DETERMINATION OF JUMPS OF DISTRIBUTIONS BY DIFFERENTIATED MEANS

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Abstract. Differentiated means are defined in order to find formulas for jumps of distributions. We analyze two types of jumps occurring in the notions of distributional jump behavior and symmetric jump behavior. We start by defining what we call Riesz differentiated means for numerical series, then the differentiated means are extended to distributional evaluations for the Schwartz class of tempered distributions. The jumps of tempered distributions are completely determined by the differentiated means of the Fourier transform. We also find formulas for the jumps in terms of the asymptotic behavior of partial derivatives of harmonic representations and harmonic conjugate functions. Applications to Fourier series are given.

1. Introduction

A classical result of L. Fejér ([11],[47, Vol.I, p.107]) states that if f is a 2π-periodic function of bounded variation having Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{1.1}$$

then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=-N}^{N} n c_n e^{inx_0} = \frac{1}{i\pi}\left(f(x_0^+) - f(x_0^-)\right) \ , \tag{1.2}$$

at every point $x = x_0$ where f has a simple discontinuity. Therefore, the limit (1.2) involving the differentiated Fourier series determines the

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2000 *Mathematics Subject Classification.* Primary 40C05, 40C99, 40H05, 42A24, 42A50. Secondary 46F10, 46F20, 40A05, 40D25.

Key words and phrases. Jumps, Fourier series, summability methods, asymptotic behavior of generalized functions.

The first author gratefully acknowledges support by the Louisiana State Board of Regents grant LEQSF(2005-2007)-ENH-TR-21.

jumps of the function. If one writes (1.1) in the sine-cosine form, i.e.,

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\ , \tag{1.3}$$

then (1.2) takes the form

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}n(b_n\cos nx-a_n\sin nx)=\frac{1}{\pi}\left(f(x_0^+)-f(x_0^-)\right)\ . \tag{1.4}$$

Relation (1.4) is an example of what we call a *differentiated mean.* A. Zygmund studied a more general problem in [46] (see also [47]), under an extended notion of symmetric jump related to the notion of *de la Vallée Poussin generalized derivatives*, he obtained formulas for the jump in terms of Cesàro versions of (1.4).

The study of the jump behavior and the determination of jumps by different types of means has become an important area because of its applications in edge detection from spectral data [12, 13]. Results of this kind are important in applied mathematics because they have direct consequences in computational algorithms (consult references in [12]). Recently, it has attracted the attention of many authors and some generalization of classical results have been given [1, 7, 12, 13, 14, 22, 23, 24, 25, 28, 29, 30, 34, 35, 39, 41, 44, 45]. Basically, we could say that these generalizations go in three directions: extensions of the notion of jump, enlargement of the class of functions, and the use of different means to determine the jump.

In the present article we provide results of a general character. We leave the usual classes of classical functions, and obtain results for very general distributions and tempered distributions. The usual notions for jumps are extended to distributional notions for pointwise jumps, the jump behavior and the symmetric jump behavior [7, 39, 40, 41]. The distributional jumps include those of classical functions. In order to determine the pointwise jumps of distributions, we define a new type of means, the differentiated means in the Cesàro and Riesz sense; these means are applicable to Fourier series and to the Fourier transform of tempered distributions. We then obtain formulas of type (1.2) in terms of the differentiated means of the Fourier transform of tempered distributions. Our results are applicable to Fourier series, we therefore generalize some of the results mentioned above. The approach taken in this article also has a numerical advantage with respect to other approaches; in the case of the jump occurring in the jump behavior, our formulas only use partial part of the spectral data (either positive or negative part). For the case of symmetric jumps, we recover some results from [46, 47]. When we deal only with distributions in $\mathcal{D}'(\mathbb{R})$, thus we do not have the Fourier transform available, we can still use differentiated *Abel-Poisson* means in order to determine the jump, that is, the jump can be calculated in terms of the asymptotic behavior of partial derivatives of harmonic representations and harmonic conjugates.

2. Preliminaries and notation

In this section we explain the notions of jumps to be considered in this article. We also give a summary of the tools and notions that we will use from the theory of asymptotic analysis on spaces of distributions [10, 31, 43].

We will assume the reader is familiar with the basic notions from the theory of summability of numerical series, specifically with the summability methods by Cesàro, Riesz and Abel means. There is a very rich and extensive literature on the subject; for instance, we refer to [4, 10, 18, 19, 20, 47].

We denote the Schwartz spaces of smooth compactly supported test functions and smooth rapidly decreasing test functions by $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$, respectively; they are equipped with the usual Schwartz topologies [33]. Their corresponding dual spaces, the spaces of distributions and tempered distributions, are denoted by $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$. The space $\mathcal{E}(\mathbb{R})$ denotes the space of all smooth functions over $\mathbb{R}$ with its usual Fréchet space structure; its dual, $\mathcal{E}'(\mathbb{R})$, is then the space of distributions with compact support. We refer the reader to [33] for the well known properties of these spaces.

We will make use of the Fourier transform on $\mathcal{S}'(\mathbb{R})$. On test functions of $\mathcal{S}(\mathbb{R})$, we use the following Fourier transform

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \phi(t) e^{-ixt} \mathrm{d}t \ , \tag{2.1}$$

and as usual we extend it to $\mathcal{S}'(\mathbb{R})$ by transposition.

Let us define the notions of *jump behavior* and *symmetric jump behavior* of distributions at points [7, 39, 40, 41]. We begin with the jump behavior.

Definition 1. *A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to have a distributional jump behavior (or jump behavior) at $x = x_0 \in \mathbb{R}$ if it satisfies the following distributional asymptotic relation*

$$f\left(x_0 + \varepsilon x\right) = \gamma_{-} H(-x) + \gamma_{+} H(x) + o(1) \ , \tag{2.2}$$

as $\varepsilon \to 0^+$ *in* $\mathcal{D}'(\mathbb{R})$*, where* H *is the Heaviside function, i.e., the characteristic function of* $(0,\infty)$*, and* $\gamma_\pm$ *are constants. The jump (or saltus) of* f *at* $x = x_0$ *is defined then as the number* $[f]_{x=x_0} = \gamma_+ - \gamma_-$ *.*

The meaning of (2.2) is in the weak topology of $\mathcal{D}'(\mathbb{R})$, in the sense that for each $\phi \in \mathcal{D}(\mathbb{R})$,

$$\lim_{\varepsilon\to 0^+} \langle f(x_0+\varepsilon x), \phi(x)\rangle = \gamma_- \int_{-\infty}^{0} \phi(x)\,\mathrm{d}x + \gamma_+ \int_0^\infty \phi(x)\,\mathrm{d}x \ . \tag{2.3}$$

Observe that when $\gamma_+ = \gamma_-$ we recover the usual Łojasiewicz notion of the value of a distribution at a point [26]. It should be noticed that our notion includes the jump of ordinary functions; indeed, if a locally integrable function has a discontinuity of the first kind, that is, the right and left limits $f(x_0^\pm)$ exist, then it satisfies (2.3) with $\gamma_\pm = f\left(x_0^\pm\right)$. In particular, jumps of functions of local bounded variation are distributional jump behaviors. We provide more examples of classical notions for jumps in Examples 1 and 2 below.

We now define the symmetric jump behavior.

Definition 2. *A distribution* $f \in \mathcal{D}'(\mathbb{R})$ *is said to have a distributional symmetric jump behavior (or symmetric jump behavior) at* $x = x_0 \in \mathbb{R}$ *if the jump distribution* $\psi_{x_0}(x) = f(x_0+x) - f(x_0-x)$ *has jump behavior at* $x = 0$ *. In such a case, we define the jump (or saltus) of* f *at* $x = x_0$ *as the number* $[f]_{x=x_0} = [\psi_{x_0}]_{x=0}/2$ *.*

It is easy to see that the jump behavior of the jump distribution in Definition 2 must be of the form

$$\psi_{x_0}(\varepsilon x) = [f]_{x=x_0} \operatorname{sgn} x + o(1) \quad \text{as } \ \varepsilon \to 0^+ \ \text{ in } \ \mathcal{D}'(\mathbb{R}) \ , \tag{2.4}$$

where $\operatorname{sgn} x$ is the signum function, i.e., $H(x) - H(-x)$.

