leq n_0 < n$ are truncated cones $\pi^*_{n-4}\mathcal{O}$, $\mathcal{O}$ an ovoid of $Q(6,3)$.
- If $p \notin \mathcal{B}$ and $|p^\perp \cap \mathcal{B}| = q^{n-1} + q^{n-4}$ then the points of $p^\perp \cap \mathcal{B}$ are projected from p onto a truncated cone $\pi^*_{n-5}\mathcal{O}$, $\mathcal{O}$ an ovoid of $Q(6,3)$.
- If $L \cap \mathcal{B} = \emptyset$ and $|L^\perp \cap \mathcal{B}| = q^{n-2} + q^{n-5}$ then the points of $p^\perp \cap \mathcal{B}$ are the points of a truncated cone $\pi^*_{n-6}\mathcal{O}$.

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The final step

- If $p \notin \mathcal{B}$ and $|p^\perp \cap \mathcal{B}| = q^{n-1} + q^{n-4}$ then the points of $p^\perp \cap \mathcal{B}$ are the points of a truncated cone $\pi^*_{n-5}\mathcal{O}$, $\mathcal{O}$ an ovoid of $Q(6,3)$.

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Results on ovoids

- $Q^+(7,2)$ has a unique ovoid, lies in 7 dimensions
- $Q^+(9,2)$ has no ovoid, hence $Q^+(2n+1,2)$, $n \geq 5$ has no ovoid
- $Q^+(5,2)$ has minimal blocking sets of size q^2+2 (Blokhuis, O'Keefe, Payne, Storme and Wilbrink).

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- We use the blocking set of $Q^+(5,2)$ of size q^2+2 to find lines of $Q^+(7,2)$ containing 2 points of $\mathcal{B}$.
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Results on ovoids

- $Q^+(7,4)$ has a unique ovoid, lies in 7 dimensions
- Existence of ovoids of $Q^+(9,4)$ is open
- There are small minimal blocking sets of $Q^+(5,4)$: size q^2+2, q^2+3 and q^2+4.
- Because q is still small, it may be possible to prove the the smallest minimal blocking sets *different from an ovoid* are truncated cones. Because the existence of ovoids of $Q^+(9,4)$ is open, we cannot extend the result to higher dimensions.

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References

Thas01 J. A. Thas. Ovoids, spreads and *m*-systems of finite classical polar spaces. In *Surveys in combinatorics, 2001 (Sussex)*, volume 288 of *London Math. Soc. Lecture Note Ser.*, pages 241–267. Cambridge Univ. Press, Cambridge, 2001.