Non-existence of some Moore Cayley digraphs

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based on joint work with
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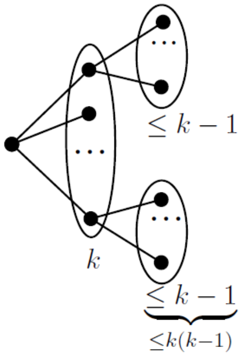
Moore bound

Let Γ be an undirected graph:

- regular of degree k;
- of diameter D;
- on N vertices.



$$N \leq 1 + k + k(k-1) + \ldots + k(k-1)^{D-1}$$

Moore graphs

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Then:

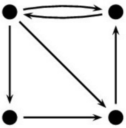
$$N \leq 1 + k + k(k-1) + \ldots + k(k-1)^{D-1},$$

and if equality attains (Damerell, Bannai&Ito):

Diameter D	Valency k	Moore graph	Transitivity
1	k	K_{k+1}	$\surd$
D	2	C_{2D+1}	$\surd$
2	3	Petersen	$\surd$
2	7	Hoffman-Singleton	$\surd$
2	57	?	✗

Digraphs = Mixed graphs = Partially directed graphs

Digraphs may have arcs as well as (undirected) edges:



An analogue of the Moore bound for digraphs can be derived, but its general form is quite complicated. In fact:

Theorem (Nguyen, Miller, Gimbert, 2007)

There are no Moore digraphs with diameter > 2.

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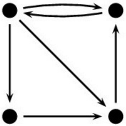

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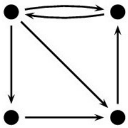

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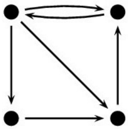

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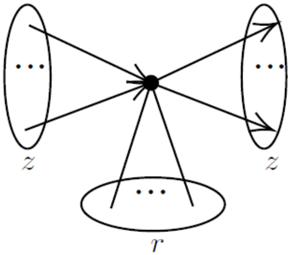

Moore digraphs

Theorem (Bosák, 1979)

Let Δ be a Moore digraph of diameter 2 with degrees (r, z). Then the number n of vertices of Δ is

$$n = (r + z)^2 + z + 1$$

and exactly one of the following cases occurs:

- $z = 1, r = 0$ (a directed 3-cycle);
- $z = 0, r = 2$ (an undirected 5-cycle);
- there exists an odd positive integer c such that

$$c \text{ divides } (4z - 3)(4z + 5) \text{ and } r = \frac{1}{4}(c^2 + 3).$$

Admissible values of r: $1, 3, 7, 13, 21, \ldots,$

For given r: infinitely many admissible values of z.

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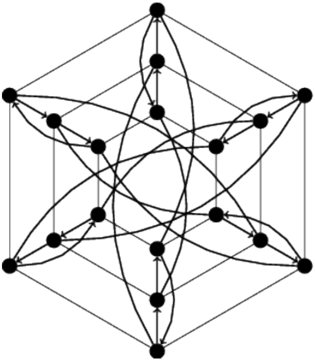
Known Moore digraphs

- $r = 1$: only Moore digraphs are the Kautz digraphs.
  (Gimbert, 2001)

  They are the line graphs of complete digraphs.

- $r > 1$: only three examples are known:
  - the Bosák graph on 18 vertices, $(r, z) = (3, 1)$;



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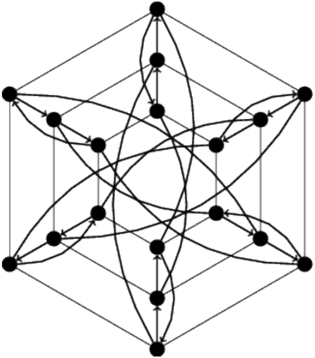

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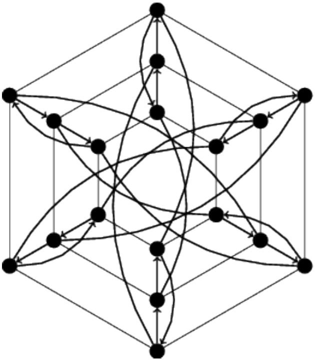