We now discuss two examples of particular types of jump behavior related to classical functions. It is not difficult to see that both examples are particular cases of our distributional notions for jumps.

Example 1. *Lebesgue jumps.* Let f be a locally (Lebesgue) integrable function, then we say that f has a Lebesgue jump behavior if there are two numbers $\gamma_\pm$ such that

$$\lim_{h\to 0^\pm} \frac{1}{h} \int_{x_0}^{x_0+h} |f(x) - \gamma_\pm|\,\mathrm{d}x = 0 \ . \tag{2.5}$$

We say that f has a symmetric Lebesgue jump behavior if there is a numbers $d = [f]_{x_0}$ such that

$$\lim_{h\to 0^+} \frac{1}{h} \int_0^h |f(x_0+x) - f(x_0-x) - d|\,\mathrm{d}x = 0 \ . \tag{2.6}$$

Example 2. *Jump behavior of the first order.* Let μ be a Radon measure [37]. Then we say that μ has a jump behavior of the first order if there exist $\gamma_\pm$ such that

$$\lim_{h\to 0^\pm} \frac{1}{h}\int_{x_0}^{x_0+h} \mathrm{d}\mu(x) = \gamma_\pm \ . \tag{2.7}$$

We say that μ has a symmetric jump behavior of the first order if there exists $d = [f]_{x=x_0}$ such that

$$\lim_{h\to 0^+} \frac{1}{h}\left(\int_{x_0}^{x_0+h} \mathrm{d}\mu(x) - \int_{x_0-h}^{x_0} \mathrm{d}\mu(x)\right) = d \ . \tag{2.8}$$

A particular case is obtained if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. Moreover, the first order jump behavior and symmetric jump behavior can still be defined by an integral expression even if f is not locally (Lebesgue) integrable but just Denjoy locally integrable [16]. For instance, in such a case the existence of the jump behavior of the first order is equivalent to the existence of the limits

$$\lim_{h\to 0^\pm} \frac{1}{h}\int_{x_0}^{x_0+h} f(x)\,\mathrm{d}x = \gamma_\pm \ , \tag{2.9}$$

where the last integral is taken in the Denjoy sense, and similarly for the symmetric jump, $\lim_{h\to 0^+}(1/h)\int_0^h (f(x_0+x) - f(x_0-x))\,\mathrm{d}x = \gamma_\pm$.

The notions of Lebesgue jump and symmetric jump behaviors have been widely used in Fourier series by many authors [11, 27, 47]. While the use of first order jump and symmetric jump behaviors have become popular recently [28, 29, 30, 44] for locally integrable functions.

We can consider more general asymptotic behaviors of distributions at points than (2.2); indeed, one could try to look for a representation of the form

$$f(x_0+\varepsilon x) = \varepsilon^\alpha g(x) + o(\varepsilon^\alpha)\ , \quad \text{as } \varepsilon \to 0^+ \ , \tag{2.10}$$

in the space $\mathcal{D}'(\mathbb{R})$. Needless to say that (2.10) holds under evaluation at each test functions of $\mathcal{D}(\mathbb{R})$. One can then show that g has to be homogeneous of degree α [10, 31, 43]. The asymptotic relation (2.10) is an example of the so called *quasiasymptotic behaviors of distributions* [31, 38, 42, 43]. Suppose now that $f \in \mathcal{S}'(\mathbb{R})$. If we write $\mathcal{S}'(\mathbb{R})$ instead of $\mathcal{D}'(\mathbb{R})$ in (2.10), we mean that it holds after evaluation at each test function of $\mathcal{S}(\mathbb{R})$. Because of the results of [42], one has that if it is assumed that a tempered distribution satisfies (2.10) just in $\mathcal{D}'(\mathbb{R})$, then it will automatically have the same behavior in $\mathcal{S}'(\mathbb{R})$. In particular this is true for the jump and symmetric jump behaviors.

We have the analog to (2.10) at infinity, in such a case the dilation parameter is taken to infinity. Another useful concept to study the asymptotic behavior of distributions at infinity is that of *the Cesàro behavior* of distributions, it is studied by using the order symbols $O(x^{\alpha})$ and $o(x^{\alpha})$ in the Cesàro sense [6, 10]. If $f \in \mathcal{D}'(\mathbb{R})$ and $\alpha \in \mathbb{R} \setminus \{-1, -2, -3, \ldots\}$, we say that $f(x) = O(x^{\alpha})$ as $x \to \infty$ in the Cesàro sense and write

$$f(x) = O(x^{\alpha}) \qquad (\mathrm{C})\ , \text{ as } x \to \infty\ , \tag{2.11}$$

if there exists $m \in \mathbb{N}$ such that each primitive F of order m of f, i.e., $F^{(m)} = f$, is an ordinary function for large arguments and satisfies the ordinary order relation

$$F(x) = p(x) + O(x^{\alpha+m})\ , \qquad \text{as } x \to \infty\ , \tag{2.12}$$

for a suitable polynomial p of degree $m-1$ at the most. To emphasize the order of the Cesàro behavior, it is convenient sometimes to write,

$$f(x) = O(x^{\alpha}) \qquad (\mathrm{C}, m)\ , \text{ as } x \to \infty\ , \tag{2.13}$$

in this case we call m the order of the Cesàro behavior. A similar definition applies to the little o symbol. The definitions when $x \to -\infty$ are clear.

One can use these ideas to define the limit of a distribution at infinity in the Cesàro sense; indeed, we say that

$$\lim_{x\to\infty} f(x) = \gamma \qquad (\mathrm{C})\ , \tag{2.14}$$

if $f(x) = \gamma + o(1)$ (C), as $x \to \infty$. We can even define distributional evaluations in the Cesàro sense [6, 10]. Let f be a distribution with support bounded on the left and let $\phi \in \mathcal{E}(\mathbb{R})$, we say the evaluation $\langle f(x), \phi(x)\rangle$ has a value γ in the Cesàro sense, and write

$$\langle f(x), \phi(x)\rangle = \gamma \qquad (\mathrm{C}) \tag{2.15}$$

if the first order primitive G of $g = \phi f$, with support bounded on the left, satisfies $\lim_{x\to\infty} G(x) = \gamma$ (C). If f has support bounded at the right then $\langle f(x), \phi(x)\rangle$ (C) exists if and and only if $\langle f(-x), \phi(-x)\rangle$ (C) exists and we have that $\langle f(x), \phi(x)\rangle = \langle f(-x), \phi(-x)\rangle$ (C) .

3. Differentiated Riesz and Cesàro means

In this section we shall define a new type of means, the *differentiated Riesz and Cesàro means.* They will be the main tool of the next section when finding formulas for jumps of distributions. We begin with the case of series.

Definition 3. *Let* $\{\lambda_n\}_{n=0}^{\infty}$ *be an increasing sequence of non-negative numbers such that* $\lim_{n\to\infty}\lambda_n=\infty$*. Let* k *and* $m\in\mathbb{N}$*. We say that a series* $\sum_{n=0}^{\infty}c_n$ *is summable to* γ *by the* k*-differentiated Riesz means of order* m*, relative to* $\{\lambda_n\}_{n=0}^{\infty}$*, if*

$$\lim_{x\to\infty} k\binom{m+k}{m}\sum_{\lambda_n<x}c_n\left(\frac{\lambda_n}{x}\right)^k\left(1-\frac{\lambda_n}{x}\right)^m=\gamma\ . \tag{3.1}$$

In such a case, we write

$$\text{d.m.}\sum_{n=0}^{\infty}c_n=\gamma \qquad \left(\mathrm{R}^{(k)},\{\lambda_n\},m\right)\ . \tag{3.2}$$

When $\lambda_n=n$*, we simply write* $\left(\mathrm{C}^{(k)},m\right)$ *for* $\left(\mathrm{R}^{(k)},\{n\},m\right)$*, and say that the series is is summable by the* k*-differentiated Cesàro means of order* m*.*

Notice that if $k=0$, the means are trivial. So from now on, we assume that k is always a positive integer, while m might be equal to 0. Observe also that it is possible to take non-integral values for k and m; however, we will only use the integral case in this article and thus we shall always take $k,m\in\mathbb{N}$. When we do not want to make reference to m, we simply write $\left(\mathrm{C}^{(k)}\right)$ or $\left(\mathrm{R}^{(k)},\{\lambda_n\}\right)$, respectively.

