# EXTERIOR EULER SUMMABILITY

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Abstract. We define and study a summability procedure that is similar to Euler summability but applied in the exterior of a disc, not in the interior. We show that the procedure is well defined and that it actually has many interesting properties. We use these ideas to study some problems of analytic continuation and to give a construction of the convex support of an analytic functional. We also use this exterior summability to study Mittag-Leffler developments.

## 1. Introduction

In a recent article, Amore [1] introduced a procedure for the convergence acceleration of series, illustrating it with several very interesting examples. This procedure is similar, and actually related to the procedure introduced by Euler many years ago [14, Chp. VIII]. Basically, while Euler's scheme works in the *interior* of a disc, Amore's procedure works in the *exterior* of a disc.

The purpose of this article is to study the corresponding exterior *summability,* that is, to apply this analysis in order to assign sums to several divergent series. The study of the Euler interior summability is due initially to Knopp [16, 17], and it is clearly explained in [14]; examples of Euler's procedure for the convergence acceleration of series can be found in [7]. As we show, the exterior Euler summability is not only different from the usual Euler summability, but it actually provides many new interesting and useful results, particularly in the study of analytic continuations and in the construction of the convex support of analytic functionals.

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The plan of the article is the following. We define two notions of exterior Euler summability in Section 2, one that applies to power series and one that applies to numerical series, and prove that both definitions are equivalent. In Section 3 we prove that if one starts with a series $\sum_{n=0}^{\infty} a_n/z^{n+1}$ which converges for $|z| > R$ for some $R < \infty$, then there exists a compact convex set $K_{\mathrm{cv}}$ such that the series is exterior Euler summable for $z \notin K_{\mathrm{cv}}$ but is never exterior Euler summable if $z \in K_{\mathrm{cv}}$ is not an extreme point of this set; both summability and non-summability are possible at the extreme points. We interpret these results in terms of the analytic continuation of the function $f_0(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ defined in $|z| > R$, showing that it has an analytic continuation $f_{\mathrm{cv}}$ to $\overline{\mathbb{C}} \setminus K_{\mathrm{cv}}$, and that $K_{\mathrm{cv}}$ is the smallest compact convex subset of $\mathbb{C}$ for which such a continuation exists.

The exterior Euler summability is based upon a series transformation in terms of a parameter $\lambda$. In Section 4 we prove that the sum of the transformed series, if it exists, is independent of this parameter; the uniqueness is rather clear if $z \notin K_{\mathrm{cv}}$, but it involves the study of the boundary behavior of $f_{\mathrm{cv}}(\omega)$ as $\omega \to z$ if $z$ is an extreme point of $K_{\mathrm{cv}}$.

Section 5 gives an account of the relationship between the exterior Euler summability and some results about analytic functionals, hyperfunctions, and distributions. The use of the Cauchy representation [11] allows one to relate an analytic function to an analytic functional, and the series transformation required in the Euler exterior summability becomes a change of variables in the corresponding analytic functional.

We present several illustrations in Section 6. In particular we show that the geometric series $\sum_{n=0}^{\infty} \omega^n$ is exterior Euler summable, to the sum $(1-\omega)^{-1}$, for *all* complex numbers $\omega \neq 1$, and this gives an idea of the power of the procedure. Remarkably, the standard interior Euler procedure, the Borel methods, and the Mittag-Leffler method give a much smaller region of summability for the geometric series [14]. In Section 7 we explain how not only convergent, but actually Abel summable series can be transformed by the exterior Euler procedure, and illustrate the ideas with the series $\sum_{n=1}^{\infty} (-1)^n n^{-s}$, which is Abel summable for any complex number $s$.

The exterior Euler expansion of functions given by Mittag-Leffler developments is studied in Section 8, where we show that the region of summability is rather large, and where we give several examples of these "double series" manipulations.

## 2. Definition of Exterior Euler Summability

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. Suppose that the series $\sum_{n=0}^{\infty} a_n/z^{n+1}$ converges for $|z|>R$ for some $R<\infty$. Let $\xi \in \mathbb{C}$. We say that the series $\sum_{n=0}^{\infty} a_n/\xi^{n+1}$ is exterior Euler summable to $S = S(\xi)$, and write

$$\sum_{n=0}^{\infty} \frac{a_n}{\xi^{n+1}} = S \qquad \text{(Ex)}\ , \tag{2.1}$$

if there exists $\lambda \in \mathbb{C}$ such that

$$\sum_{n=0}^{\infty} \frac{a_{n,\lambda}}{(\xi+\lambda)^{n+1}} = S\,, \tag{2.2}$$

where

$$a_{n,\lambda} = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} a_j\,. \tag{2.3}$$

When $\lambda = 0$ it reduces to ordinary convergence.

One may think that it is possible, in principle, that the sum value $S$ given by (2.2) depends on $\lambda$, $S = S_\lambda$. However, we shall show in Section 4 that if the corresponding series converge at $\lambda_1$ and at $\lambda_2$, then $S_{\lambda_1} = S_{\lambda_2}$.

Our definition makes it clear that the exterior Euler summability is to be applied to a *power series* in $\xi^{-1}$. One may give a corresponding definition for *numerical series.* We say that the series $\sum_{n=0}^{\infty} a_n$ is (Ex') summable to $S$ if $\sum_{n=0}^{\infty} a_n/\xi^{n+1} = S$ (Ex) when $\xi = 1$. Fortunately the two summability procedures are equivalent.

**Lemma 2.1.** *Let $\{a_n\}_{n=0}^{\infty}$ be a numerical sequence and let $\xi \in \mathbb{C}$. Then*

$$\sum_{n=0}^{\infty} \frac{a_n}{\xi^{n+1}} = S \qquad \text{(Ex')}\ , \tag{2.4}$$

*if and only if*

$$\sum_{n=0}^{\infty} \frac{a_n}{\xi^{n+1}} = S \qquad \text{(Ex)}\ . \tag{2.5}$$

*Proof.* Let $A_n(\xi) = a_n/\xi^{n+1}$. Then $\sum_{n=0}^{\infty} a_n/\xi^{n+1} = S$ (Ex'), if and only if $\sum_{n=0}^{\infty} A_n(\xi)/\omega^{n+1} = S$ (Ex), for $\omega = 1$, and this, in turn, is equivalent to the existence of $\lambda \in \mathbb{C}$ such that

$$\sum_{n=0}^{\infty} A_{n,\lambda}(\xi)\,/\,(1+\lambda)^{n+1} = S\,,$$

where

$$\begin{aligned} A_{n,\lambda}(\xi) &= \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} A_j(\xi) \\ &= \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} \frac{a_j}{\xi^{j+1}} \\ &= \frac{1}{\xi^{n+1}} \sum_{j=0}^{n} \binom{n}{j} (\lambda\xi)^{n-j} a_j \\ &= \frac{1}{\xi^{n+1}} a_{n,\lambda\xi} \,. \end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} A_{n,\lambda}(\xi) / (1+\lambda)^{n+1} = S$ is equivalent to the convergence of $\sum_{n=0}^{\infty} a_{n,\lambda\xi} / (\xi+\lambda\xi)^{n+1}$ to $S$, and this means exactly that $\sum_{n=0}^{\infty} a_n/\xi^{n+1} = S$ (Ex). □

## 3. The Region of Summability

We now study the set of points where a power series of the type $\sum_{n=0}^{\infty} a_n/z^{n+1}$ is exterior Euler summable. Naturally, there is a disc $D = \mathbb{D}(0, r_0)$ such that the series converges for $z \notin \overline{D}$ and diverges for $z \in D$. In the case of (Ex) summability we shall show that there exists a compact convex set $K$ such that $\sum_{n=0}^{\infty} a_n/z^{n+1}$ is exterior Euler summable if $z \notin K$ while it is not exterior Euler summable if $z \in \operatorname{Int} K$. As in the convergence case, both summability and non-summability can occur if $z \in \partial K$. In fact we shall show that

$$K = \bigcap_{\lambda \in \mathbb{C}} \overline{D}_{\lambda} \,, \tag{3.1}$$

where $D_\lambda = \mathbb{D}(-\lambda, r_\lambda)$ is the disc centered at $-\lambda$ outside of where the series $\sum_{n=0}^{\infty} a_{n,\lambda} / (z+\lambda)^{n+1}$ is convergent.

