Characterization results on arbitrary (weighted) minihypers and linear codes meeting the Griesmer bound

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Blocking sets

Definition

Consider the projective plane $\mathrm{PG}(2,q)$. A set B of points of $\mathrm{PG}(2,q)$, different from a line, is called a *blocking set* if any line of $\mathrm{PG}(2,q)$ contains at least one point of B.

Definition

A blocking set B of $\mathrm{PG}(2,q)$ is called *minimal* if it does not contain a smaller blocking set as a subset.

Examples:

- The projective triangle
- A Baer subplane
- A Hermitian curve

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More blocking sets

Definition

Consider the projective space $\mathrm{PG}(n, q)$. A set B of points of $\mathrm{PG}(2, q)$, different from a line, is called a *blocking set* if any hyperplane of $\mathrm{PG}(n, q)$ contains at least one point of B. A blocking set B of $\mathrm{PG}(n, q)$ is called *minimal* if it does not contain a smaller blocking set as a subset.

Definition

Consider the projective plane $\mathrm{PG}(2, q)$. A set B of points of $\mathrm{PG}(2, q)$ is called a *t-fold blocking set* if any line of $\mathrm{PG}(2, q)$ contains at least t points of B. A t-fold blocking set B of $\mathrm{PG}(2, q)$ is called *minimal* if it does not contain a smaller t-fold blocking set as a subset.

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Minihypers

Definition

Consider the projective space $\mathrm{PG}(n,q)$. A weighted $\{f,m;n,q\}$-*minihyper*, $f \geq 1$, $n \geq 2$, is a pair (F,w), where F is a subset of the point set of $\mathrm{PG}(n,q)$ and where w is a weight function w: $\mathrm{PG}(n,q) \to \mathbb{N}$: $x \mapsto w(x)$, satisfying:

1. $w(x) > 0 \iff x \in F$,
2. $\sum_{x \in F} w(x) = f$, and
3. $\min\{\sum_{x \in H} w(x) \| H \in \mathcal{H}\} = m$, where $\mathcal{H}$ is the set of hyperplanes of $\mathrm{PG}(n,q)$.

Constructions . . .

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Linear codes

Definition

A *linear* $[n, k, d]$*-code* C over the finite field $\mathrm{GF}(q)$ is a k-dimensional subspace of the n-dimensional vector space $V(n, q)$, where d is the *minimum distance* of C.

Theorem

Suppose that C is a linear $[n, k, d]$ code. The Griesmer bound states that

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil = g_q(k, d),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x

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Linear codes meeting the Griesmer bound and minihypers

Suppose that C is a linear $[n, k, d]$ code. Then we can write d in an unique way as $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \epsilon_i q^{\lambda_i}$ such that $\theta \geq 1$ and $0 \leq \epsilon_i < q$. Then the Griesmer bound for an $[n, k, d]$-code can be expressed as:

$$n \geq \theta v_k - \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i + 1}$$

where $v_l = (q^l - 1)/(q - 1)$, for any integer $l \geq 0$.

Linear codes meeting the Griesmer bound and minihypers

Theorem

(Hamada and Helleseth) There is a one-to-one correspondence between the set of all non-equivalent $[n, k, d]$*-codes meeting the Griesmer bound and the set of all projectively distinct* $\{\sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i+1}, \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i}; k-1, q\}$*-minihypers* (F, w)*, such that* $1 \leq w(p) \leq \theta$ *for every point* $p \in F$*.*

The link is described explicitly

Linear codes meeting the Griesmer bound and minihypers

Let $G = (g_1 \cdots g_n)$ be a generator matrix for a linear $[n, k, d]$-code, meeting the Griesmer bound. We look at a column of G as being the coordinates of a point in $PG(k-1, q)$. Let the point set of $PG(k-1, q)$ be $\{s_1, \ldots, s_{v_k}\}$. Let $m_i(G)$ denote the number of columns in G defining s_i. Let $m(G)$ be the maximum value in $\{m_i(G) \mid i = 1, 2, \ldots, v_k\}$. Then $\theta = m(G)$ is uniquely determined by the code C and we call it the *maximum multiplicity* of the code. Define the weight function $w : PG(k-1, q) \to \mathbb{N}$ as $w(s_i) = \theta - m_i(G)$, $i = 1, 2, \ldots, v_k$. Let $F = \{s_i \in PG(k-1, q) \mid w(s_i) > 0\}$, then (F, w) is a $\{\sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i+1}, \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i}; k-1, q\}$-minihyper with weight function w.