The first surprising fact about our means is that these methods of summation are not *regular* [18]; that is, if $\sum_{n=0}^{\infty}c_n$ is convergent to γ, we do not necessarily have that $\sum_{n=0}^{\infty}c_n$ is $\left(\mathrm{R}^{(k)},\{\lambda_n\},m\right)$-summable to γ. However, our method is what Hardy calls $\mathfrak{I}_c$ [18, p.43], it means that it sums convergent series but not necessarily to the same value of convergence. That fact is presented in the next proposition: Indeed, our method of differentiated Riesz means sums all convergent series to 0.

Proposition 1. *Suppose that* $\sum_{n=0}^{\infty}c_n$ *is convergent to some value* γ*, then*

$$\text{d.m.}\sum_{n=0}^{\infty}c_n=0 \qquad \left(\mathrm{R}^{(k)},\{\lambda_n\},m\right)\ . \tag{3.3}$$

Proof. We assume that $m\geq 1$, when $m=0$ the proof is similar. Define $s(x)=\sum_{\lambda_n<x}c_n$. We have that $s(x)\to\gamma$ as $x\to\infty$. So,

$$\sum_{\lambda_n<x}c_n\frac{\lambda_n^k}{x^k}\left(1-\frac{\lambda_n}{x}\right)^m=\int_0^x\left(\frac{t}{x}\right)^k\left(1-\frac{t}{x}\right)^m\mathrm{d}s(t)$$

$$=\int_0^1\left((m+k)t-k\right)t^{k-1}(1-t)^{m-1}s(xt)\,\mathrm{d}t\ ,$$

and the last term converges to

$$\gamma \left((m+k) \int_0^1 t^k (1-t)^{m-1} \mathrm{d}t - k \int_0^1 t^{k-1} (1-t)^{m-1} \mathrm{d}t \right) = 0 \ ,$$

as required. □

The fact that the differentiated Riesz means sum convergent series to 0 will be reflected in their ability to detect the jump of Fourier series.

We now generalize Definition 3 to distributional evaluations.

Definition 4. *Let $g \in \mathcal{D}'(\mathbb{R})$ be a distribution with support bounded on the left and let $\phi \in \mathcal{E}(\mathbb{R})$. We say that the evaluation $\langle g(x), \phi(x) \rangle$ has a value γ in the k-differentiated Cesàro sense (at order m) and write*

$$\text{d.m.} \ \langle g(x), \phi(x) \rangle = \gamma \qquad \left(\mathrm{C}^{(k)}, m \right) \ , \tag{3.4}$$

if

$$x^k \phi(x) g(x) = \gamma x^{k-1} + o\left(x^{k-1}\right) \qquad (\mathrm{C}, m+1) \ , \quad x \to \infty \ . \tag{3.5}$$

A similar definition applies if g has support bounded on the right; notice that unlike the (C) sense, where $\langle f(-x), \phi(x) \rangle = \langle f(x), \phi(-x) \rangle$ (C), in this case we have that d.m. $\langle f(-x), \phi(x) \rangle = -$d.m. $\langle f(x), \phi(-x) \rangle$ $\left(\mathrm{C}^{(k)}\right)$. Again, if we do not want to make reference to m, we simply write $\left(\mathrm{C}^{(k)}\right)$. Observe that one readily verifies that

$$\text{d.m.} \sum_{n=0}^{\infty} c_n = \gamma \qquad \left(\mathrm{R}^{(k)}, \{\lambda_n\}, m \right) \tag{3.6}$$

if and only if

$$\text{d.m.} \left\langle \sum_{n=0}^{\infty} c_n \delta(x - \lambda_n), 1 \right\rangle = \gamma \qquad \left(\mathrm{C}^{(k)}, m \right) \ . \tag{3.7}$$

More generally, if μ is a Radon measure concentrated on $[0, \infty)$, one writes instead of (3.4)

$$\text{d.m.} \int_0^{\infty} \phi(t) \mathrm{d}\mu(t) = \gamma \qquad \left(\mathrm{C}^{(k)}, m \right) \ . \tag{3.8}$$

Hence (3.8) holds if and only if

$$\lim_{x \to \infty} k \binom{m+k}{m} \int_0^x \phi(t) \left(\frac{t}{x} \right)^k \left(1 - \frac{t}{x} \right)^m \mathrm{d}\mu(t) = \gamma \ . \tag{3.9}$$

We want to define the k-differentiated Cesàro distributional evaluations for the case of unrestricted supports.

Lemma 1. *If* $g \in \mathcal{E}'(\mathbb{R})$ *then for any* $k > 0$, $m \geq 0$ *and* $\phi \in \mathcal{E}(\mathbb{R})$, *one has that* d.m. $\langle g(x), \phi(x) \rangle = 0 \ \left(\mathrm{C}^{(k)}, m\right)$.

Proof. Since $\phi(x)g(x) \in \mathcal{E}'(\mathbb{R})$, one can assume that $\phi \equiv 1$. It is enough to show the result for $m = 0$. Next, let $G \in \mathcal{D}'(\mathbb{R})$ be a distribution with support bounded at the left such that $G'(x) = x^k g(x)$, since G' vanishes in a neighborhood of infinity, then G is constant in that neighborhood of infinity, consequently, for x large enough $G(x) = o(x^k)$, as $x \to \infty$, in the ordinary sense, as required. □

We can now define the k-differentiated Cesàro distributional evaluations for distributions with unrestricted support.

Definition 5. *Let* $g \in \mathcal{D}'(\mathbb{R})$ *and let* $\phi \in \mathcal{E}(\mathbb{R})$. *Let* $g = g_1 + g_2$ *be a decomposition of* g *where* $g_1(x)$ *and* $g_2(-x)$ *have supports bounded on the left. We say that* d.m. $\langle g(x), \phi(x) \rangle = \gamma \ \left(\mathrm{C}^{(k)}\right)$ *if both* d.m. $\langle g_i(x), \phi(x) \rangle = \gamma_i \ \left(\mathrm{C}^{(k)}\right)$ *exist and* $\gamma = \gamma_1 + \gamma_2$.

Observe that because of Lemma 1 the last definition is independent of the decomposition of f.

We also have the analog to Proposition 1 for distributions.

Proposition 2. *Let* $f \in \mathcal{D}'(\mathbb{R})$ *and let* k *be a positive integer. If* $\langle g(x), \phi(x) \rangle = \gamma$ (C), *for some* γ, *then* d.m. $\langle g(x), \phi(x) \rangle = 0 \ \left(\mathrm{C}^{(k)}\right)$.

Proof. It is enough to assume that g has support bounded on one side, say on the left, and that $\phi \equiv 1$. The condition, together with the assumption on the support, implies that

$$g(\lambda x) = \gamma \frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right),$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. Hence multiplying by $(\lambda x)^k$, we see that

$$(\lambda x)^k g(\lambda x) = o\left(\lambda^{k-1}\right) \quad \text{as} \quad \lambda \to \infty,$$

in $\mathcal{S}'(\mathbb{R})$. Hence, since the support of g is bounded on the left, we can apply [10, Lemma 6.5.4] to conclude that $x^k g(x) = o\left(x^{k-1}\right)$ (C). □

We were not precise in the order of summability in Proposition 2. If we want to obtain information about the order, then it requires a more elaborated argument.