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Cayley digraphs

Given a finite group G and a subset $S \subseteq G \setminus \{1\}$,
with $S = S_1 \cup S_2$, $S_1 = S_1^{-1}$, and $S_2 \cap S_2^{-1} = \emptyset$,
the **Cayley** (di-)graph $Cay(G, S)$ has:

- the vertex set G;
- an arc $g \longrightarrow gs$ for every $g \in G$, $s \in S$;
- the undirected degree $r = |S_1|$;
- the directed degree $z = |S_2|$.

Moore digraphs of diameter 2 are defined by the property:

$$\text{for every pair } (x, y) \text{ of vertices of } \Delta,$$
$$\text{there is a unique trail } x \longrightarrow \ldots \longrightarrow y \text{ of length at most 2.}$$

If Δ is a Moore Cayley digraph $Cay(G, S)$, then:

- for $g \in S$, $\nexists$ a pair $(s_1, s_2) \in S \times S$ such that $g = s_1 s_2$;
- for $g \notin S$, $\exists!$ a pair $(s_1, s_2) \in S \times S$ such that $g = s_1 s_2$.

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Moore Cayley digraphs on at most 486 vertices, 1

n	r	z	Exist	Transitive	Cayley
18	3	1	!		Yes
40	3	3	No[1]		
54	3	4	No[1]		
84	7	2	No[1]		
88	3	6	?	?	No[2]
108	3	7	≥ 2		Yes
150	7	5	?	?	No[2]
154	3	9	?	?	No[2]
180	3	10	?	?	No[2]

[1]: López, Miret, Fernández: *Non-existence of some mixed Moore graphs of diameter* 2 *using SAT* (2016).

[2]: Erskine: *Mixed Moore Cayley graphs* (2017).

Moore Cayley digraphs on at most 486 vertices, 2

n	r	z	Exist	Transitive	Cayley
204	7	7	?	?	No[2]
238	3	12	?	?	No[2]
270	3	13	?	?	No[2]
294	13	4	?	?	No[2]
300	7	10	?	?	No[2]
340	3	15	?	?	No[2]
368	13	6	?	?	No[2]
374	7	12	?	?	No[2]
378	3	16	?	?	No[2]
460	3	18	?	?	No[2]
486	21	1	?	No[3]	

[3]: Jørgensen: talk in Pilsen (2018).

The adjacency algebra of Δ

The **adjacency matrix** $\mathsf{A} = \mathsf{A}(\Delta) \in \mathbb{R}^{V \times V}$:

$$(\mathsf{A})_{x,y} := \begin{cases} 1 & \text{if } x \to y, \\ 0 & \text{otherwise.} \end{cases}$$

As for every pair (x, y) of vertices of Δ, there is a unique trail $x \longrightarrow \ldots \longrightarrow y$ of length at most 2:

$$\mathsf{I} + \mathsf{A} + \mathsf{A}^2 = r\mathsf{I} + \mathsf{J}, \quad \text{and} \quad \mathsf{JA} = \mathsf{AJ} = k\mathsf{J},$$

so A is diagonalizable with 3 eigenspaces with eigenvalues $k = r + z$, and $\sigma_1, \sigma_2 \in \mathbb{Z}$, which are expressed in n, r, z.

