Power sum polynomials and discrete tomography

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Finite Geometry & Friends

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Outline

Motivation

Definition

Let $P = (p_1, \ldots, p_{n+1})$ be a point of $\mathrm{PG}(n, q)$. The corresponding Rédei factor is the linear polynomial $P \cdot \mathbf{X} = p_1 X_1 + \ldots + p_{n+1} X_{n+1}$.

The zeros of $P \cdot \mathbf{X}$ are (the Plücker coordinates of) the hyperplanes through P.

Definition

Let $S = \{P_i : i = 1, \ldots, |S|\} \subseteq \mathrm{PG}(n, q)$ be a point set. The Rédei polynomial of S is defined as

$$R^S(X_1, \ldots, X_n) := \prod_{i=1}^{|S|} P_i \cdot \mathbf{X}.$$

Definition

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$$G^S(X_1, \dots, X_{n+1}) := \sum_{i=1}^{|S|} (P_i \cdot \mathbf{X})^{q-1}.$$

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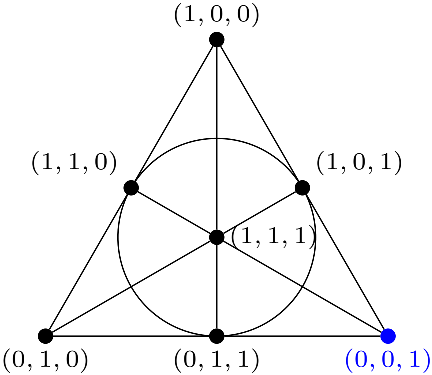
P. Sziklai wrote:

> The advantage of the power sum polynomial (compared to the Rédei-polynomial) is that it is of lower degree if $|S| \geq q$. The disadvantage is that while the Rédei-polynomial contains the complete information of the point set (S can be reconstructed from it), the power sum polynomial of two different point sets may coincide. This is a hard task in general to classify all the point sets belonging to one given power sum polynomial.

Example

In PG(2, 2)



$S = \{(0, 0, 1)\}$

$G^S(X, Y, Z) = Z$

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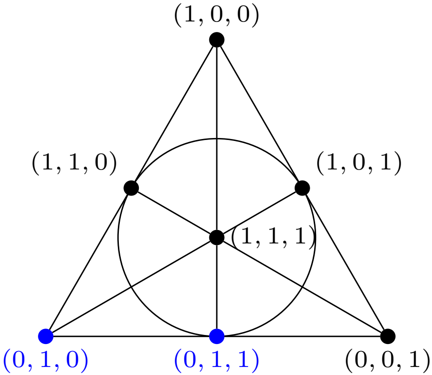
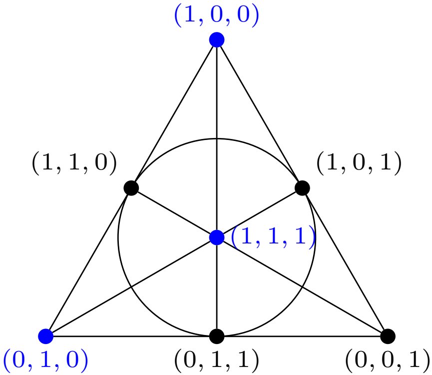


$$S = \{(0,1,0),(0,1,1)\}$$

$$\begin{aligned} G^S(X,Y,Z) &= (Y)+(Y+Z) \\ &= Z \end{aligned}$$

Example

In PG$(2,2)$

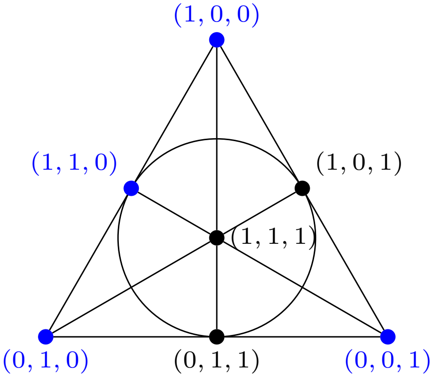


$$S = \{(1,0,0),(0,1,0),(1,1,1)\}$$

$$G^S(X,Y,Z) = Z$$

Example

In PG(2, 2)