Theorem 1. *Let* $f \in \mathcal{D}'(\mathbb{R})$ *and* k *be a positive integer. If* $\langle g(x), \phi(x) \rangle = \gamma$ (C, m), *for some* γ, *then* d.m. $\langle g(x), \phi(x) \rangle = 0 \ \left(\mathrm{C}^{(k)}, n\right)$, *for* $n \geq m$.

Proof. We may assume that $\operatorname{supp} g$ is bounded at the left, $\phi \equiv 1$ and $n = m$. Let G be the $(m+1)$-primitive of g with support bounded at the left, then

$$G(x) \sim \frac{\gamma}{m!}\, x^m \ , \quad x \to \infty \ , \tag{3.10}$$

We now calculate the $(m+1)$-primitive of $x^k g(x)$ with support bounded at the left. In the well known formula

$$\phi h^{(m+1)} = \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \left(\phi^{(j)} h\right)^{(m+1-j)} \ , \tag{3.11}$$

valid for $\phi \in \mathcal{E}(\mathbb{R})$ and $h \in \mathcal{D}'(\mathbb{R})$, we take $h = G$ and $\phi(x) = x^k$. This shows that

$$F(x) = \sum_{j=0}^{m+1} (-1)^j \frac{C(k,j)}{(j-1)!} \binom{m+1}{j} \int_0^x (x-t)^{j-1} t^{k-j} G(t) \mathrm{d}t \ ,$$

where $C(k,j) = k(k-1)\dots(k-j+1)$, is the desired $(m+1)$-primitive of $x^k g(x)$. Then, (3.10) implies

$$\begin{aligned} F(x) &= \frac{\gamma}{m!} \sum_{j=0}^{m+1} (-1)^j \frac{C(k,j)}{(j-1)!} \binom{m+1}{j} \int_0^x (x-t)^{j-1} t^{k-j+m} \mathrm{d}t + o(x^{m+k}) \\ &= \frac{\gamma}{(m!)^2} \int_0^x (x-t)^m t^k \frac{d^{m+1}}{dt^{m+1}} (t^m) \mathrm{d}t + o(x^{m+k}) = o(x^{m+k}) \ , \end{aligned}$$

as $x \to \infty$, here we have used again (3.11) but now with $h(x) = x^m$. □

4. Determining the jumps of tempered distributions by differentiated Cesàro means

In this section we determine the jump, for the jump behavior and symmetric jump behavior, of general tempered distributions. This is done in two ways, in terms of the asymptotic behavior of its Fourier transform, and in terms of differentiated Cesàro means.

Theorem 2. *Let $f \in \mathcal{S}'(\mathbb{R})$ have the distributional jump behavior at $x = x_0$,*

$$f\left(x_0 + \varepsilon x\right) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \quad \text{as} \ \ \varepsilon \to 0^+ \ . \tag{4.1}$$

Let k be a positive integer. Then for any decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$, with $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ *and* $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$*, one has that*

$$\text{d.m.} \left\langle \hat{f}_\pm(x), e^{i x_0 x} \right\rangle = \frac{1}{i} [f]_{x=x_0} \qquad \left(\mathrm{C}^{(k)}\right) \tag{4.2}$$

In particular, $\text{d.m.} \left\langle \hat{f}(x), e^{ix_0x} \right\rangle = (2/i)\,[f]_{x=x_0} \quad \left(\mathrm{C}^{(\mathrm{k})}\right)$, *and*

$$(4.3)\quad x^k e^{i\lambda x_0 x}\hat{f}_{\pm}(\lambda x) = (\pm 1)^{k-1}\frac{1}{\lambda i}\,[f]_{x=x_0}\,x_{\pm}^{k-1} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda\to\infty\ ,$$

where the last quasiasymptotic relation holds in the sense of weak convergence in $\mathcal{S}'(\mathbb{R})$.

Proof. Differentiating (4.1) k-times, one has that

$$(4.4)\qquad f^{(k)}\left(x_0+\varepsilon x\right) = [f]_{x=x_0}\frac{\delta^{(k-1)}(x)}{\varepsilon^k} + o\left(\frac{1}{\varepsilon^k}\right)\ ,$$

as $\varepsilon\to 0^+$ in $\mathcal{D}'(\mathbb{R})$. If we take Fourier transform in (4.4), we obtain the asymptotic behavior,

$$(4.5)\quad (\lambda x)^k e^{i\lambda x_0 x}\hat{f}(\lambda x) = \frac{1}{i}\,[f]_{x=x_0}\,(\lambda x)^{k-1} + o\left(\lambda^{k-1}\right) \quad \text{as } \lambda\to\infty\ ,$$

in $\mathcal{S}'(\mathbb{R})$. Therefore $x^k e^{ix_0x}\hat{f}(x)$ has quasiasymptotic behavior at infinity with respect to λ^{k-1}, and hence the structural theorem for quasiasymptotic behaviors (see [38, Thm.2.6] or the decomposition theorem in [43, p.134]) applied to (4.5) yields (4.2) and (4.3). □

A particular case is obtained when $\hat{f}$ is a Radon measure. Notice that this class of distributions includes the so called pseudofunctions [15].

Corollary 1. *Let* $f\in\mathcal{S}'(\mathbb{R})$ *have the distributional jump behavior (4.1). Suppose that its Fourier transform is given by a Radon measure* μ. *Then for each positive integer* k *there exists* $m\in\mathbb{N}$ *such that for any decomposition of* $\mu=\mu_-+\mu_+$ *as two Radon measures concentrated on* $(-\infty,0]$ *and* $[0,\infty)$, *respectively, one has that*

$$(4.6)\qquad \text{d.m.}\int_0^\infty e^{\pm ix_0t}\mathrm{d}\mu_{\pm}(\pm t) = \pm\frac{1}{i}\,[f]_{x=x_0} \qquad \left(\mathrm{C}^{(k)},m\right)\ ,$$

or which amounts to the same,

(4.7)

$$\lim_{x\to\infty} ik\begin{pmatrix} m+k \\ m\end{pmatrix}\int_0^x e^{\pm ix_0t}\left(\frac{t}{x}\right)^k\left(1-\frac{t}{x}\right)^m \mathrm{d}\mu_{\pm}(\pm t) = \pm\,[f]_{x=x_0}\ .$$

Note that Theorem 2 and Corollary 1 provide us with formulas for the jump by only considering the spectral data of f from either the left or right side of the origin. In the case of symmetric jump behavior this is not longer possible; however, we can still recover the jump by taking symmetric means.

Theorem 3. *Suppose that $f \in \mathcal{S}'(\mathbb{R})$ has a symmetric jump at $x = x_0$. Let k be a positive integer. Then for any decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$, where* $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ *and* $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$, *we have that*

$$\text{d.m.} \left\langle e^{ix_0x} \hat{f}_+(x) - e^{-ix_0x} \hat{f}_-(-x), 1 \right\rangle = \frac{2}{i} [f]_{x=x_0} \qquad \left(\mathrm{C}^{(\mathrm{k})}\right) \ . \tag{4.8}$$

Proof. Let ψ_{x_0} be the jump distribution. It has the jump behavior at $x = 0$

$$\psi_{x_0}(\varepsilon x) = [f]_{x=x_0} \operatorname{sgn} x + o(1) \quad \text{as} \ \ \varepsilon \to 0^+ \quad \text{in} \ \ \mathcal{D}'(\mathbb{R}) \ ,$$

and so $[\psi_{x_0}]_{x=0} = 2 [f]_{x=x_0}$. Since $\hat{\psi}_{x_0}(x) = e^{ix_0x} \hat{f}(x) - e^{-ix_0x} \hat{f}(-x)$, a decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$ leads to the decomposition $\hat{\psi}_{x_0}(x) = \hat{\psi}_-(x) + \hat{\psi}_+(x)$ where

$$\hat{\psi}_\pm(x) = e^{ix_0x} \hat{f}_\pm(x) - e^{-ix_0x} \hat{f}_\mp(-x) \ ,$$

and thus Theorem 2 implies (4.8). □

When $\hat{f}$ is a Radon measure, we can give formulas of type (4.7). Depending on the parity of k, we should use the means of a Fourier type integral or a conjugate type integral. This fact is given in the next two corollaries which follow immediately from Theorem 3.

