Strongly regular graphs and substructures of finite classical polar spaces

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Strongly regular graphs

Definition

Let $\Gamma = (X, \sim)$ be a graph, it is strongly regular with parameters (n, k, λ, μ) if all of the following holds:

(i) The number of vertices is n.

(ii) Each vertex is adjacent with k vertices.

(iii) Each pair of adjacent vertices is commonly adjacent to λ vertices.

(iv) Each pair of non-adjacent vertices is commonly adjacent to μ vertices.

We exclude “trivial cases”.

Adjacency matrix

Let $\Gamma = (X, \sim)$ be a srg(n, k, λ, μ).

Definition

The adjacency matrix of Γ is the matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$

$$a_{ij} = \begin{cases} 1 & i \sim j \\ 0 & i \not\sim j \end{cases}$$

Theorem (proof: e.g. Brouwer, Cohen, Neumaier)

The matrix A satisfies

$$A^2 + (\mu - \lambda)A + (n-k)I = \mu J$$

Eigenvalues and eigenspaces

Corollary

The matrix A has three eigenvalues:

$$k, \tag{1}$$

$$r = \frac{\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2} > 0, \tag{2}$$

$$s = \frac{\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2} < 0; \tag{3}$$

and furthermore

$$\mathbb{C}^n = \langle j \rangle \perp V_+ \perp V_-.$$

Line graph of $\mathrm{PG}(3, q)$

- Vertices: lines of $\mathrm{PG}(3, q)$
- Adjacency: two vertices are adjacent iff the corresponding lines meet.

Parameters of the line graph of $\mathrm{PG}(3, q)$

- $n = (q^2 + q + 1)(q^2 + 1)$
- $k = (q + 1)^2 q.$
- $\lambda = 2q^2 + q - 1.$
- $\mu = (q + 1)^2.$
- $r = q^2 - 1.$
- $s = -1 - q^2.$

History of Cameron-Liebler line classes

- 1982: Cameron and Liebler studied irreducible collineation groups of $\mathrm{PG}(d, q)$ having equally many point orbits as line orbits
- Such a group induces a symmetrical tactical decomposition of $\mathrm{PG}(d, q)$.
- They show that such a decomposition induces a decomposition with the same property in any 3-dimensional subspace.
- They call any line class of such a tactical decomposition a "Cameron-Liebler line class"

Cameron-Liebler line classes

Definition

A spread is a set $\mathcal{S}$ of lines of $\mathrm{PG}(3, q)$ partitioning the point set of $\mathrm{PG}(3, q)$.

Definition

A Cameron-Liebler line class with parameter x is a set $\mathcal{L}$ of lines of $\mathrm{PG}(3, q)$ such that $|\mathcal{L} \cap \mathcal{S}| = x$ for any spread $\mathcal{S}$.

If $\mathcal{L}$ is a CL-line class, then for the characteristic vector of the corresponding vertex set in the line graph it holds

$$\chi_{\mathcal{L}} \in \langle j \rangle \perp V_+$$

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Existence and non-existence of CL-line classes

- “Trivial examples”
- Conjecture by Cameron and Liebler: these are the only examples
- Disproven by a construction of Bruen and Drudge
- Many (strong) non-existence results

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Existence and non-existence of CL-line classes

Theorem (A. Bruen, K. Drudge, 1999)

Let q be odd, there exists a Cameron-Liebler line class with parameter $\frac{q^2+1}{2}$.

Theorem (A.L. Gavrilyuk, K. Metsch, 2014)

Let $\mathcal{L}$ be a CL line class with parameter x. Let n be the number of lines of $\mathcal{L}$ in a plane. Then

$$\binom{x}{2} + n(n-x) \equiv 0 \pmod{q+1}$$

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- Input (Morgan Rodgers, May 2011): there exist Cameron-Liebler line classes with parameter $x = \frac{q^2-1}{2}$ for $q \in \{5, 9, 11, 17, \ldots\}$.
- They all are stabilized by a cyclic group of order $q^2 + q + 1$.
- Question: are these member of an infinite family?