We shall also establish that the analytic function

$$f_0(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} \,, \qquad |z| > r_0 \,, \tag{3.2}$$

has a *unique* analytic continuation to the region $\overline{\mathbb{C}} \setminus K$. In fact, it is known that for an analytic function as $f_0(z)$, defined in the exterior of a disc, there exists a smallest compact convex set $K_{\text{cv}}$ such that $f_0$ has an analytic continuation to $\overline{\mathbb{C}} \setminus K_{\text{cv}}$, the set $K_{\text{cv}}$ being the *conjugate indicator diagram* of the entire function $\sum_{n=0}^{\infty} a_n z^n/n!$ [4, Sect. 5.3]; we shall show that $K = K_{\text{cv}}$.

The construction of $K_{\text{cv}}$ is as follows. Consider the family $\mathcal{F}$ of analytic continuations $(f_\Omega, \Omega)$ of $(f_0, \Omega_0)$ such that $\Omega = \overline{\mathbb{C}} \setminus L$ where $L$ is a compact convex set. Then one shows by induction that the family of these sets $L$ has the finite intersection property. Hence one can define

$$K_{\text{cv}} = \bigcap_{(f,\Omega)\in\mathcal{F}} \overline{\mathbb{C}} \setminus \Omega \,, \tag{3.3}$$

and the analytic extension $f_{\text{cv}}$, defined in $\overline{\mathbb{C}} \setminus K_{\text{cv}}$.

We shall show that the power series $\sum_{n=0}^{\infty} a_n/z^{n+1}$ is exterior Euler summable to $f_{\text{cv}}(z)$ for $z \in \overline{\mathbb{C}} \setminus K_{\text{cv}}$, but the series is never exterior Euler summable if $z \in \operatorname{Int} K_{\text{cv}}$.

In our analysis we shall use the fact that any compact convex subset of $\mathbb{C}$ is the intersection of all the open discs that contain it. We now give a proof of this simple fact along with other related results.

**Lemma 3.1.** *Let $K$ be a compact convex subset of $\mathbb{C}$. Then,*

$$K = \bigcap_{\substack{K\subset D \\ D \text{ open disc}}} D \,, \tag{3.4}$$

*and also*

$$K = \bigcap_{\substack{K\subset D \\ D \text{ open disc}}} \overline{D} \,. \tag{3.5}$$

*On the other hand*

$$\bigcap_{\substack{K\subset \overline{D} \\ D \text{ open disc}}} D = \{z \in K : z \text{ is not an extreme point of } K\} \,. \tag{3.6}$$

*Proof.* To prove (3.4) it is enough to prove that if $z \notin K$ then there exists an open disc $D$ with $K \subset D$ and $z \notin D$. Since there exists an open half-plane $H$ with the property that $K \subset H$ and $z \notin H$, we can assume that $K \subset \{\omega \in \mathbb{C} : \Re e\, \omega < 0\}$ and $z > 0$. Since $K$ is compact, there exists $M > 0$ such that $K \subset \{\omega \in \mathbb{C} : -M \le \Re e\, \omega < 0,\ |\Im m\, \omega| \le M\}$. Choose $\lambda > 0$ such that

$$\lambda > \max\left\{M, \frac{M^2 - z^2}{2z}\right\} \,, \tag{3.7}$$

and $r$ such that

$$\sqrt{\lambda^2 + M^2} < r < \lambda + z \,. \tag{3.8}$$

Then $K \subset \mathbb{D}(-\lambda, r)$ and $z \notin \mathbb{D}(-\lambda, r)$.

Next we observe that clearly $K\subseteq\bigcap_{K\subset D,\ D\text{ open disc}}\overline{D}$, while if $K\subset D$, where $D=\mathbb{D}\left(-\lambda,s\right)$, then $r=\max_{z\in K}\left|z+\lambda\right|<s$, and so if $r<t<s$, then $K\subset D_1\subset\overline{D}_1\subset D$, if $D_1=\mathbb{D}\left(-\lambda,t\right)$. Hence $\bigcap_{K\subset D_1,\ D_1\text{ open disc}}\overline{D}_1\subseteq\bigcap_{K\subset D,\ D\text{ open disc}}D=K$. This gives (3.5).

Finally we establish (3.6). Observe first that $L=\bigcap_{K\subset\overline{D},\ D\text{ open disc}}D$ is a subset of $K$. If $z\in K$ is not an extreme point, then there exists $\omega_1,\omega_2\in K$ such that $z$ is in the open segment from $\omega_1$ to $\omega_2$. If $K\subset\overline{D}$, $D$ an open disc, then since $\omega_1,\omega_2\in\overline{D}$ it follows that $z\in D$; thus $z\in L$. On the other hand, if $z$ is an extreme point of $K$, then there exists an open disc $D$ with $K\subset\overline{D}$ and with $\partial D\cap K=\left\{z\right\}$, and this yields that $z\notin L$. □

We can now prove the ensuing result.

**Lemma 3.2.** *Let $f_0\left(z\right)=\sum_{n=0}^{\infty}a_n/z^{n+1}$ for $z\in\Omega_0=\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}\left(0,r\right)$, the series being convergent in $\Omega_0$. Suppose that $f_0\neq0$. Then*

$$K_{\text{cv}}=\bigcap_{\lambda\in\mathbb{C}}\overline{D}_\lambda\,, \tag{3.9}$$

*where $D_\lambda=\mathbb{D}\left(-\lambda,r_\lambda\right)$ is the disc such that $\sum_{n=0}^{\infty}a_{n,\lambda}/\left(\xi+\lambda\right)^{n+1}$ converges if $\xi\notin\overline{D}_\lambda$ and diverges if $\xi\in D_\lambda$.*

*Proof.* Let $K_{\text{e}}=\bigcap_{\lambda\in\mathbb{C}}\overline{D}_\lambda$. If $\mathcal{F}_1$ is a subfamily of the family $\mathcal{F}$ of analytic continuations $(f_\Omega,\Omega)$ of $(f_0,\Omega_0)$ such that $\Omega=\overline{\mathbb{C}}\setminus L$ where $L$ is a compact convex set, then

$$K_{\text{cv}}\subseteq\bigcap_{(f,\Omega)\in\mathcal{F}_1}\overline{\mathbb{C}}\setminus\Omega\,,$$

and so $K_{\text{cv}}\subseteq K_{\text{e}}$.

To prove the reverse inclusion, let us observe that if $K_{\text{cv}}\subset D$, where $D=\mathbb{D}\left(-\lambda,s\right)$ is a disc with center at $-\lambda$, then $s>r_\lambda$ and actually $\sum_{n=0}^{\infty}a_{n,\lambda}/\left(\xi+\lambda\right)^{n+1}$ converges if $\xi\in\overline{\mathbb{C}}\setminus D$. If we now use (3.4), we obtain

$$K_{\text{e}}=\bigcap_{\lambda\in\mathbb{C}}\overline{D}_\lambda\subseteq\bigcap_{\substack{K_{\text{cv}}\subset D\\ D\text{ open disc}}}D=K_{\text{cv}}\,, \tag{3.10}$$

as required. □

Returning to exterior Euler summability, we can give the following result.

**Theorem 3.3.** *Let* $f_0(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ *for* $z \in \Omega_0 = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}(0,r)$, *the series being convergent in* $\Omega_0$. *Suppose that* $f_0 \neq 0$. *Then*

$$\sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} = f_{\mathrm{cv}}(z) \qquad (\mathrm{Ex}) , \tag{3.11}$$

*if* $z \in \overline{\mathbb{C}} \setminus K_{\mathrm{cv}}$. *The series* $\sum_{n=0}^{\infty} a_n/z^{n+1}$ *is not exterior Euler summable if* $z \in K_{\mathrm{cv}}$ *is not an extreme point of* $K_{\mathrm{cv}}$.