Corollary 2. *Let $f \in \mathcal{S}'(\mathbb{R})$ have a distributional symmetric jump behavior at $x = x_0$. Suppose that its Fourier transform is a Radon measure μ. Let $2k - 1$ be a positive odd integer. Then there exists $m \in \mathbb{N}$ such that*

$$\lim_{x \to \infty} \frac{i(2k-1)}{2x^{2k-1}} \binom{m + 2k - 1}{m} \int_{-x}^{x} t^{2k-1} e^{ix_0t} \left(1 - \frac{|t|}{x}\right)^m \mathrm{d}\mu(t) = [f]_{x=x_0} \, . \tag{4.9}$$

Corollary 3. *Let $f \in \mathcal{S}'(\mathbb{R})$ have a distributional symmetric jump behavior at $x = x_0$. Suppose that its Fourier transform is a Radon measure μ. Let $2k$ be a positive even integer. Then there exists $m \in \mathbb{N}$ such that for any decomposition $\mu = \mu_- + \mu_+$, as two Radon measures concentrated on $(-\infty, 0]$ and $[0, \infty)$, respectively, one has that*

$$\lim_{x \to \infty} \frac{ik}{x^{2k}} \binom{m + 2k}{m} \int_{-x}^{x} t^{2k} e^{ix_0t} \left(1 - \frac{|t|}{x}\right)^m \mathrm{d}\sigma(t) = [f]_{x=x_0} \, , \tag{4.10}$$

where $\sigma = \mu_+ - \mu_-$.

Sometimes is possible to single out a measure σ in (4.10). For certain distributions one can talk about a unique Hilbert transform [9], say $\tilde{f}$, in such a case one may take $\sigma = i\hat{\tilde{f}}$. Actually, this will be done in Section 6 for the case of periodic distributions.

5. Jumps and local boundary behavior of derivatives of harmonic and analytic functions.

In this section, we determine the jump of a distribution in terms of the asymptotic behavior of derivatives of analytic representations; we also find formulas for the jump in terms of partial derivatives of harmonic and harmonic conjugate functions.

We start with the jump behavior and analytic representations. Recall that given $f \in \mathcal{D}'(\mathbb{R})$, we may see f as a hyperfunction, that is, $f(x) = F(x+i0) - F(x-i0)$, where F is analytic for $\Im m\, z \neq 0$; moreover, this representation holds distributionally in the sense that

$$f(x) = \lim_{y\to 0^+} \left(F(x+iy) - F(x-iy)\right) \ , \tag{5.1}$$

where the last limit is taken in the weak topology of $\mathcal{D}'(\mathbb{R})$ [3]. In such a case, we say that F is an analytic representation of f. Note that, initially, we are not assuming that $F(x \pm i0)$ belong to $\mathcal{D}'(\mathbb{R})$ separately, but that their difference does; however, it is shown in [5, Section 5] that the existence of the distributional jump of F across the real axis implies the existence of $F(x \pm i0)$, separately, in $\mathcal{D}'(\mathbb{R})$.

Given $0 < \eta \leq \pi/2$ and $x_0 \in \mathbb{R}$, we define the subset of the upper half-plane $\Delta^{+}_{\eta}(x_0)$ as the set of those z such that $\eta \leq \arg(z-x_0) \leq \pi-\eta$, similarly, we define the subset of the lower half-plane $\Delta^{-}_{\eta}(x_0)$ as the set of those z such that $\eta - \pi \leq \arg(z-x_0) \leq -\eta$.

Theorem 4. *Let $f \in \mathcal{D}'(\mathbb{R})$ have the jump behavior at $x = x_0$*

$$f\left(x_0 + \varepsilon x\right) = \gamma_{-} H(-x) + \gamma_{+} H(x) + o(1) \quad \textit{as} \ \ \varepsilon \to 0^+ \ . \tag{5.2}$$

Suppose that F is an analytic representation of f on $\Im m\, z \neq 0$, then for each positive integer k and $0 < \eta \leq \pi/2$, one has that

$$\lim_{z\to z_0,\, z\in\Delta^{\pm}_{\eta}(x_0)} \left(z-x_0\right)^k F^{(k)}(z) = (-1)^k \frac{(k-1)!}{2\pi i} \left[f\right]_{x=x_0} \ . \tag{5.3}$$

Proof. We first show that if (5.3) holds for one analytic representation, then it holds for any analytic representation of f. In fact by the very well known edge of the wedge theorem, any two such analytic representations differ by an entire function, and for entire functions (5.3) gives 0. Next, we prove that we may assume that $f \in \mathcal{S}'(\mathbb{R})$. Indeed we can decompose $f = f_1 + f_2$ where f_2 is zero in a neighborhood of x_0 and $f_1 \in \mathcal{S}'(\mathbb{R})$. Let F_1 and F_2 be analytic representations of f_1 and f_2, respectively; then F_2 can be continued across a neighborhood of x_0 (edge of the wedge theorem once again), hence $F_2(z) = F_2(x_0) + O\left(|z-x_0|\right) = O\left(1\right)$ as $z \to x_0$. Additionally, f_1 has the same jump behavior as f. Thus, we may assume that

$f \in \mathcal{S}'(\mathbb{R})$. Consider the following analytic representation [3, p.83], where $\hat{f} = \hat{f}_- + \hat{f}_+$ is a decomposition as in Theorem 2,

$$F(z) = \begin{cases} \frac{1}{2\pi}\left\langle \hat{f}_+(t), e^{izt}\right\rangle, & \Im m\ z > 0\ , \\ -\frac{1}{2\pi}\left\langle \hat{f}_-(t), e^{izt}\right\rangle, & \Im m\ z < 0\ , \end{cases}$$

Keep the number z on a compact subset of $\Delta_\eta^\pm(x_0)$, then

$$\begin{aligned} F^{(k)}\left(x_0 + \frac{z}{\lambda}\right) &= \pm\frac{i^k}{2\pi}\lambda^{k+1}\left\langle t^k e^{i\lambda x_0 t}\hat{f}_\pm(\lambda t), e^{izt}\right\rangle \\ &= \pm\frac{(\pm i)^{k-1}}{2\pi}[f]_{x=x_0}\lambda^k\int_0^\infty t^{k-1}e^{\pm izt}\mathrm{d}t + o\left(\lambda^k\right) \\ &= (-1)^k\frac{(k-1)!}{2\pi i}[f]_{x=x_0}\left(\frac{\lambda}{z}\right)^k + o\left(\lambda^k\right), \end{aligned}$$

as $\lambda \to \infty$, where we have used (4.3). □

Next, we determine the jump, occurring in jump behavior, by finding the local boundary asymptotic behavior of partial derivatives of harmonic and harmonic conjugate functions. We say that $U(z)$, harmonic on $\Im m\ z > 0$, is a harmonic representation of $f \in \mathcal{D}'(\mathbb{R})$ [3] if

$$\lim_{y\to 0^+} U(x+iy) = f(x) \qquad \text{in } \mathcal{D}'(\mathbb{R})\ . \tag{5.4}$$

Observe that, because of the results from [8] and [5, Section 5], one has that if a harmonic function on the upper half-plane admits distributional boundary values, then any harmonic conjugate to it admits distributional boundary values.

