# The Independence Number of the Orthogonality Graph

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Finite Geometry and Friends

## The Orthogonality Graph

**Graph $\Gamma$:** $X = \{-1, 1\}^n$, $x \sim y \Leftrightarrow \langle x, y \rangle = 0$.
**Alternative:** $X = \{0, 1\}^n$, $x \sim y$ if $d(x, y) = n/2$.

What is $\alpha(\Gamma)$?

1. $n$ odd: $\alpha(\Gamma) = 2^n$ (no edges).
2. $n \equiv 2 \pmod 4$: $\alpha(\Gamma) = 2^{n-1}$ (bipartite).

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2. $n \equiv 2 \pmod 4$: $\alpha(\Gamma) = 2^{n-1}$ (bipartite).
3. $n \equiv 0 \pmod 4$: Interesting!

**Example $n = 4$:** 0000, 0001, 1110, 1111.

**Exercise:** Show that $\alpha(\Gamma) = 4$.

**Hint:** Use a Hadamard matrix of size 4.

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**Example** $n = 8$**:**

00000000, 00000010, 00000100, 00001000, 00010000, 00100000, 01000000, 10000000,
00000001, 00000011, 00000101, 00001001, 00010001, 00100001, 01000001, 10000001,
11111110, 11111100, 11111010, 11110110, 11101110, 11011110, 10111110, 01111110,
11111111, 11111101, 11111011, 11110111, 11101111, 11011111, 10111111, 01111111.

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**Size:** 32.
**Exercise:** Show that $\alpha(\Gamma) = 32$.

**Hint:** Use a Hadamard matrix of size 8.

**Question:** Classification?

# Independent Sets, General

**Example** $n = 4$**:** 0000, 0001, 1110, 1111.

What is the **construction** behind **examples**?

Set

$$Y = \{(c_1, \ldots, c_n) \in X : \\ |\{i : 1 \leq i \leq n-1, c_i = 1\}| < \tfrac{1}{4}n \text{ or } \geq \tfrac{3}{4}n\}.$$

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We have

$$a_n := |Y| = 4 \sum_{i=0}^{n/4-1} \binom{n-1}{i}.$$

Hence, $\alpha(\Gamma) \ge a_n$.

# Conjecture

Recall: $\alpha(\Gamma)$ is at least

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## Conjecture

We have $\alpha(\Gamma) = a_n$.

**Conjecture** due to Frankl (1986/1987),[1] Galliard for $n = 2^k$ (2001), Newman (2004).

[1] A 1987 paper by Frankl and Rödl contains a reference to a 1986 paper by Frankl together with the claim that there this conjecture is made. The 1986 paper does not contain this conjecture, but an argument for $\alpha(\Gamma) \geq a_n$.

# What is known?

**Conjecture**

We have $\alpha(\Gamma) = a_n$.

Results:

- Frankl (1986): $\alpha(\Gamma) = a_n$ if $n = 4p^k$, $p$ odd prime.

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- Frankl (1986): $\alpha(\Gamma) = a_n$ if $n = 4p^k$, $p$ odd prime.
- Frankl-Rödl (1987): $\alpha(\Gamma) \leq 1.99^n$.
- De Klerck-Pasechnik (2005): $\alpha(\Gamma) = a_n$ for $n = 16$.

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- I-Tanaka + referee (2019): $\alpha(\Gamma) = a_n$ for $n = 24$.

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Galliard, Tapp, Wolf et al. ($\sim 2000$): interest for $n = 2^k$ due to **quantum-telepathy games** in **quantum information theory**.

# Small Cases

**How** got the **small** cases solved?

**Lemma**

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**Method:** Schrijver's semidefinite programming bound.

**Theorem**

$\alpha(\Gamma) = a_n$ *for* $n = 24$.

**Method:** "2nd level" of Schrijver's SDP bound.

Suggested in **I-Tanaka (2019)**, calculations done by **referee**.

# The Proof for $n = 2^k$ (I)

**Theorem (I-Tanaka (2019))**

$\alpha(\Gamma) = a_n$ *for* $n = 2^k$.

**Proof:** As Frankl for $n = 4p^k$, $p$ odd prime, with **one** difference.

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**First Idea:** Reduce the problem to 4 problems on the hypercube on $n-1$ coordinates.

**Recall** $n = 4$ example: 0000, 0001, 1110, 1111.

**Not too hard!**

# The Proof for $n = 2^k$ (II)

**Theorem (I-Tanaka (2019))**

$\alpha(\Gamma) = a_n$ *for* $n = 2^k$.

Recall

$$a_n = 4 \sum_{i=0}^{n/4-1} \binom{n-1}{i}.$$

**Observation:** Eigenspaces $V_0, V_1, \ldots$ of the orthogonality graph on $n-1$ coordinates have dimensions:

$$\binom{n-1}{0}, \binom{n-1}{1}, \binom{n-1}{2}, \binom{n-1}{3}, \ldots$$

**Second Idea:** Bound the problem by dimension of eigenspaces.

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**In Detail:** Show that **independent set** $Y$ of size $4\alpha$ **corresponds to a subspace** of $V_0 + V_1 + \ldots + V_{n/4-1}$ of dimension at least $\alpha$.

**Frankl's Method:** Replace distance $\xi$ by $\binom{\xi-1}{n/4-1}$.
**Works** for $n = 4p^k$, but not $p$ even.

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**Frankl's Method:** Replace distance $\xi$ by $\binom{\xi-1}{n/4-1}$.
**Works** for $n = 4p^k$, but not $p$ even.

**I-Tanaka:** Replace[2] distance $\xi$ by $\binom{\xi/2-1}{n/4-1}$.
**Works** also for $n = 2^k$.

[2] This hides intermediate research such as $\frac{1}{16}\xi^3 - \frac{3}{2}\xi^2 + \frac{49}{4}\xi - 33$ for $k = 4$.

Thank you for your attention!