*Proof.* Notice that the series $\sum_{n=0}^{\infty} a_n/z^{n+1}$ is exterior Euler summable if $z \notin K_{\mathrm{e}} = \bigcap_{\lambda\in\mathbb{C}} \overline{D}_\lambda$, thus the Lemma 3.2 yields (3.11).

Suppose now that $z \in K_{\mathrm{cv}}$ and $\sum_{n=0}^{\infty} a_n/z^{n+1}$ is exterior Euler summable. Then there exists $\lambda \in \mathbb{C}$ such that $z \in \overline{\mathbb{C}} \setminus D_\lambda$. But $K_{\mathrm{cv}} \subset \overline{D}_\lambda$, so $z \notin \bigcap_{K_{\mathrm{cv}}\subset\overline{D},\ D \text{ open disc}} D$, and from (3.6), we obtain that $z$ must be an extreme point of $K_{\mathrm{cv}}$. ☐

Let us remark that the function $f_{\mathrm{cv}}$ may or may not have an analytic extension across the flat portions of $\partial K_{\mathrm{cv}}$, but the extreme points of $K_{\mathrm{cv}}$ are natural boundary points for the analytic continuation. In particular, if $\partial K_{\mathrm{cv}}$ is strictly convex, so that it does not have any flat sections, then $\partial K_{\mathrm{cv}}$ is a natural boundary for the analytic continuation of $f_{\mathrm{cv}}$. When $\partial K_{\mathrm{cv}}$ is a natural boundary for the analytic continuation of $f_{\mathrm{cv}}$ then $\left(f_{\mathrm{cv}}, \overline{\mathbb{C}} \setminus K_{\mathrm{cv}}\right)$ is the maximal analytic continuation of $f_0$, not only among the ones defined in the complement of a convex set, but among *all* analytic continuations. Interestingly, the series $\sum_{n=0}^{\infty} a_n/z^{n+1}$ could be exterior Euler summable at the extreme points of $\partial K_{\mathrm{cv}}$ but it never is in the flat sections of $\partial K_{\mathrm{cv}}$.

## 4. Uniqueness of the Sum

Implicit in our definition of exterior Euler summability is the fact that for a fixed $z \in \mathbb{C}$ the sum of the series $\sum_{n=0}^{\infty} a_{n,\lambda}/(z+\lambda)^{n+1}$, if convergent, is independent of $\lambda$. We now show that this is the case.

If $z \in \overline{\mathbb{C}} \setminus K_{\mathrm{cv}}$, then (3.11) shows that $\sum_{n=0}^{\infty} a_{n,\lambda}/(z+\lambda)^{n+1} = f_{\mathrm{cv}}(z)$ whenever the series is convergent, and thus the sum of the series is independent of $\lambda$. When $z \in K_{\mathrm{cv}}$, however, a proof of the uniqueness of the sum value is required. Let us start with some preliminary results.

**Lemma 4.1.** *Let* $z \in K_{\mathrm{cv}}$. *If the series* $\sum_{n=0}^{\infty} a_{n,\lambda}/(z+\lambda)^{n+1}$ *converges, then*

$$\lim_{t\to 0^+} f_{\mathrm{cv}}(z+t\omega) = \sum_{n=0}^{\infty} \frac{a_{n,\lambda}}{(z+\lambda)^{n+1}} , \tag{4.1}$$

*whenever*

$$\Re e\left(\frac{\omega}{z+\lambda}\right)>0\,, \tag{4.2}$$

*uniformly on compacts of this open half-plane* $\mathbb{H}_\lambda$.

*If* $\mu\in\mathbb{C}$ *let*

$$h_\mu(\omega)=\sum_{n=2}^{\infty}\frac{a_{n,\mu}}{(n-1)\,n\,(\omega+\mu)^{n-1}}\,, \tag{4.3}$$

*for* $\omega\in\mathbb{C}\setminus\overline{D}_\mu$. *Then* $h_\lambda(\omega)$ *admits a continuous extension to* $\mathbb{C}\setminus D_\lambda$, *while* $(\omega-z)^2\,h_{-z}(\omega)$ *admits a continuous extension to* $\mathbb{C}\setminus D_\lambda$ *that vanishes at* $\omega=z$.

*Proof.* The limit formula (4.1) when (4.2) is satisfied follows easily from the Abel limit theorem. For the second part, observe that if $\sum_{n=0}^{\infty}a_{n,\lambda}/\left(z+\lambda\right)^{n+1}$ converges, then the series defining $h_\lambda(\omega)$ is absolutely convergent if $|\omega+\lambda|\geq|z+\lambda|$, and this yields the continuity of $h_\lambda(\omega)$ in $\mathbb{C}\setminus D_\lambda$. The result about $(\omega-z)^2\,h_{-z}(\omega)$ follows by writing this function in terms of $h_\lambda(\omega)$, observing that

$$h_{\mu_2}(\omega)-h_{\mu_1}(\omega)=(a_0\omega-a_1)\ln\left(\frac{\omega+\mu_1}{\omega+\mu_2}\right)+a_0\left(\mu_2-\mu_1\right)\,, \tag{4.4}$$

for $\omega\in\mathbb{C}\setminus\left(\overline{D}_{\mu_1}\cup\overline{D}_{\mu_2}\right)$. □

Observe that, in particular,

$$\lim_{t\to0^+}f_{\mathrm{cv}}\left(z+t\left(z+\lambda\right)\right)=\sum_{n=0}^{\infty}\frac{a_{n,\lambda}}{(z+\lambda)^{n+1}}\,, \tag{4.5}$$

if the series converges.

**Theorem 4.2.** *Let* $f_0(z)=\sum_{n=0}^{\infty}a_n/z^{n+1}$ *for* $z\in\Omega_0=\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}(0,r)$, *the series being convergent in* $\Omega_0$. *Suppose that* $f_0\neq0$. *Let* $z\in\mathbb{C}$. *Suppose the series*

$$S_\lambda=\sum_{n=0}^{\infty}\frac{a_{n,\lambda}}{(z+\lambda)^{n+1}} \tag{4.6}$$

*converges for two complex numbers* $\lambda_1$ *and* $\lambda_2$. *Then*

$$S_{\lambda_1}=S_{\lambda_2}\,. \tag{4.7}$$

*Proof.* It remains to give the proof when $z\in K_{\mathrm{cv}}$. Observe first that unless $z$ belongs to the segment from $-\lambda_1$ to $-\lambda_2$, then the two half-planes $\mathbb{H}_{\lambda_1}$ and $\mathbb{H}_{\lambda_2}$ cannot be disjoint, and thus if $\omega\in\mathbb{H}_{\lambda_1}\cap\mathbb{H}_{\lambda_2}$, the

Lemma 4.1 yields that $\lim_{t\to 0^+} f_{\mathrm{cv}}(z+t\omega)$ should be equal to both $S_{\lambda_1}$ and $S_{\lambda_2}$, and (4.7) follows.

If $z$ belongs to the segment from $-\lambda_1$ to $-\lambda_2$, then $K_{\mathrm{cv}} = \{z\}$, and thus $f_{\mathrm{cv}}(\omega) = g\left((\omega - z)^{-1}\right)$ for some entire function $g$ with $g(0) = 0$. Using the Lemma 4.1, the function $(\omega - z)^2 h_{-z}(\omega)$ is continuous in $\mathbb{C}\setminus D_{\lambda_1}$, and continuous in $\mathbb{C}\setminus D_{\lambda_2}$. But $(\mathbb{C}\setminus D_{\lambda_1})\cup(\mathbb{C}\setminus D_{\lambda_2}) = \mathbb{C}$, and thus $(\omega - z)^2 h_{-z}(\omega)$ is continuous in all $\mathbb{C}$. Moreover, $h_{-z}$ is analytic at $\infty$, with $h_{-z}(\infty) = 0$, and therefore $h_{-z}(z + 1/\xi) = A\xi + B\xi^2$ for some constants $A$ and $B$. It follows that $g$ is a polynomial of degree 3 at the most. However, if $g$ is a polynomial, then $\lim_{t\to 0^+} f_{\mathrm{cv}}(z+t\omega) = \infty$ for any $\omega \neq 0$, which implies that $S_{\lambda_1} = S_{\lambda_2} = \infty$, and thus the series $\sum_{n=0}^{\infty} a_{n,\lambda}/(z+\lambda)^{n+1}$ diverges for both $\lambda = \lambda_1$ and $\lambda = \lambda_2$. □

Our analysis gives the behavior of $f_{\mathrm{cv}}(\omega)$ as $\omega$ approaches an extreme point of $K_{\mathrm{cv}}$ where the series is exterior Euler summable.