Theorem 5. *Let $f \in \mathcal{D}'(\mathbb{R})$ have the distributional jump behavior (5.2) at $x = x_0$. Let U be a harmonic representation of f on $\Im m\ z > 0$. Let V be a harmonic conjugate to U. Suppose that k is a positive integer, then*

$$\frac{\partial^k U}{\partial x^k}(z) = \frac{(k-1)!}{(-1)^k\pi}[f]_{x=x_0}\Im m\frac{1}{(z-x_0)^k} + o\left(|z-x_0|^{-k}\right)\ , \tag{5.5}$$

and

$$\frac{\partial^k V}{\partial x^k}(z) = \frac{(k-1)!}{(-1)^{k+1}\pi}[f]_{x=x_0}\Re e\frac{1}{(z-x_0)^k} + o\left(|z-x_0|^{-k}\right)\ , \tag{5.6}$$

as $z \to x_0$ on any sector of the form $\Delta_+^\eta(x_0)$, $0 < \eta \le \pi/2$.

Proof. Notice that, since harmonic conjugates differ from each other by a constant, we may use any specific V we want. We now show that we may work with any harmonic representation U of f. Suppose that U and U_1 are two harmonic representations of f, then $U_2 = U - U_1$ represents the zero distribution. Then by applying the reflection principle to the real and imaginary parts of U [2, Section 4.5], [36, Section 3.4], we have that U_2 admits a harmonic extension to a (complex) neighborhood of x_0. Consequently, if V and V_1 are harmonic conjugates to U and U_1, we have that $V_2 = V - V_1$ is harmonic conjugate to U_2, and thus it admits a harmonic extension to a (complex) neighborhood of x_0 as well. Therefore $\dfrac{\partial^k U_2}{\partial x^k}(z)$, $\dfrac{\partial^k V_2}{\partial x^k}(z) = O(1)$ in a neighborhood of x_0; consequently, we have that U and V satisfy (5.5) and (5.6) if and only if U_1 and V_1 do it.

Let F be an analytic representation of f. We may assume that $U(z) = F(z) - F(\bar{z})$ and $V(z) = -i\,(F(z) + F(\bar{z}))$. Notice that $\dfrac{\partial^k U}{\partial x^k}(z) = F^{(k)}(z) - F^{(k)}(\bar{z})$ and $\dfrac{\partial^k V}{\partial x^k}(z) = -i\left(F^{(k)}(z) + F^{(k)}(\bar{z})\right)$, and then an application of (5.3) gives (5.5) and (5.6). □

Observe that when k is odd it is possible to recover the jump from the radial asymptotic behavior of $\dfrac{\partial^k U}{\partial x^k}$ but not from the one of $\dfrac{\partial^k V}{\partial x^k}$; similarly, when k is even we recover the jump from the radial behavior of $\dfrac{\partial^k V}{\partial x^k}$, but not from the one of $\dfrac{\partial^k U}{\partial x^k}$. This is also true for the symmetric jump behavior.

Theorem 6. *Let $f \in \mathcal{D}'(\mathbb{R})$ have symmetric jump at $x = x_0$. Let k be a positive integer. Suppose that U is a harmonic representation of f on $\Im m\, z > 0$ and V is a harmonic conjugate to U. Then,*

$$\lim_{y\to 0^+} y^{2k-1}\frac{\partial^{2k-1}U}{\partial x^{2k-1}}\,(x_0+iy) = (-1)^{k+1}\frac{(2k-2)!}{\pi}\,[f]_{x=x_0}\ , \tag{5.7}$$

and

$$\lim_{y\to 0^+} y^{2k}\frac{\partial^{2k}V}{\partial x^{2k}}\,(x_0+iy) = (-1)^{k+1}\frac{(2k-1)!}{\pi}\,[f]_{x=x_0}\ . \tag{5.8}$$

Proof. We apply our results to the jump distribution ψ_{x_0}. Let U be a harmonic representation of f and V be a harmonic conjugate. We have that $U\left(x_0+z\right) - U\left(x_0-\bar{z}\right)$ and $V\left(x_0+z\right) + V\left(x_0-\bar{z}\right)$ are a harmonic representation and a harmonic conjugate for ψ_{x_0}. The result now follows from Theorem 5 and the fact $\left[\psi_{x_0}\right]_{x=0} = 2\,[f]_{x=x_0}$. □

We remark that for distributions the radial behavior of its harmonic representations can be considered as Abel-Poisson means, while the radial behavior of harmonic conjugate functions can be considered as conjugate Abel-Poisson means; hence, one can say that Theorem 6 gives the jump in terms of *differentiated Abel-Poisson means*. We will apply this useful observation to Fourier series in the next section. We also want to point out that Theorem 4 and Theorem 5 are much stronger than Theorem 6, and in the context of Fourier series, as we shall see, can be used to express the jump as differentiated Abel-Poisson means of only a partial part of the spectrum.

If we assume that f is the boundary value of an analytic function on the upper half-plane, we can get a better result than Theorem 6. This is the content of the next theorem.

Theorem 7. *Let F be analytic in the upper half-plane, with distributional boundary values $f(x) = F(x+i0)$. Suppose f has a distributional symmetric jump behavior at $x = x_0$. Then, for any $0 < \eta \leq \pi/2$*

$$F^{(k)}(z) \sim \frac{(k-1)![f]_{x=x_0}}{(-1)^k i\pi (z-x_0)^k} \qquad \textit{as} \quad z \in \Delta_\eta^+(x_0) \to x_0 \ . \tag{5.9}$$

Proof. Let ψ_{x_0} be the jump distribution at $x = x_0$. Then ψ_{x_0} has a jump behavior at $x = 0$ and $[\psi_{x_0}]_{x=0} = 2[f]_{x=x_0}$. Observe that $U(z) = F(x_0+z) - F(x_0 - \bar{z})$ is a harmonic representation of ψ_{x_0} and $V(z) = -i\left(F(x_0+z) + F(x_0-\bar{z})\right)$ is a harmonic conjugate. Hence, we can apply Theorem 5 to U and V to obtain that

$$F^{(k)}(x_0+z) = (-1)^k F(x_0-\bar{z}) + 2(-1)^k \frac{(k-1)!}{\pi} [f]_{x=x_0} \Im m \frac{1}{z^k} + o\left(|z|^{-k}\right)$$

and

$$F^{(k)}(x_0+z) = (-1)^{k+1} F(x_0-\bar{z}) + 2(-1)^k \frac{(k-1)!}{i\pi} [f]_{x=x_0} \Re e \frac{1}{z^k} + o\left(|z|^{-k}\right)$$

as $z \in \Delta_\eta^+(0) \to 0$; and therefore (5.9) follows. □

6. Applications to Fourier series

This section is dedicated to applications of our results to Fourier series. We determine the jump of 2π-periodic distributions in terms of differentiated Cesàro-Riesz and Abel-Poisson means.

Throughout this section f is a 2π-periodic distribution with Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \ , \tag{6.1}$$

where the series converges in $\mathcal{S}'(\mathbb{R})$.

6.1. **Jump behavior and Fourier series.** Notice that the Fourier transform of f is given by

$$\hat{f}(x) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(x-n) \ , \tag{6.2}$$

hence, as an immediate corollary of Theorem 2, we obtain,

Theorem 8. *If f has a jump behavior at $x = x_0$, with jump $[f]_{x=x_0}$, then for each positive integer k we have that*

$$\text{d.m.} \sum_{n=0}^{\infty} c_n e^{ix_0 n} = \frac{1}{2\pi i} [f]_{x=x_0} \qquad \left(\mathrm{C}^{(k)}\right) \ , \tag{6.3}$$

and

$$\text{d.m.} \sum_{n=1}^{\infty} c_{-n} e^{-ix_0 n} = -\frac{1}{2\pi i} [f]_{x=x_0} \qquad \left(\mathrm{C}^{(k)}\right) \ . \tag{6.4}$$

Notice that, as we have previously remarked, in our formulas we only need either the positive or the negative part of the spectral data of f, having an advantage over other approaches where the complete spectral data of f is used.

We now interpret Theorem 4 in the context of Fourier series; again notice that only one part of the spectrum is used. Observe that

$$F(z) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} c_n e^{izn}, & \Im m\ z > 0 \ , \\ \displaystyle -\sum_{n=-\infty}^{1} c_n e^{izn}, & \Im m\ z < 0 \ , \end{cases} \tag{6.5}$$

is an analytic representation of f, from where we have immediately.