The **projection matrix** E_{σ_i} onto the (right) σ_i-eigenspace:

$$\mathsf{E}_{\sigma_i} \in \langle \mathsf{A}, \mathsf{I}, \mathsf{J} \rangle.$$

Duval (1988); Jørgensen (2003); Godsil, Hobart, Martin (2007)

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The Higman-Benson observation

- $G \leq \mathrm{Aut}(\Delta)$;
- $g \in G$: $g \mapsto \mathsf{X}_g$, a permutation matrix;
- $\mathsf{X}_g\mathsf{A} = \mathsf{A}\mathsf{X}_g$, and as $\mathsf{E}_{\sigma_i} \in \langle \mathsf{A}, \mathsf{I}, \mathsf{J} \rangle \Rightarrow \mathsf{X}_g\mathsf{E}_{\sigma_i} = \mathsf{E}_{\sigma_i}\mathsf{X}_g$;
- By using this, one can show that

$$\mathrm{Tr}(\mathsf{E}_{\sigma_i}\mathsf{X}_g) \in \mathbb{Z};$$

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$$\in \mathbb{Q}, \text{ but often } \notin \mathbb{Z}.$$

- Now:

$$\mathrm{Tr}(\mathsf{I}\mathsf{X}_g) = \#\{v \in \Delta \mid v = v^g\},$$
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Application to Moore Cayley digraphs

- Δ: a Moore Cayley digraph over G with degrees (r, z).

  Recall: $\exists$ an odd positive c which divides $(4z-3)(4z+5)$, and
  $$r = \frac{1}{4}(c^2+3).$$
- $G \leq \mathrm{Aut}(\Delta)$ is a regular subgroup;
- The Higman-Benson observation shows that:
  $$\left(-\frac{1}{c}\mathrm{Tr}(\mathsf{AX}_g) + \frac{c^2-4c+4z+5}{4c} \right) \in \mathbb{Z}$$
  for any automorphism $g \in G$, where
  $$\mathrm{Tr}(\mathsf{AX}_g) = \#\{v \in \Delta \mid v \longrightarrow v^g\}.$$
- For certain orders $n = |G|$ and $|g|$, this implies that $\mathrm{Tr}(\mathsf{AX}_g)$ is "too large" so that it contradicts:

  for every pair (x, y) of vertices of Δ,
  there is a unique trail $x \longrightarrow \ldots \longrightarrow y$ of length at most 2.

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- Δ: a Moore Cayley digraph over G with degrees (r, z).

  Recall: $\exists$ an odd positive c which divides $(4z-3)(4z+5)$, and
  $$r = \frac{1}{4}(c^2+3).$$
- $G \leq \mathrm{Aut}(\Delta)$ is a regular subgroup;
- The Higman-Benson observation shows that:
  $$\left(-\frac{1}{c}\mathrm{Tr}(\mathsf{AX}_g) + \frac{c^2-4c+4z+5}{4c}\right) \in \mathbb{Z}$$
  for any automorphism $g \in G$, where
  $$\mathrm{Tr}(\mathsf{AX}_g) = \#\{v \in \Delta \mid v \longrightarrow v^g\}.$$
- For certain orders $n = |G|$ and $|g|$, this implies that $\mathrm{Tr}(\mathsf{AX}_g)$ is "too large" so that it contradicts:

  for every pair (x, y) of vertices of Δ,
  there is a unique trail $x \longrightarrow \ldots \longrightarrow y$ of length at most 2.

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Results

n	r	z	Cayley (by Erskine)	Cayley (by Higman)
40	3	3	No	No
54	3	4	No	
84	7	2	No	No
88	3	6	No	No
150	7	5	No	
154	3	9	No	No
180	3	10	No	

Results

n	r	z	Cayley (by Erskine)	Cayley (by Higman)
204	7	7	No	No
238	3	12	No	No
270	3	13	No	
294	13	4	No	
300	7	10	No	
340	3	15	No	No
368	13	6	No	No
374	7	12	No	No
378	3	16	No	
460	3	18	No	No
486	21	1		

It rules out 29 out of 58 feasible parameter sets for $v \leq 2000$.

Although it does not cover all results by Erskine, the proof is computer-free.

Question

n	r	z	Cayley (by Erskine)	Cayley (by Higman)
40	3	3	No	No
54	3	4	No	
84	7	2	No	No
88	3	6	No	No
150	7	5	No	
154	3	9	No	No
180	3	10	No	

$$150 = 3\times\text{Hoffman-Singleton} + \text{arcs?}$$