**Theorem 4.3.** *Let $f_0(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ for $z \in \Omega_0 = \overline{\mathbb{C}}\setminus\overline{\mathbb{D}}(0,r)$, the series being convergent in $\Omega_0$. Suppose that $f_0 \neq 0$. Let $z \in K_{\mathrm{cv}}$ be a point where*

$$\sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} = S \qquad (\mathrm{Ex}) \tag{4.8}$$

*exists. Then there exists an open arc $I$ of $|\xi| = 1$ with $|I| \geq \pi$ such that*

$$\lim_{t\to 0^+} f_{\mathrm{cv}}\left(z + te^{i\theta}\right) = S\,, \tag{4.9}$$

*if $e^{i\theta} \in I$, uniformly over compacts of $I$. There may be rays $z + te^{i\theta}$, $t > 0$, of $\mathbb{C}\setminus K_{\mathrm{cv}}$ for which the limit of $f_{\mathrm{cv}}\left(z + te^{i\theta}\right)$ as $t \to 0^+$ does not exist.*

*Proof.* Indeed, we just need to take $I$ as the set of numbers $\xi$ with $|\xi| = 1$ that belong to some half-plane $\mathbb{H}_\lambda$ given be (4.2) for which the series (4.1) converges.

If we take

$$f_{\mathrm{cv}}(\omega) = \omega^2 e^{-1/\omega} - \omega^2 + \omega - \frac{1}{2}\,, \tag{4.10}$$

then

$$f_{\mathrm{cv}}(\omega) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(n+3)!\,\omega^{n+1}} \qquad (\mathrm{Ex})\,, \tag{4.11}$$

for $\omega \neq 0$, so that $K_{\mathrm{cv}} = \{0\}$. The series is exterior Euler summable at $\omega = 0$ to $S = -1/2$ (cf. Example 4 below). Here $f_{\mathrm{cv}}(t\omega) \to S$ as $t \to 0^+$ if $\Re e\, \omega \geq 0$, but the limit does not exist if $\Re e\, \omega < 0$. □

Observe that while $f_{\mathrm{cv}}$ is analytic, and thus continuous in $\overline{\mathbb{C}} \setminus K_{\mathrm{cv}}$, its extension to the set of points where the series is exterior Euler summable, a subset of $(\overline{\mathbb{C}} \setminus K_{\mathrm{cv}}) \cup \partial K_{\mathrm{cv}}$, will not be continuous, in general.

## 5. Analytic Functionals and Exterior Summability

There is a close connection between the exterior Euler summability and some results about analytic functionals and hyperfunctions [3, 18, 19, 20], as we now explain.

Let $U$ be an open set in $\mathbb{C}$. We denote by $\mathfrak{O}(U)$ the space of analytic functions defined on $U$. The topology of $\mathfrak{O}(U)$ is that of uniform convergence on compact subsets of $U$, i.e., the topology generated by the family of seminorms $\|\varphi\|_K = \max\{|\varphi(z)| : z \in K\}$, for $K$ a compact subset of $U$ and $\varphi \in \mathfrak{O}(U)$. Since we can find a sequence of compact subsets of $U$, $\{K_n\}_{n=1}^{\infty}$, with $K_n \subset \mathrm{int}(K_{n+1})$, $\bigcup_{n=1}^{\infty} K_n = U$, it follows that $\mathfrak{O}(U)$ is a Fréchet space, actually a strict projective limit of Banach spaces.

A subset $S$ of a topological space $X$ is called locally closed if each $x \in S$ has a neighborhood in $X$, $V_x$, such that $S \cap V_x$ is closed in $V_x$. It can be shown that $S$ is locally closed in $X$ if and only if there exist an open set $U$ and a closed set $F$ such that $S = U \cap F$. If $S$ is locally closed in $X$, we say that $U \supseteq S$ is an open neighborhood of $S$ if $U$ is open in $X$ and $S$ is closed in $U$. We denote the set of open neighborhoods of $S$ as $\mathsf{N}(S)$.

If $S$ is locally closed in $\mathbb{C}$ then $\mathfrak{O}(S)$ is the space of germs of analytic functions defined on $S$. That is, a function $\varphi$ defined on $S$ belongs to $\mathfrak{O}(S)$ if and only if there exists $U \in \mathsf{N}(S)$ and an analytic function $\widetilde{\varphi} \in \mathfrak{O}(U)$ such that $\pi_S^U(\widetilde{\varphi}) = \varphi$, where $\pi_S^U$ is the restriction operator from $U$ to $S$. The system of topological vector spaces $\{\mathfrak{O}(U)\}_{U \in \mathsf{N}(S)}$ with operators $\pi_V^U : \mathfrak{O}(U) \longrightarrow \mathfrak{O}(V)$ for $U \supseteq V$ is actually a directed system and thus we can give $\mathfrak{O}(S)$ the inductive limit topology. When $K$ is compact, then $\mathfrak{O}(K)$ is a strict limit of Banach spaces. If $S \subseteq \mathbb{R}$ is open then $\mathfrak{O}(S)$ is the space of real analytic functions on $S$, while if $S \subseteq \mathbb{R}$ is locally closed then $\mathfrak{O}(S)$ is the space of germs of real analytic functions on $S$.

If $S \subseteq \mathbb{C}$ is locally closed, then the dual space $\mathfrak{O}'(S)$ is called the space of *analytic functionals* on $S$. When $K \subseteq \mathbb{R}$ is compact then

$\mathfrak{O}'(K)$ is actually isomorphic to the space $\mathfrak{B}(K)$ of hyperfunctions defined on $K$, although hyperfunctions are usually constructed by using a different approach [18]. Observe that if $K \subseteq \mathbb{R}$ then the space of distributions $T \in \mathcal{D}'(\mathbb{R})$ whose support is contained in $K$, the space $\mathcal{E}'[K]$, is a subspace of $\mathfrak{B}(K)$.

If $K$ is a compact subset of $\mathbb{C}$, and $T \in \mathfrak{O}'(K)$ then its Cauchy or analytic representation, denoted as $f(z) = \mathcal{C}\{T(\omega); z\}$, is the analytic function $f \in \mathfrak{O}\left(\overline{\mathbb{C}} \setminus K\right)$ given by

$$f(z) = \mathcal{C}\{T(\omega); z\} = \frac{1}{2\pi i}\left\langle T(\omega), \frac{1}{\omega - z}\right\rangle . \tag{5.1}$$

Notice that the analytic representation satisfies

$$\lim_{z\to\infty} f(z) = 0 . \tag{5.2}$$

According to a theorem of Silva [18], the operator $\mathcal{C}$ is an isomorphism of the space $\mathfrak{O}'(K)$ onto the subspace $\mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K\right)$ of $\mathfrak{O}\left(\overline{\mathbb{C}} \setminus K\right)$ formed by those analytic functions that satisfy (5.2). When $K \subseteq \mathbb{R}$ then the operator $\mathcal{C}$ becomes an isomorphism of the space of hyperfunctions $\mathfrak{B}(K)$ onto $\mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K\right)$.

The inverse operator $\mathcal{C}^{-1}$ is given as follows. Let $\varphi \in \mathfrak{O}(K)$, and let $\widetilde{\varphi} \in \mathfrak{O}(U)$ be an analytic extension to some region $U \in \mathsf{N}(K)$; let $C$ be a closed curve in $U$ such that the index of any point of $K$ with respect to $C$ is one. Then if $f \in \mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K\right)$ we define $T = \mathcal{C}^{-1}\{f\} \in \mathfrak{O}'(K)$ by specifying its action on $\varphi$ as

$$\langle T(\omega), \varphi(\omega)\rangle = -\oint_C f(\xi)\,\widetilde{\varphi}(\xi)\;\mathrm{d}\xi . \tag{5.3}$$

Clearly $T = \mathcal{C}^{-1}\{f\}$ is also defined if $f \in \mathfrak{O}\left(\overline{\mathbb{C}} \setminus K\right)$, but in this space $\mathcal{C}^{-1}$ has a non-trivial kernel, namely, the constant functions.