Theorem 9. *If f has a jump behavior at $x = x_0$, with jump $[f]_{x=x_0}$, then for each positive integer k we have that for $0 < \eta \le \pi/2$,*

$$\lim_{z \to x_0,\ z \in \Delta_\eta^+(x_0)} (z - x_0)^k \sum_{n=0}^{\infty} n^k c_n e^{inz} = -\frac{(k-1)!}{2\pi(-i)^{k+1}} [f]_{x=x_0} \ , \tag{6.6}$$

and

$$\lim_{z \to x_0,\ z \in \Delta_\eta^-(x_0)} (z - x_0)^k \sum_{n=-\infty}^{1} n^k c_n e^{inz} = \frac{(k-1)!}{2\pi(-i)^{k+1}} [f]_{x=x_0} \ . \tag{6.7}$$

Remark 1. We remark that we may also consider non-harmonic series and obtain analog results. Indeed, suppose that $\{\lambda_n\}_{n=0}^{\infty}$ is an increasing sequence such that $0 \leq \lambda_0$ and $\lim_{n\to\infty} \lambda_n = \infty$, let

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\lambda_{|n|}x} \ , \tag{6.8}$$

convergent in $\mathcal{S}'(\mathbb{R})$. Then if g has a distributional jump behavior at $x = x_0$, we have that for each positive integer k

$$\text{d.m.} \sum_{n=0}^{\infty} c_n e^{ix_0\lambda_n} = \frac{1}{2\pi i} [g]_{x=x_0} \qquad \left(\mathrm{R}^{(k)}, \{\lambda_n\}\right) \ , \tag{6.9}$$

$$\text{d.m.} \sum_{n=1}^{\infty} c_{-n} e^{ix_0\lambda_n} = -\frac{1}{2\pi i} [g]_{x=x_0} \qquad \left(\mathrm{R}^{(k)}, \{\lambda_n\}\right) \ , \tag{6.10}$$

$$\lim_{z\to x_0,\ z\in\Delta_\eta^{+}(x_0)} (z - x_0)^k \sum_{n=0}^{\infty} \lambda_n^k c_n e^{i\lambda_n z} = -\frac{(k-1)!}{2\pi(-i)^{k+1}} [g]_{x=x_0} \ , \tag{6.11}$$

and

$$\lim_{z\to x_0,\ z\in\Delta_\eta^{-}(x_0)} (z - x_0)^k \sum_{n=1}^{\infty} \lambda_n^k c_{-n} e^{i\lambda_n z} = \frac{(k-1)!}{2\pi\,(-i)^{k+1}} [g]_{x=x_0} \ . \tag{6.12}$$

6.2. **Symmetric jump behavior and Fourier series.** As usual, we define the conjugate distribution of f as

$$\tilde{f}(x) = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{inx} \ , \tag{6.13}$$

with $\tilde{c}_n = -i \operatorname{sgn} n\, c_n$, $\tilde{c}_0 = 0$. Notice that $\tilde{f}$ is the Hilbert transform of f [9]. Since we will use symmetric means, it is convenient to use the sines and cosines series for f, i.e.,

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \ , \tag{6.14}$$

where $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$, then $\tilde{c}_n = \left(-b_{|n|} - i \operatorname{sgn} n a_{|n|}\right)/2$ and

$$\tilde{f}(x) = \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \ . \tag{6.15}$$

We obtain from Theorem 3.

Theorem 10. *Let k be a positive integer. If f has a symmetric jump at $x = x_0$, then*

$$\text{d.m.}\sum_{n=1}^{\infty}(a_n \sin nx_0 - b_n \cos nx_0) = -\frac{1}{\pi}\,[f]_{x=x_0} \qquad \left(\mathrm{C}^{(k)}\right) \ . \tag{6.16}$$

Proof. Observe that $\hat{f} = \hat{f}_- + \hat{f}_+$, where

$$\hat{f}_+(x) = a_0\pi\,\delta(x) + \pi\sum_{n=1}^{\infty}(a_n - ib_n)\,\delta(x-n)\ ,$$

and

$$\hat{f}_-(x) = \pi\sum_{n=1}^{\infty}(a_n + ib_n)\,\delta(x+n)\ .$$

Thus, an easy calculation gives that $e^{ix_0x}\hat{f}_+(x) - e^{-ix_0x}\hat{f}_-(-x)$ is equal to

$$a_0\pi\,\delta(x) + 2\pi i\sum_{n=1}^{\infty}(a_n \sin nx_0 - b_n \cos nx_0)\,\delta(x-n)\ ,$$

and therefore (6.16) is a direct consequence of Theorem 3. □

Relation (6.16) can also be written in terms of the Fourier coefficients $\{c_n\}$ and $\{\tilde{c}_n\}$. By direct computation, or by applying Corollaries 2 and 3, one obtains the following corollary.

Corollary 4. *Let k be a positive integer. If f has a symmetric jump at $x = x_0$, then*

$$\lim_{x\to\infty}(2k-1)\binom{m+2k-1}{m}\sum_{-x<n<x}c_n e^{ix_0n}\left(\frac{n}{x}\right)^{2k-1}\left(1-\frac{|n|}{x}\right)^m = \frac{1}{i\pi}\,[f]_{x=x_0}\ , \tag{6.17}$$

and

$$\lim_{x\to\infty}2k\binom{m+2k}{m}\sum_{-x<n<x}\tilde{c}_n e^{ix_0n}\left(\frac{n}{x}\right)^{2k}\left(1-\frac{|n|}{x}\right)^m = -\frac{1}{\pi}\,[f]_{x=x_0}\ . \tag{6.18}$$

We now express the jump in terms of differentiated Abel-Poisson means.

Theorem 11. *If f has symmetric jump behavior at $x = x_0$, then for any positive k we have that*

$$\sum_{n=1}^{\infty}(a_n \sin nx_0 - b_n \cos nx_0)\,n^k r^n \sim -\frac{(k-1)!\,[f]_{x=x_0}}{\pi(1-r)^k}\ , \tag{6.19}$$

as $r \to 1^-$.

Proof. Notice that

$$U(z) = \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)\,e^{izn} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + ib_n)\,e^{-i\bar{z}n}$$

and

$$V(z) = -\frac{1}{2}\sum_{n=1}^{\infty}(ia_n + b_n)\,e^{izn} + \frac{1}{2}\sum_{n=1}^{\infty}(ia_n - b_n)\,e^{-i\bar{z}n}$$

are a harmonic representation of f and a harmonic conjugate. If k is odd, we obtain that

$$\frac{\partial^k U}{\partial x^k}(x_0 + iy) = i^{k+1}\sum_{n=1}^{\infty}(a_n \sin nx_0 - b_n \cos nx_0)\,n^k e^{-ny}\ ,$$

on the other hand if k is even, we have that

$$\frac{\partial^k V}{\partial x^k}(x_0 + iy) = i^{k}\sum_{n=1}^{\infty}(a_n \sin nx_0 - b_n \cos nx_0)\,n^k e^{-ny}\ .$$

So, in any case we obtain from Theorem 6 that for each positive integer

$$\lim_{y\to 0^+} y^k \sum_{n=1}^{\infty}(a_n \sin nx_0 - b_n \cos nx_0)\,n^k e^{-ny} = -\frac{(k-1)!}{\pi}\,[f]_{x=x_0}\ .$$

□

We end this section with a direct corollary of Theorem 7.