$$S = \{(0,0,1),(1,0,0),(0,1,0),(1,1,0)\}$$

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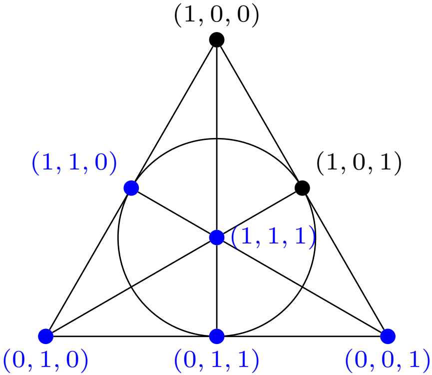

$$S = \{(0,0,1),(0,1,0),\\ (0,1,1),(1,1,0),(1,1,1)\}$$

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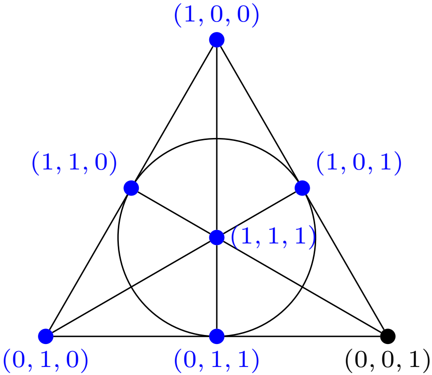
Example

In $\mathrm{PG}(2,2)$



$S = \mathrm{PG}(2,2) \setminus \{(0,0,1)\}$

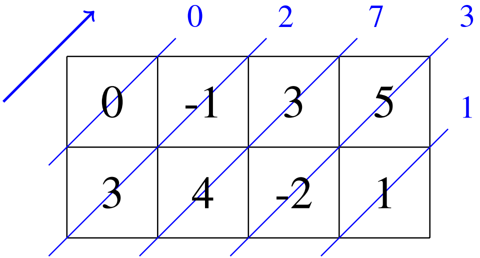
$G^S(X, Y, Z) = Z$

Discrete tomography

Tomography is concerned with the reconstruction of the internal of an object from the knowledge of its projections taken along given directions.

In discrete tomography the object (image) is a set of pixels and directions have rational slope.

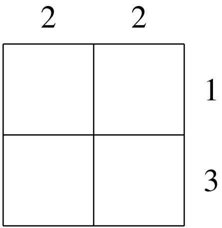
Usual assumption: the image is confined in a given finite grid.



Uniqueness of reconstruction

One of the main tasks of tomography is to ensure that the reconstructed image equals the original one.

In general, it is not achievable.



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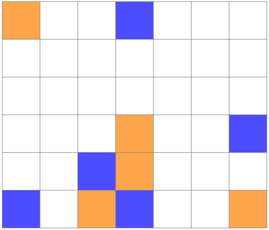
$$\begin{array}{|c|c|}\hline 2 & -1 \\ \hline 0 & 3 \\ \hline \end{array} \quad + \quad \alpha \begin{array}{|c|c|}\hline -1 & 1 \\ \hline 1 & -1 \\ \hline \end{array}$$

Switching components and ghosts

Definition

A switching component w.r.t. a set U of directions is a pair of sets of pixels, having the same projections along the directions in U. The union of the elements of a switching component (suitably weighted) constitute the support of a ghost, which is a nonzero image with null projections along U.

Example: a switching component (left) and a ghost (right) w.r.t. the coordinate directions.