The construction of the infinite family

- We are looking for a vector χ_T such that

$$(\chi_T - \frac{x}{q^2+1}j)A = (q^2-1)(\chi_T - \frac{x}{q^2+1}j)$$

The construction of the infinite family

- Not containing the trivial examples:

$$(\chi'_T - \frac{x}{q^2-1} j')A' = (q^2-1)(\chi'_T - \frac{x}{q^2-1} j')$$

The construction of the infinite family

- Using the cyclic group of order $q^2 + q + 1$:

$$(\chi'_T - \frac{x}{q^2 - 1} j')B = (q^2 - 1)(\chi'_T - \frac{x}{q^2 - 1} j')$$

- Assume that $q \not\equiv 1 \pmod 3$ then all orbits have length $q^2 + q + 1$, this induces a *tactical decomposition* of A'

The construction of the infinite family

Definition

Let $A = (a_{ij})$ be a matrix A partition of the row indices into $\{R_1, \ldots, R_t\}$ and the column indices into $\{C_1, \ldots, C_{t'}\}$ is a *tactical decomposition* of A if the submatrix $(a_{p,l})_{p \in R_i, l \in C_j}$ has constant column sums c_{ij} and row sums r_{ij} for every (i, j).

- the matrix $B = (c_{ij})$.

The construction of the infinite family

Theorem (Higman–Sims, Haemers (1995))

Suppose that A can be partitioned as

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix}$$

with each A_{ii} square and each A_{ij} having constant column sum c_{ij}. Then any eigenvalue of the matrix $B = (c_{ij})$ is also an eigenvalue of A.

The construction of the infinite family

- Assuming that $q \equiv 1 \pmod 4$, we have control on the entries of the matrix B, and, it turns out that B is a *block circulant* matrix!
- Now we have the eigenvector we are looking for, and also yields the full symmetry group of the tight set.

The infinite family

Theorem (JDB, J. Demeyer, K. Metsch, M. Rodgers)

There exist a CL line class of $\mathrm{PG}(3,q)$, $q \equiv 5, 9 \pmod{12}$ *with a symmetry group of order* $3\frac{q-1}{2}(q^2+q+1)$.

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Finite classical polar spaces

- $V(d+1,q)$: $d+1$-dimensional vector space over the finite field $\mathrm{GF}(q)$.
- f: a non-degenerate sesquilinear or non-singular quadratic form on $V(d+1,q)$.

Definition

A *finite classical polar space* associated with a form f is the geometry consisting of subspaces of $\mathrm{PG}(d,q)$ induced by the totally isotropic sub vector spaces with relation to f.

An easy example

- The Klein correspondence maps lines of $\mathrm{PG}(3, q)$ to points of $\mathrm{PG}(5, q)$ through their Plücker coordinates.
- These points satisfy the equation $X_0X_1 + X_2X_3 + X_4X_5 = 0$.
- This is a polar space of *rank 3*, denoted as $\mathrm{Q}^+(5, q)$
- A Cameron-Liebler line class with parameter x is an x-tight set of $\mathrm{Q}^+(5, q)$.

Geometrical definition

- $\mathcal{S}$: a finite classical polar space of rank r over $\mathrm{GF}(q)$.
- $\theta_n(q) := \frac{q^n-1}{q-1}$ the number of points in an $n-1$-dimensional projective space.

Definition

An *i-tight set* is a set $\mathcal{T}$ of points such that

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i\theta_{r-1}(q) + q^{r-1} & \text{if } P \in \mathcal{T} \\ i\theta_{r-1}(q) & \text{if } P \notin \mathcal{T} \end{cases}$$

Definition

An *m-ovoid* is a set $\mathcal{O}$ of points such that every generator of $\mathcal{S}$ meets $\mathcal{O}$ in exactly m points.

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Back to eigenspaces

- If $\mathcal{T}$ is an i-tight set, then

$$\chi_{\mathcal{T}} \in \langle j \rangle \perp V_+$$

- If $\mathcal{O}$ is an m-ovoid, then

$$\chi_{\mathcal{O}} \in \langle j \rangle \perp V_-$$

Possible applications

Theorem

Let $\mathcal{O}$ be a weighted m-ovoid. Let $\mathcal{T}$ be a weighted i-tight set. Then

$$\chi_{\mathcal{O}} \cdot \chi_{\mathcal{T}} = mi.$$

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