Corollary 5. *Suppose that the 2π-periodic distribution f is the boundary value of analytic function, i.e., its Fourier series expansion is of the form*

$$f(x) = \sum_{n=0}^{\infty} c_n e^{ixn}\ . \tag{6.20}$$

If f has symmetric jump behavior at $x = x_0$, then for any positive integer k and $0 < \eta \leq \pi/2$, one has that

$$\sum_{n=0}^{\infty} n^k c_n e^{izn} \sim (-1)^k \frac{(k-1)![f]_{x=x_0}}{\pi i^{k+1}(z-x_0)^k} \qquad \textit{as} \quad z \in \Delta_{\eta}^{+}(x_0) \to x_0\ . \tag{6.21}$$

7. A characterization of differentiated Cesàro means

In this section we provide a characterization of the summability method by differentiated Cesàro means in terms of the Cesàro behavior of the sequence $\left\{n^k c_n\right\}_{n=1}^{\infty}$. This equivalence is stated in the next theorem. The proof adapts an argument from the proof of [18, Thm.58, p.113] to our context; G. Hardy attributes the main argument to A.E. Ingham [21]. One may also adapt M. Riesz's original proof of the equivalence between the $(\mathrm{R},\{n\})$ and (C) methods of summation [20, 32].

Theorem 12. *Let $\{c_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. Let k be a positive integer. Then*

$$\text{d.m.} \sum_{n=0}^{\infty} c_n = \gamma \qquad \left(\mathrm{C}^{(k)}, m\right) \tag{7.1}$$

if and only if

$$n^k c_n = \gamma n^{k-1} + o\left(n^{k-1}\right) \qquad (\mathrm{C}, m+1)\ . \tag{7.2}$$

Proof. Set $a_n = n^k c_n - \gamma n^{k-1}$, since

$$\lim_{x\to\infty} \binom{m+k}{m} \frac{k}{x^k} \sum_{0<j<x} j^{k-1}\left(1-\frac{j}{x}\right)^m = k\binom{m+k}{m}\int_0^1 t^{k-1}(1-t)^m \mathrm{d}t$$
$$= 1\ ,$$

we have that (7.1) holds if and only if

$$T_m(x) := \sum_{0<j<x} a_j (x-j)^m = o\left(x^{m+k}\right)\ , \qquad x\to\infty\ . \tag{7.3}$$

Set

$$A_{m+1}(n) = \sum_{j=0}^{n} \binom{m+j}{m} a_{n-j}\ . \tag{7.4}$$

Observe that relation (7.2) is equivalent to

$$A_{m+1}(n) = o(n^{m+k})\ , \qquad n\to\infty\ . \tag{7.5}$$

Therefore, we shall show that (7.3) and (7.5) are equivalent.

Assume first that $A_{m+1}(n) = o\left(n^{m+k}\right)$. Set $x = n+\vartheta$, where n is an integer and $0\le\vartheta<1$. Since $T_m(x) = \sum_{j=0}^{n}\left(n-j+\vartheta\right)^m a_j$, we have that for $|z| < 1$

$$\sum_{n=0}^{\infty} T_m(x) z^n = \sum_{n=0}^{\infty} (n+\vartheta)^m z^n \quad \sum_{n=0}^{\infty} a_n z^n$$
$$= (1-z)^{m+1} \sum_{n=0}^{\infty} (n+\vartheta)^m z^n \quad \sum_{n=0}^{\infty} A_{m+1}(n) z^n \ .$$

Now, it is easy to see [18, p.113] that

$$(1-z)^{m+1} \sum_{n=0}^{\infty} (n+\vartheta)^m z^n = \sum_{j=0}^{m} c_j(\vartheta) z^j \ ,$$

where the coefficients $c_j(\vartheta)$ are polynomials in ϑ of degree m. Thus,

$$T_m(x) = \sum_{j=0}^{m} c_j(\vartheta) A_{m+1}(n-j) = o\left(x^{m+k}\right) \ , \qquad x \to \infty \ .$$

We now assume that $T_m(x) = o(x^{m+k})$. We take $m+1$ numbers $0 < \vartheta_0 < \vartheta_1 < \ldots, < \vartheta_m$. The equation

$$\binom{n+m}{m} = \sum_{j=0}^{m} b_j (n+\vartheta_j)^m$$

can be written as a system of $m+1$ equations with non-zero determinant, then it has unique solutions $b_0, \ldots, b_m$. Hence, we obtain

$$A_{m+1}(n) = \sum_{j=0}^{n} \binom{n-j+m}{m} a_j = \sum_{j=0}^{m} b_j T_m(n+\vartheta_j) = o\left(n^{m+k}\right) \ ,$$

as $n \to \infty$, as required. □

It is convenient to spell out what (7.2) says. Recall the definition of the Cesàro means [18, 47] of a sequence $\{b_n\}_{n=0}^{\infty}$. Given $l \geq 0$, the Cesàro mean of order l of the sequence (not to be confused with the means of a series) are

$$C_l \left\{b_j; n\right\} := \frac{l!}{n^l} \sum_{j=0}^{n} \binom{j+l-1}{l-1} b_{n-j} \ .$$

Notice that

$$\sum_{j=0}^{n} \binom{m+j}{m} (n-j)^{k-1} \sim \frac{1}{m!} \sum_{j=1}^{n} j^m (n-j)^{k-1}$$
$$\sim \frac{(k-1)!}{(m+k)!} n^{m+k} , \quad n \to \infty .$$

Therefore if we define

$$C_m^{(k)} \{c_j \, ; n\} := \frac{(m+k)!}{(k-1)! \, n^{m+k}} \sum_{j=1}^{n} \binom{n-j+m}{m} j^k c_j \tag{7.6}$$
$$= \frac{k}{m+1} \binom{m+k}{m} \frac{C_{m+1} \left\{ j^k c_j \, ; \, n \right\}}{n^{k-1}} ,$$

we have then that (7.2) means

$$\lim_{n\to\infty} C_m^{(k)} \{c_j \, ; n\} = \gamma . \tag{7.7}$$

So alternatively, we could use (7.7) to define the k-differentiated Cesàro means instead of the means originally used in Definition 3. This also justifies the switch of notation from $\left(\mathrm{R}^{(k)}, \{n\}\right)$ to $\left(\mathrm{C}^{(k)}\right)$ in Definition 3.

Theorem 12 has an interesting distributional consequence which is presented in the next corollary. We denote the integral part of a number x by $[x]$. Given a sequence $\{a_n\}_{n=0}^{\infty}$, we denote by $a_{[x]}$ the piecewise constant function equal to a_n for $n \leq x < n+1$.

Corollary 6. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers and let k be a non-negative integer. Then,*

$$\sum_{n=0}^{\infty} a_n \delta(x-n) = \gamma x^k + o\left(x^k\right) \qquad (\mathrm{C}, m) \ , \quad x \to \infty , \tag{7.8}$$

if and only if

$$a_n = \gamma n^k + o\left(n^k\right) \qquad (\mathrm{C}, m) \ , \quad n \to \infty , \tag{7.9}$$

and, in turn, if and only if

$$a_{[x]} = \gamma x^k + o\left(x^k\right) \qquad (\mathrm{C}, m) \ , \quad x \to \infty . \tag{7.10}$$

On combining Theorem 8 and Theorem 12, we obtain new formulas for the jump of Fourier series occurring in the jump behaviors.

Corollary 7. *Let f be a 2π-periodic distribution having Fourier series*

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \ .$$

If f has a jump behavior at $x = x_0$, then

$$\lim_{n\to\infty} nc_n e^{inx_0} = \frac{1}{2\pi i}[f]_{x=x_0} \qquad \text{(C)} \ , \tag{7.11}$$

and

$$\lim_{n\to\infty} nc_{-n} e^{-inx_0} = -\frac{1}{2\pi i}[f]_{x=x_0} \qquad \text{(C)} \ . \tag{7.12}$$

We end this article with a corollary that can be tracked down to the work of A. Zygmund [17, 46], of course he stated it in a very different form; at that time distribution theory did not even exist! The proof follows immediately from Theorem 10 and Theorem 12.

Corollary 8. *Let f be a 2π-periodic distribution having Fourier series*

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin x) \ .$$

If f has a symmetric jump behavior at $x = x_0$, then

$$\lim_{n\to\infty} n\left(b_n \cos nx_0 - a_n \sin nx_0\right) = \frac{1}{\pi}[f]_{x=x_0} \qquad \text{(C)} \ . \tag{7.13}$$

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