Some characterizations

Theorem

A weighted t-fold blocking set B of $\mathrm{PG}(2, q)$, $q \geq 4$, $2 \leq t < \sqrt{q} + 1$ *containing no line, has at least* $tq + \sqrt{tq} + 1$ *points.*

Theorem

A weighted t-fold blocking set B of $\mathrm{PG}(2, q)$ *containing at least one point of weight one, of size* $|B| = t(q + 1) + r$, $t + r \leq \delta_0$, *contains a line.*

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Table for δ_0

p	h	δ_0
p	even	$\leq \sqrt{q}$
p	$h = 1$	$\leq (p+1)/2$
p	3	$\leq p^2$
2	$6m+1, m \geq 1$	$\leq 2^{4m+1} - 2^{4m} - 2^{2m+1}/2$
> 2	$6m+1, m \geq 1$	$\leq p^{4m+1} - p^{4m} - p^{2m+1}/2 + 1/2$
2	$6m+3, m \geq 1$	$< 2^{4m+5/2} - 2^{4m+1} - 2^{2m+1} + 1$
> 2	$6m+3, m \geq 1$	$\leq p^{4m+2} - p^{2m+2} + 2$
≥ 5	$6m+5, m \geq 0$	$< p^{4m+7/2} - p^{4m+3} - p^{2m+2}/2 + 1$

Some characterizations

Theorem

A weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; k-1, q\}$*-minihyper* (F, w)*,* $k \geq 4$*, with* $\epsilon_1 + \epsilon_0 \leq \delta_0$*, is a sum of* ϵ_1 *lines and* ϵ_0 *points.*

Higher dimensions: needed results

Theorem

(Hamada and Helleseth) A μ-dimensional subspace intersects a weighted $\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k-1, q\}$-minihyper, $\sum_{i=0}^{k-2} \epsilon_i = \delta \leq q$, $(\epsilon_0, \ldots, \epsilon_{k-2}) \in E_{ext}(k-1, q)$, in a weighted $\{\sum_{i=0}^{\mu} m_i v_{i+1}, \sum_{i=0}^{\mu} m_i v_i; \mu, q\}$-minihyper, where $\sum_{i=0}^{\mu} m_i \leq \delta$.

Theorem

Let F be a $\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=1}^{k-2} \epsilon_i v_i; k-1, q\}$-minihyper where $t \geq 2$, $q > h$, $0 \leq \epsilon_i \leq q-1$, $\sum_{i=0}^{k-2} \epsilon_i = h$.
Then a plane of $\mathrm{PG}(k-1, q)$ is either contained in F or intersects F in an $\{m_0 + m_1(q+1), m_1; 2, q\}$-minihyper with $m_0 + m_1 \leq h$.

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Let F be a $\{\sum_{i=0}^{k-2}\epsilon_i v_{i+1}, \sum_{i=1}^{k-2}\epsilon_i v_i; k-1, q\}$-minihyper where $t \geq 2$, $q > h$, $0 \leq \epsilon_i \leq q-1$, $\sum_{i=0}^{k-2}\epsilon_i = h$.
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Characterizations using planes

Theorem

A weighted $\{\epsilon_2(q^2+q+1)+\epsilon_1(q+1)+\epsilon_0, \epsilon_2(q+1)+\epsilon_1; k-1, q\}$-minihyper (F, w), with $\epsilon_2+\epsilon_1+\epsilon_0 \le \delta_0$, is a sum of ϵ_2 planes, ϵ_1 lines, and ϵ_0 points.

Theorem

A weighted $\{\sum_{i=0}^{t} \epsilon_i v_{i+1}, \sum_{i=1}^{t} \epsilon_i v_i; k-1, q\}$-minihyper, with $\sum_{i=0}^{t} \epsilon_i \le \delta_0$, is the sum of ϵ_t t-dimensional subspaces, ϵ_{t-1} $(t-1)$-dimensional subspaces,$\dots$,ϵ_1 lines and ϵ_0 points.

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A weighted $\{\sum_{i=0}^{t} \epsilon_i v_{i+1}, \sum_{i=1}^{t} \epsilon_i v_i; k-1, q\}$*-minihyper, with* $\sum_{i=0}^{t} \epsilon_i \leq \delta_0$*, is the sum of* ϵ_t t*-dimensional subspaces,* ϵ_{t-1} $(t-1)$*-dimensional subspaces,. . . ,*ϵ_1 *lines and* ϵ_0 *points.*

Linear Codes . . .

Theorem

A union of ϵ_{k-2} $(k-2)$-dimensional spaces, ϵ_{k-3} $(k-3)$-dimensional spaces, ..., ϵ_1 lines, and ϵ_0 points, which all are pairwise disjoint, exists in $\mathrm{PG}(k-1,q)$, if and only if there exists a linear $[v_k - \sum_{i=0}^{k-2} \epsilon_i v_{i+1}, k, q^{k-1} - \sum_{i=0}^{k-2} \epsilon_i q^i]$-code meeting the Griesmer bound.