If $K_1$ and $K_2$ are compact subsets of $\mathbb{C}$ with $K_1 \subset K_2$ and $K_1$ has no holes, that is, $\overline{\mathbb{C}} \setminus K_1$ is connected, then any functional $T \in \mathfrak{O}'(K_1)$ can be considered as an analytic functional of the space $\mathfrak{O}'(K_2)$, so that we have a canonical injection $\mathfrak{O}'(K_1) \hookrightarrow \mathfrak{O}'(K_2)$. This injection corresponds to the injection $\mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K_1\right) \hookrightarrow \mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K_2\right)$ provided by the restriction to a smaller region. In general not all analytic functionals $T \in \mathfrak{O}'(K_2)$ are in the image of $\mathfrak{O}'(K_1)$, that is, in general they do not admit an "extension" to $\mathfrak{O}'(K_1)$. An extension exists precisely when the Cauchy representation $f = \mathcal{C}\{T\} \in \mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K_2\right)$ admits an analytic continuation to $\overline{\mathbb{C}} \setminus K_1$.

If $T\in\mathfrak{O}'(K)$, then the power series expansion of its Cauchy representation at infinity takes the form

$$\mathcal{C}\{T(\omega);z\}=-\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\mu_n(T)}{z^{n+1}}\,,\qquad |z|>\rho\,, \tag{5.4}$$

where

$$\mu_n(T)=\langle T(\omega),\omega^n\rangle\ , \tag{5.5}$$

are the moments of $T$ and where $\rho=\max\{|z|:z\in K\}$. Observe that $\mathcal{C}\{T(\omega);z\}$ is defined if $z\in\overline{\mathbb{C}}\setminus K$, but the series in (5.4) will generally be divergent if $|z|<\rho$, and could be divergent if $|z|=\rho$.

Our results about exterior Euler summability yield several corresponding results on analytic functionals.

**Theorem 5.1.** *Let $K$ be a compact convex subset of $\mathbb{C}$ and let $T\in\mathfrak{O}'(K)$ with Cauchy representation $f=\mathcal{C}\{T\}\in\mathfrak{O}_0\left(\overline{\mathbb{C}}\setminus K\right)$. Then*

$$\sum_{n=0}^{\infty}\frac{\mu_n(T)}{z^{n+1}}=-2\pi i f(z)\qquad\text{(Ex)}\ , \tag{5.6}$$

*for all $z\in\mathbb{C}\setminus K$.*

Analytic functionals do not have a support, like distributions do, but they have *carriers.* If $T\in\mathfrak{O}'(K)$ then $L\subset K$ is a carrier if $T$ admits an extension to $\mathfrak{O}'(L)$. However, carriers are not unique. It can be shown [3, Sect. 1.3] that when $\Omega$ is a bounded convex open set of $\mathbb{C}$ and $T\in\mathfrak{O}'\left(\overline{\Omega}\right)$ then $T$ has a smallest compact convex carrier, $\widetilde{K}(T)\subset\overline{\Omega}$, called the *convex support* of $T$. Our results immediately give the following,

**Theorem 5.2.** *Let $f_0(z)=\sum_{n=0}^{\infty}a_n/z^{n+1}$ for $z\in\Omega_0=\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}(0,r)$, the series being convergent in $\Omega_0$. Suppose that $f_0\neq 0$. Let $T_0\in\mathfrak{O}'\left(\overline{\mathbb{D}}(0,r)\right)$ be the analytic functional $T_0=\mathcal{C}^{-1}\{f_0\}$. Then $K_{\mathrm{cv}}$ is the smallest compact convex subset $K\subset\overline{\mathbb{D}}(0,r)$ such that $T_0$ admits an extension to $T\in\mathfrak{O}'(K)$, and the analytic representation of $T$ is $f_{\mathrm{cv}}$.*

In other words, $K_{\mathrm{cv}}$, the complement of the region of exterior Euler summability of $\sum_{n=0}^{\infty}a_n/z^{n+1}$, is precisely the convex support of $T_0=\mathcal{C}^{-1}\{f_0\}$.

The solution of the moment problem in the space of distributions $\mathcal{E}'[I]$, where $I$ is an interval of the form $[-a,a]$, was given in [10] (see also [12, Thm. 7.3.1]). Here we can give the solution of the moment problem in $\mathfrak{O}'(K)$ if $K$ is a compact convex subset of $\mathbb{C}$.

**Theorem 5.3.** *Let $\{\mu_n\}_{n=0}^{\infty}$ be a sequence of complex numbers and let $K$ be a compact convex subset of $\mathbb{C}$. Then the moment problem*

$$\langle T(\omega), \omega^n\rangle = \mu_n\,, \qquad n \in \mathbb{N}\,, \tag{5.7}$$

*has a solution $T \in \mathfrak{O}'(K)$ if and only if the series $\sum_{n=0}^{\infty} \mu_n / z^{n+1}$ is exterior Euler summable for all $z \in \mathbb{C} \setminus K$, in fact, $-2\pi i T$ is then the inverse Cauchy representation of the analytic function given by the Euler exterior sum of this series. If there is a solution, it is unique.*

## 6. Special Cases

We shall now give several examples of the exterior Euler summability.

**Example 1.** Let $\omega \in \mathbb{C}$ be *fixed* and consider the Dirac delta function at $\omega$, the analytic functional $\delta_\omega \in \mathfrak{O}'(\{\omega\})$, given by

$$\langle \delta_\omega(z), \varphi(z)\rangle = \varphi(\omega)\ , \tag{6.1}$$

for $\varphi \in \mathfrak{O}(\{\omega\})$. In this case the moments are given by $\mu_n = \omega^n$, $n \in \mathbb{N}$, while the Cauchy representation is $f(z) = (2\pi i)^{-1}(\omega - z)^{-1}$, so that we obtain

$$\sum_{n=0}^{\infty} \frac{\omega^n}{z^{n+1}} = \frac{1}{z-\omega} \qquad (\mathrm{Ex})\ , \tag{6.2}$$

for all $z \neq \omega$.

The power of a summation method is many times measured [14] by the set of points $\omega$ where the geometric series $\sum_{n=0}^{\infty} \omega^n$ is summable to $(1-\omega)^{-1}$; a reason for this is the Okada theorem [5, Thm. 5.2.8] that says that if the geometric series is summable in a star shaped region $\Omega$, with $\mathbb{D} \subset \Omega$, then the Taylor series of any function analytic at the origin is summable, to the function, in its substar subset of $\Omega$. For instance for convergence this set is the open disc $|\omega| < 1$, while for Cesàro summability it is the set $|\omega| \leq 1$, $\omega \neq 1$. We now study this question for (Ex') summability. It should be noticed that standard summability procedures employed in analytic continuation [14, Chp. VIII], such as Borel, Mittag-Leffler, or Lindelöf summability, provide a much smaller region of summability for the geometric series.

**Example 2.** The geometric series $\sum_{n=0}^{\infty} \omega^n$ is (Ex') summable if and only if the series $\sum_{n=0}^{\infty} \omega^n / z^{n+1}$ is (Ex) summable at $z = 1$; the previous example shows that this is the case precisely when $1 = z \neq \omega$. Therefore,

$$\sum_{n=0}^{\infty} \omega^n = \frac{1}{1-\omega} \qquad (\mathrm{Ex'})\ , \qquad \text{for all } \omega \neq 1\,. \tag{6.3}$$

It is easy to see that one can take the derivative of an exterior Euler summation formula in the region $\mathbb{C}\setminus K_{\mathrm{cv}}$. This yields the following formulas.

