

Lower and upper bounds of maximal partial ovoids of orthogonal polar spaces

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Geometric and algebraic combinatorics 4

Quadrics

- in $\text{PG}(2n, q)$, $n \geq 2$,
 $Q(2n, q) : X_0^2 + X_1X_2 + \dots X_{2n-1}X_{2n} = 0.$
- in $\text{PG}(2n + 1, q)$, $n \geq 2$,
 $Q^-(2n + 1, q) : f(X_0, X_1) + X_2X_3 + \dots X_{2n}X_{2n+1} = 0,$
 $f(X_0, X_1)$: irreducible, homogeneous, of degree 2.
- in $\text{PG}(2n + 1, q)$, $n \geq 2$,
 $Q^+(2n + 1, q) : X_0X_1 + X_2X_3 + \dots X_{2n}X_{2n+1} = 0.$



Rank

- $Q(2n, q)$: rank n .
- $Q^-(2n + 1, q)$: rank n .
- $Q^+(2n + 1, q)$: rank $n + 1$.

Ovoids

Let \mathcal{P} be a finite classical polar space.

Definition

ovoid: every generator meets \mathcal{O} in exactly one point.

First question: existence?

Existence of ovoids (low rank)

$Q^-(5, q)$	no (J.A. Thas)
$Q(4, q)$	yes
$Q^+(3, q)$	yes

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Existence of ovoids (high rank)

$Q^-(7, q)$ $Q^-(2n+1, q), n \geq 2$	no no (slicing)
$Q^+(5, q)$	yes
$Q(6, q)$	no, $q > 3$ odd prime (S. Ball, P. Govaerts, L. Storme)
$Q(6, q)$	examples known when $q = 3^h$
$Q(6, q)$	no, q even (J.A. Thas)
$Q(8, q)$ $Q(2n, q), n \geq 4$	no (A. Gunawardena, E. Moorhouse) no (slicing)

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Existence of ovoids (high rank)

$Q^+(2n+1, q)$, $n \geq 3$, $q = p^h$: no, if

$$p^n > \binom{2n+p}{2n+1} - \binom{2n+p-2}{2n+1}$$

$Q^+(7, q)$: yes if q is odd prime or $q \equiv 0$ or $2 \pmod{3}$

Questions

- If there is no ovoid, what is the (size of) the *largest* partial ovoid?
- If there are ovoids, what is the (size of) the *largest* maximal partial ovoid different from an ovoid?
- What is the (size of) the *smallest* maximal partial ovoids?

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Definition

Let \mathcal{P} be a finite classical polar space.

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partial ovoid: every generator meets \mathcal{O} in at most one point.

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How to obtain a lower bound

$n_i :=$ number of points of $Q(d, q)$ collinear with i points of \mathcal{O} .

$$\sum n_i = |Q^\pm(2n+1, q)| - w$$

$$\sum in_i = wq|Q^\pm(2n-1, q)|$$

$$\sum i(i-1)n_i = w(w-1)|Q^\pm(2n-1, q)|$$

$$\sum i(i-1)(i-2)n_i = w(w-1)(w-2)|Q(2n-2, q)|$$

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Counting



Using the equations ...

$$\begin{aligned}
 0 &\leq \sum_i n_i (i-1)(i-a)(i-a-1) \\
 &= \sum_i n_i i(i-1)(i-2) - (2a-1) \sum_i n_i i(i-1) \\
 &\quad + (a^2 + a) \sum_i n_i (i-1)
 \end{aligned}$$

Resulting bounds for $Q^-(5, q)$ and $Q^+(5, q)$

Theorem

$$Q^-(5, q) : q \geq 4 \Rightarrow w \geq 2q + 2, q < 4 \Rightarrow w \geq 2q + 1$$

$$Q^-(2n + 1, q) : n \geq 3 \Rightarrow w \geq 2q + 1$$

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Resulting bounds for $Q(2n, q)$

Theorem

$$Q(4, q) : q \text{ odd} \Rightarrow w \geq 1.419q$$

$$Q(6, q) : q \in \{3, 5, 7\} \Rightarrow w \geq 2q, q \geq 9 \Rightarrow w \geq 2q - 1$$

$$Q(8, 3) : w \geq 2q$$

$$Q(2n, q) : n \geq 4 \Rightarrow w \geq 2q + 1$$

A lower bound for $Q(2n, q)$, q even

- $Q(2n, q)$, q even has a *nucleus*
- Projecting from this nucleus yields the *symplectic* polar space $W(2n - 1, q)$.

Theorem

The smallest maximal partial ovoids of $W(2n - 1, q)$ are the hyperbolic lines

Theorem

The smallest maximal partial ovoids of $Q(2n, q)$ are conics whose nucleus coincides with the nucleus of $Q(2n, q)$.

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An upper bound for $Q^+(2n + 1, q)$

- $|\mathcal{O}| = q^n + 1 - \delta.$
- $|P^\perp \cap \mathcal{O}| \geq q^{n-1} + 1 - \delta$
- $n_i = 0$ for $i < q^{n-1} + 1 - \delta$ and $i > q^{n-1} + 1$
- $0 \leq \sum_i n_i (i - q^{n-1} - 1)(i - q^{n-1})(i - q^{n-1} - 1 + \delta)$

Theorem

A maximal partial ovoid \mathcal{O} of $Q^+(2n + 1, q)$, that is not an ovoid, has at most $q^n - q^{(n-1)/2}$ points.

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An upper bound for $Q(2n, q)$, q odd non prime

Lemma

Consider $Q(2n, q) \subseteq Q^+(2n+1, q)$, $n \geq 3$, q not a prime, and suppose that $Q^+(2n+1, q)$ has an ovoid with $q^n + 1 - \delta$, $\delta > 0$, points in $Q(2n, q)$. Then $\delta \geq 2(q^{n-2} + q^{n-3} + \dots + q + 1) + 1$.

Theorem (A. Gács and JDB)

$Q(4, q)$ has no maximal partial ovoids when q is odd and non prime.

Corollary

$Q(6, q)$, q not a prime, does not have a maximal partial ovoid of size $q^3 + 1 - \delta$ with $0 < \delta < q + 1$.

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This dualizes to an upper bound for partial ovoids of $Q^-(5, q)$.

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Inductive bounds

\mathcal{P}_n denotes a finite classical polar space of rank n .

Theorem

If partial ovoids of \mathcal{P}_r have deficiency ϵ_r , then partial ovoids of \mathcal{P}_{r+1} have deficiency at least q^{ϵ_r} .