$\longleftrightarrow$

1	0	0	−1	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	1	0	0	−1
0	0	−1	1	0	0	0
−1	0	1	−1	0	0	1

Looking for connections & master plan

Discrete tomography		**Projective geometry**
lattice grid	$\longleftrightarrow$	$\mathrm{PG}(2, q)$
image	$\longleftrightarrow$	S
projections along U	$\longleftrightarrow$	G^S
ghost w.r.t. U	$\longleftrightarrow$	S such that $G^S \equiv \mathbf{0}$

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How to compute the zeros of G^S: look at the intersection of S with lines.

In fact, if $|S \cap \ell| = m$ for a line ℓ, then $G^S(\ell) = |S| - m$.

Some general results

Let $q = p^h$, p prime.

Lemma

Let $S \subseteq \mathrm{PG}(2,q)$ be a ghost. Then $\mathrm{PG}(2,q)\backslash S$ is a ghost.

Proof

For every line ℓ, $G^S(\ell) = |S| - m = 0 \mod p$. Then

$$\begin{aligned} G^{\mathrm{PG}(2,q)\backslash S}(\ell) &= |\mathrm{PG}(2,q)\backslash S| - |(\mathrm{PG}(2,q)\backslash S) \cap \ell| \\ &= q^2 + q + 1 - |S| - (q + 1 - m) \\ &= m - |S| = 0 \mod p. \end{aligned}$$

□

$\Longrightarrow$ $\emptyset$ and $\mathrm{PG}(2,q)$ are ghosts.

Some general results

Lemma

A partial pencil $\mathcal{P}$ of $\lambda p + 1$ lines, $\lambda = 0, \dots, p^{h-1}$, is a ghost. Consequently, a set of $q - \lambda p$ lines through a point P minus P is a ghost.

In particular, it results that a line is a ghost, as well as every affine plane contained in PG$(2, q)$.

Proof

Every line ℓ meets $\mathcal{P}$ in either one, $\lambda p + 1$ or $q + 1$ points. In all cases $m = 1 \mod p$ and

$$G^{\mathcal{P}}(\ell) = (\lambda p + 1)q + 1 - m = 0 \mod p.$$

□

Blocking sets

Definition

A blocking set (for the lines) of $\mathrm{PG}(2,q)$ is a set of points meeting every line of $\mathrm{PG}(2,q)$ and not containing a line.

Definition

A Baer subplane of $\mathrm{PG}(2,q)$, q square, is a subplane of order $\sqrt{q}$.

A Baer subplane is a minimal blocking set for lines (with minimum size) and a ghost (each line intersects it in either 1 or $\sqrt{q}+1$ points).

Definition

A unital of $\mathrm{PG}(2,q)$, q square, is a set of $q\sqrt{q}+1$ points meeting every line of $\mathrm{PG}(2,q)$ in either 1 or $\sqrt{q}+1$ points.

A unital is a minimal blocking set with maximum cardinality and a ghost.

Cardinalities

To summarize, there are ghosts with size:

$$0,\ q+1,\ pq,\ (p+1)q+1,\ 2pq,$$
$$\dots,\ q^2-(p-1)q+1,\ q^2,\ q^2+q+1.$$

Moreover, if q is a square, there exist ghosts with size:

$$q+\sqrt{q}+1,\ q\sqrt{q}+1,\ q^2-(\sqrt{q}-1)q,\ q^2-\sqrt{q}.$$

Operation between sets

In discrete tomography, a ghost can be added to an image.
How to move among sets with the same power sum polynomial?
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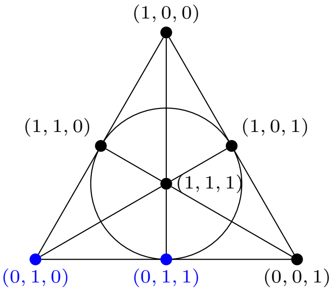

Conclusions and further work

How to go on:

- further investigate the connections between discrete tomography and power sum polynomials;
- find a “basis” for the ghosts (lines?);
- (dis)prove that also blocking sets can be seen as (multi)union of lines;
- find other kinds of ghosts, which are neither blocking sets nor partial pencils.

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Thank you for your attention!