**Example 3.** If $k\in\mathbb{N}$, $k\geq 1$, differentiation of (6.2) yields

$$\sum_{n=k-1}^{\infty}\binom{n}{k-1}\frac{\omega^{n+1-k}}{z^{n+1}}=\frac{1}{(z-\omega)^{k}}\qquad \text{(Ex)}\ ,\qquad \text{for } z\neq\omega\,. \tag{6.4}$$

Hence, we also obtain

$$\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}\omega^{n}=\frac{1}{(1-\omega)^{k}}\qquad \text{(Ex')}\ ,\qquad \text{for all } \omega\neq 1\,. \tag{6.5}$$

**Example 4.** If $g$ is an entire function with $g(0)=0$, then the function $f(z)=g\left((z-\omega)^{-1}\right)$ is analytic in $\overline{\mathbb{C}}\setminus\{\omega\}$, and we obtain

$$\sum_{n=0}^{\infty}\frac{a_n}{z^{n+1}}=f(z)\qquad \text{(Ex)}\ ,\qquad \text{for } z\neq\omega\,, \tag{6.6}$$

an expansion that is convergent for $|z|>|\omega|$. The coefficients $a_n$ in this development are the same coefficients in the convergent Taylor expansion

$$h(\xi)=g\left(\frac{\xi}{1-\xi\omega}\right)=\sum_{n=0}^{\infty}a_n\xi^{n+1}\,,\qquad |\xi|<|\omega|^{-1}\,, \tag{6.7}$$

so that $a_n=h^{(n+1)}(0)/(n+1)!$ . Notice that the series $\sum_{n=1}^{\infty}a_n\xi^{n+1}$ is actually (Ex') summable to $h(\xi)$ for all $\xi\neq 1/\omega$.

The example considered in the proof of the Theorem 4.3, namely,

$$f(z)=z^{2}e^{-1/z}-z^{2}+z-\frac{1}{2}\,, \tag{6.8}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(n+3)!z^{n+1}}=f(z)\qquad \text{(Ex)}\ ,\qquad z\neq 0\,, \tag{6.9}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(n+3)!z^{n+1}}=-\frac{1}{2}\qquad \text{(Ex)}\ ,\qquad z=0\,, \tag{6.10}$$

shows that when $g$ is not a polynomial, it is possible for the series to be exterior Euler summable *everywhere.* Actually, a series that is exterior Euler summable everywhere can only arise in this way. In the case of (6.10) the series $\sum_{n=0}^{\infty}a_{n,\lambda}/(z+\lambda)^{n+1}$ converges, to $-1/2$, for $z=0$ whenever $\lambda>0$, because, in fact, the extension of $f(z)$ to the circle $|z+\lambda|\geq\lambda$ obtained by assigning the value $-1/2$ to $z=0$, is continuous in $\mathbb{C}\setminus D_\lambda$, while on the circle $|z+\lambda|=\lambda$ this extension, namely $f\left(-\lambda+\lambda e^{i\theta}\right)$, is a differentiable function of $\theta$, even at $\theta=0$. Interestingly, if $\lambda<0$ then $f\left(-\lambda+\lambda e^{i\theta}\right)$ is also a differentiable function of $\theta$, but the series $\sum_{n=0}^{\infty} a_{n,\lambda}/\left(z+\lambda\right)^{n+1}$ does not converge when $z=0$.

**Example 5.** Consider the series

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)\,z^{n+1}}=\ln\left(\frac{z}{z-1}\right)\,,\qquad |z|>1\,. \tag{6.11}$$

Here we obtain that $K_{\text{cv}}=[0,1]$ and that the series is exterior Euler summable for any $z$ that is not in the interval $[0,1]$. This, in turn, yields the formula

$$\sum_{n=1}^{\infty}\frac{\omega^n}{n}=\ln\left(\frac{1}{1-\omega}\right)\qquad (\text{Ex'})\,,\qquad \text{for } \omega\notin[1,\infty)\,. \tag{6.12}$$

## 7. Convergence Acceleration

It is important to emphasize that the exterior Euler summability is also a devise for the convergence acceleration of slowly convergent series, or even divergent series which are Cesàro or Abel summable.

Let us consider an Abel summable series, which we write in the form $\sum_{n=0}^{\infty}(-1)^{n+1}a_n$. Let $S$ be the sum of the series,

$$\sum_{n=0}^{\infty}(-1)^{n+1}a_n=S\qquad (\text{A})\,. \tag{7.1}$$

Our aim is to find a rapidly convergent representation of $S$.

The Abel summability implies that the function

$$f(z)=\sum_{n=0}^{\infty}\frac{a_n}{z^{n+1}}\,, \tag{7.2}$$

is analytic in the region $\{z\in\mathbb{C}:|z|>1\}$. Let $T=\mathcal{C}^{-1}\{f\}$, an analytic functional in the closed unit disc $\overline{\mathbb{D}}$. *Our key assumption* is that $T$ admits an extension to $\mathfrak{O}'(K)$, where $K$ is a compact convex subset of $\overline{\mathbb{D}}$ such that $-1\notin K$. If we take $K$ minimal with this property, then $K=K_{\text{cv}}$, $f$ admits an analytic continuation $f_{\text{cv}}$ to $\overline{\mathbb{C}}\setminus K_{\text{cv}}$, a region that contains the point $z=-1$, and

$$S=f_{\text{cv}}(-1)\,. \tag{7.3}$$

Observe that, in general, the series in (7.1) is not convergent, not even Cesàro summable. However, if $a_n=O\left(n^{\beta}\right)$ for some $\beta\geq 0$, then the series should be Cesàro summable of some order since in that case the Fourier series $\sum_{n=0}^{\infty} a_n e^{-i(n+1)\theta}$ is a distribution in $\mathcal{D}'(\partial\mathbb{D})$ [2, 6, 9], and since $f_{\text{cv}}$ is analytic at $-1$, this Fourier series represents a continuous function in a neighborhood of $-1$ in the circle, and thus the series is Cesàro summable in that neighborhood [8, 22, 23].

Nevertheless, whether the series (7.1) is convergent or not, then our assumptions yield that

$$\sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} = S \qquad (\text{Ex}) , \quad \text{if } \ z = -1 , \tag{7.4}$$

so that there exist complex numbers $\lambda$ such that

$$S = \sum_{n=0}^{\infty} \frac{a_{n,\lambda}}{(-1+\lambda)^{n+1}} , \tag{7.5}$$

where

$$a_{n,\lambda} = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} a_j \tag{7.6}$$

is a convergent series. Actually, if $-\lambda \in (0,\infty)$, the series (7.5) is *exponentially convergent.* If a power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R$, and $|z| < R$, we say that the series converges like $(|z|/R)^n$; if $R = \infty$ we say that the series converges like an entire function. For example, if $K_{\text{cv}} = [0,1]$, by taking $-\lambda = 1/2$ then the series converges like $(1/3)^n$, while if $-\lambda = 1$ then the convergence is like $(1/2)^n$. See [1] for an analysis of the best way to choose $\lambda$ in order to minimize the error when using a partial sum, with a fixed number of terms, of the series (7.5); [1] also has several very interesting numerical evaluations of series.

Let us illustrate this procedure with the series $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}$, which is Abel summable for *all* values of $s \in \mathbb{C}$; it converges when $\Re e\, s > 0$. We have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \left(1 - 2^{s-1}\right) \zeta(s) \qquad (\text{A}) , \tag{7.7}$$

where $\zeta(s)$ is the Riemann zeta function. The convergence acceleration of $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}$ thus provides a procedure for the numerical evaluation of $\zeta(s)$ (the numerical evaluation when $s > 0$ is given in [1]).

In this case there exist distributions $T_s(x)$ for $s \in \mathbb{C}$, with Cauchy representations $f_s(z)$, such that

$$2\pi i f_s(-1) = \left\langle T_s(x), \frac{1}{x+1} \right\rangle = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}. \tag{7.8}$$

As we shall see, $\operatorname{supp} T_s = [0,1]$ for $s \in \mathbb{C} \setminus \{-0, -1, -2, \ldots\}$, while for $q = 0, 1, 2, \ldots$ we have $\operatorname{supp} T_{-q} = \{1\}$. In order to construct the distributions $T_s(x)$ we use the well-known formula

$$\int_0^\infty t^{s-1} e^{-kt} \mathrm{d}t = \frac{\Gamma(s)}{k^s}, \tag{7.9}$$

and make a change of variables to obtain

$$\int_0^1 x^n \ln^{s-1}(1/x)\, \mathrm{d}x = \frac{\Gamma(s)}{(n+1)^s}. \tag{7.10}$$

The function $\widetilde{T}_s(x) = \chi_{(0,1)}(x) \ln^{s-1}(1/x)$ is locally integrable in $\mathbb{R} \setminus \{1\}$, and at $x = 1$ it behaves like $\widetilde{T}_s(x) \sim (1-x)^{s-1}$ as $x \to 1^-$. It follows that $\widetilde{T}_s(x)$ defines a distribution for $s \neq 0, -1, -2, \ldots$, analytic as a function of $s$, with simple poles at the negative integers. Therefore, if we define

$$T_s(x) = \frac{\widetilde{T}_s(x)}{\Gamma(s)}, \tag{7.11}$$

then the distribution $T_s(x)$ is an entire function of $s$, with moments

$$\mu_{s,n} = \langle T_s(x), x^n \rangle = \frac{1}{(n+1)^s}. \tag{7.12}$$

Our results give the exterior Euler expansions

$$\sum_{n=1}^{\infty} \frac{1}{n^s z^n} = -2\pi i f_s(z) \qquad (\mathrm{Ex}), \qquad \text{for } z \notin [0,1], \tag{7.13}$$

if $s \neq 0, -1, -2, \ldots$, and

$$\sum_{n=1}^{\infty} \frac{n^q}{z^n} = -2\pi i f_{-q}(z) \qquad (\mathrm{Ex}), \qquad \text{for } z \neq 1, \tag{7.14}$$

if $q = 0, 1, 2, \ldots$. If we now take $z = -1$ and use the scheme given by (7.5) and (7.6) we obtain

$$\left(1 - 2^{1-s}\right) \zeta(s) = -\sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} \frac{\lambda^{n-j}}{(-1+\lambda)^{n+1} (1+j)^s} \tag{7.15}$$

(that reduces to formula (19) of [1] if we replace $\lambda$ by $-\lambda$). If $\lambda=-1$ we obtain the formula [15, 21]

$$\left(1-2^{1-s}\right)\zeta\left(s\right)=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{j=0}^{n}\binom{n}{j}\frac{\left(-1\right)^{j}}{\left(1+j\right)^{s}}\,, \tag{7.16}$$

that converges like $\left(1/2\right)^{n}$ if $s\neq0,\,-1,\,-2,\,\ldots,$ and that reduces to a finite sum in case $s=0,-1,-2,\ldots.$ If $-\lambda=1/2$, then the convergence in (7.15) is like $\left(1/3\right)^{n}$ for any $s\in\mathbb{C}$.

## 8. Mittag-Leffler Expansions

We shall now consider the exterior Euler sum representation of some Mittag-Leffler expansions.

Let us start with a series $\sum_{n=1}^{\infty}a_n$ that is Cesàro summable [12, 14]. Then the series of distributions

$$T\left(x\right)=\sum_{n=1}^{\infty}a_{n}\delta\left(x-\frac{1}{n}\right)\,, \tag{8.1}$$

is Cesàro summable in the space $\mathcal{D}'\left(\mathbb{R}\right)$ (actually in the space $\mathcal{E}'\left(\mathbb{R}\right)$). Indeed, if $\sum_{n=1}^{\infty}a_n$ is (C) summable, then so is $\sum_{n=1}^{\infty}a_n n^{-\beta}$ for any $\beta>0$. Also, there exists $N\in\mathbb{N}$ such that $a_n=O\left(n^{N}\right)$ as $n\to\infty$. If $\phi\in\mathcal{E}\left(\mathbb{R}\right)$ then we can write

$$\phi\left(x\right)=\sum_{j=0}^{N+1}\frac{\phi^{(j)}\left(0\right)x^{j}}{j!}+x^{N+2}\psi\left(x\right)\,, \tag{8.2}$$

for some function $\psi\in\mathcal{E}\left(\mathbb{R}\right)$, and thus we obtain

$$\begin{aligned}\left\langle T\left(x\right),\phi\left(x\right)\right\rangle &=\sum_{n=1}^{\infty}a_{n}\phi\left(\frac{1}{n}\right)\\ &=\sum_{n=1}^{\infty}a_{n}\left\{\phi\left(0\right)+\frac{\phi'\left(0\right)}{n}+\cdots+\frac{\phi^{(N+1)}\left(0\right)}{(N+1)!\,n^{N+1}}+\frac{\psi\left(\frac{1}{n}\right)}{n^{N+2}}\right\}\\ &=\sum_{j=0}^{N+1}\sum_{n=1}^{\infty}\frac{a_{n}}{n^{j}}\frac{\phi^{(j)}\left(0\right)}{j!}+\sum_{n=1}^{\infty}\frac{a_{n}}{n^{N+2}}\psi\left(\frac{1}{n}\right)\qquad\text{(C)}\,,\end{aligned}$$

as the sum of $N+1$ Cesàro summable series and a convergent one.

More generally, for a series of the type $\sum_{n=-\infty}^{\infty}a_n$ that is Cesàro summable in the *principal value* sense at infinity [12], namely, if the *symmetric* Cesàro limit $\sum_{n=-N}^{N}a_n$ exists as $N\to\infty$, then the series $\sum_{n=-\infty,n\neq0}^{\infty}a_n\delta\left(x-1/n\right)$ is likewise principal value Cesàro summable in the space $\mathcal{E}'\left(\mathbb{R}\right)$.

Observe that the analytic representation of a distribution of the type

$$T_{\alpha}(x) = \text{p.v.} \sum_{n=-\infty, n\neq 0}^{\infty} a_n \delta\left(x - \frac{\alpha}{n}\right) \qquad (\text{C}) \ , \tag{8.3}$$

for $\alpha \in \mathbb{R} \setminus \{0\}$ is given by

$$2\pi i f_{\alpha}(z) = \left\langle T_{\alpha}(x), \frac{1}{x-z} \right\rangle = \text{p.v.} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{n a_n}{\alpha - nz} \qquad (\text{C}) \ , \tag{8.4}$$

for $z \in \mathbb{C} \setminus \operatorname{supp}(T_{\alpha})$. If we now put $b_n = n a_n$, and $\omega = \alpha/z$, we obtain the following result.

**Lemma 8.1.** *If the series* $\sum_{n=-\infty, n\neq 0}^{\infty} b_n/n$ *is principal value Cesàro summable, then the series*

$$G(\omega) = \text{p.v.} \sum_{n=-\infty}^{\infty} \frac{b_n}{\omega - n} \qquad (\text{C}) \ , \tag{8.5}$$

*is also principal value Cesàro summable for all* $\omega \in \mathbb{C} \setminus \mathbb{Z}$, *the function* $G$ *is analytic in this region and has simple poles at the integers* $n \in \mathbb{N}$, *with residues* $b_n$.

Observe that the sum in (8.5) may have a term corresponding to $n = 0$. The results of Section 5 yield the following exterior Euler series representation of the Mittag-Leffler function $G$.

**Theorem 8.2.** *The analytic function* $G$ *given by (8.5) can be written as the exterior Euler summable series*

$$G(\omega) = \frac{b_0}{\omega} - \sum_{k=0}^{\infty} \xi_k \omega^k \qquad (\text{Ex'}) \ , \tag{8.6}$$

*for* $\omega \neq 0$, $\omega \notin (-\infty, -1] \cup [1, \infty)$, *where the moments are given as*

$$\xi_k = \text{p.v.} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{b_n}{n^{k+1}} \qquad (\text{C}) \ . \tag{8.7}$$

Actually one can give a related expansion which is valid in the region $\mathbb{C} \setminus ((-\infty, -N-1] \cup [N+1, \infty))$, $\omega \neq 0, \pm 1, \ldots, \pm N$, namely,

$$G(\omega) = \sum_{n=-N}^{N} \frac{b_n}{\omega - n} - \sum_{k=0}^{\infty} \xi_{k,N} \omega^k \qquad (\text{Ex'}) \ , \tag{8.8}$$

where

$$\xi_{k,N} = \text{p.v.} \sum_{n=-\infty,|n|>N}^{\infty} \frac{b_n}{n^{k+1}} \qquad \text{(C)} \; . \tag{8.9}$$

An example is provided by $G(\omega) = \pi \cot \pi\omega$, that has the principal value convergent Mittag-Leffler expansion p.v. $\sum_{n=-\infty}^{\infty} 1/(\omega - n)$ :

$$\pi \cot \pi\omega = \frac{1}{\omega} - 2\sum_{n=1}^{\infty} \zeta(2n)\, \omega^{2n-1} \qquad \text{(Ex')} \; , \tag{8.10}$$

for any complex number $\omega \neq 0$ with $\omega \notin (-\infty, -1] \cup [1, \infty)$. Here $\zeta$ is the Riemann zeta function.

We can also consider Mittag-Leffler developments that are not principal value (C) summable. We shall illustrate this with an expansion for the digamma function $\psi(\omega) = \Gamma'(\omega)/\Gamma(\omega)$, whose Mittag-Leffler development is given by

$$\psi(\omega) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+\omega} \right) , \qquad \omega \neq 0, -1, -2, \dots , \tag{8.11}$$

where $\gamma$ is Euler's constant. In this case the series of analytic functionals $\sum_{n=1}^{\infty} (1/n)\delta(\omega - \alpha/n)$ is not (C) summable for any $\alpha \in \mathbb{C}$, but the analytic functional

$$T(\omega) = \sum_{n=1}^{\infty} \frac{1}{n} \left( \delta\left(\omega - \frac{\alpha}{n}\right) - \delta\left(\omega - \frac{\beta}{n}\right) \right) , \tag{8.12}$$

is given by a convergent series for any $\alpha, \beta \in \mathbb{C}$. The moments are

$$\mu_k = \left\langle T(\omega), \omega^k \right\rangle = \left(\alpha^k - \beta^k\right) \zeta(k+1) \; , \quad k \in \mathbb{N} , \tag{8.13}$$

while its Cauchy representation $f = \mathcal{C}\{T\}$ is given by

$$\begin{aligned} 2\pi i f(z) &= \left\langle T(\omega), \frac{1}{\omega - z} \right\rangle \\ &= \sum_{n=1}^{\infty} \left( \frac{1}{\alpha - nz} - \frac{1}{\beta - nz} \right) \\ &= \frac{1}{z} \left( \psi(1 - \alpha/z) - \psi(1 - \beta/z) \right) . \end{aligned} \tag{8.14}$$

If we now employ the Theorem 5.1, taking into account that $\mu_0 = 0$, we obtain

$$\sum_{k=1}^{\infty} \frac{\left(\alpha^k - \beta^k\right) \zeta(k+1)}{z^k} = \psi\left(1 - \frac{\beta}{z}\right) - \psi\left(1 - \frac{\alpha}{z}\right) \qquad \text{(Ex)} \; , \tag{8.15}$$

as long as $z \notin K(\alpha, \beta)$, where the triangular set $K(\alpha, \beta)$ is the smallest convex set that contains $\alpha, \beta$, and 0, that is, the convex support of the analytic functional $T$.

When $1 \notin K(\alpha, \beta)$ then (8.15) yields

$$\sum_{k=1}^{\infty} \left(\alpha^k - \beta^k\right) \zeta(k+1) = \psi(1-\beta) - \psi(1-\alpha) \qquad (\text{Ex'}) \; . \tag{8.16}$$

If we take $\alpha = 0$ and use the fact that $\psi(1) = -\gamma$, we obtain

$$\sum_{k=1}^{\infty} \beta^k \zeta(k+1) = -\psi(1-\beta) - \gamma \qquad (\text{Ex'}) \; , \qquad \beta \notin [1, \infty) \, . \tag{8.17}$$

In particular, if $\beta = -1$ we find the sum of the not only exterior Euler summable but actually a Cesàro summable series,

$$\sum_{k=1}^{\infty} (-1)^{k+1} \zeta(k+1) = 1 \qquad (\text{C}) \; . \tag{8.18}$$

When $\beta = -N$, $N = 2, 3, 4, \ldots$, we obtain exterior Euler summable series that are not Abel summable,

$$\sum_{k=1}^{\infty} (-1)^{k+1} N^k \zeta(k+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{N} \qquad (\text{Ex'}) \; . \tag{8.19}$$

If we now take $\alpha = 1 - \omega$, $\beta = \omega$, and use the identity $\psi(1-\omega) - \psi(\omega) = \pi \cot \pi\omega$, then (8.16) yields that for $\omega \neq 0$, $\omega \notin (-\infty, -1] \cup [1, \infty)$,

$$\sum_{k=1}^{\infty} \left((1-\omega)^k - \omega^k\right) \zeta(k+1) = \pi \cot \pi\omega \qquad (\text{Ex'}) \; . \tag{8.20}$$

If $\omega = 1/4$ the series becomes convergent, and we recover the Flajolet-Vardi formula [13]

$$\sum_{k=1}^{\infty} \left(\left(\frac{3}{4}\right)^k - \left(\frac{1}{4}\right)^k\right) \zeta(k+1) = \pi \, , \tag{8.21}$$

considered also by Amore [1, Eqn. 4].

## References


[1] Amore, P., Convergence acceleration of series through a variational approach, *J. Math. Anal. Appl.* **323** (2006), 63–77.

[2] Beltrami, E. J. and Wohlers, M. R., *Distributions and the Boundary Values of Analytic Functions,* Academic Press, New York, 1966.

[3] Berenstein, C. and Gay, R., *Complex Analysis and Special Topics in Harmonic Analysis,* Springer, New York, 1995.

[4] Boas, R. P., *Entire functions,* Academic Press, 1954.
[5] Boos, J., *Classical and Modern Methods in Summability,* Oxford University Press, Oxford, 2000.
[6] Bremermann, H., *Distributions, Complex Variables and Fourier Transforms,* Addison-Wesley, Reading, Massachusetts, 1965.
[7] Bromwich, T. J. l'A., *An Introduction to the Theory of Infinite Series,* third edition, Chelsea, New York, 1991.
[8] Estrada, R., Characterization of the Fourier series of a distribution having a value at a point, *Proc. Amer. Math. Soc.* **124** (1996), 1205–1212.
[9] Estrada, R. and Kanwal, R. P., Distributional boundary values of harmonic and analytic functions, *J. Math. Anal. Appl.* **89** (1982), 262–289.
[10] Estrada, R. and Kanwal, R.P., Moment sequences for a certain class of distributions, *Complex Variables Theory Appl.* **9** (1987), 31–39.
[11] Estrada, R. and Kanwal, R.P., *Singular Integral Equations,* Birkhäuser, Boston, 2000.
[12] Estrada, R. and Kanwal, R.P., *A Distributional Approach to Asymptotics. Theory and Applications,* second edition, Birkhäuser, Boston, 2002.
[13] Flajolet, P. and Vardi, I., Zeta function expansions of classical functions, unpublished manuscript, 1996.
[14] Hardy, G.H., *Divergent Series,* Clarendon Press, Oxford, 1949.
[15] Hasse, H., Ein Summierungsverfaheren für die Riemannsche Zeta-Reihe, *Math Z.* **32** (1930), 458–464.
[16] Knopp, K., Über das Eulersche Summierungsverfaheren, *Math Z.* **15** (1922), 226–253.
[17] Knopp, K., Über das Eulersche Summierungsverfaheren II, *Math Z.* **18** (1923), 125–156.
[18] Morimoto, M., *Introduction to Sato's Hyperfunctions,* Amer. Math. Soc., Providence, 1993.
[19] Sato, M., Theory of hyperfunctions I, *J. Fac. Sci. Univ. Tokyo Ser. I* **8** (1959), 139–193.
[20] Sato M., Theory of hyperfunctions II, *J. Fac. Sci. Univ. Tokyo Ser. I* **9** (1960), 387–437.
[21] Sondow, J., Analytic continuation of Riemann's zeta function and values at negative integers via Euler's transformation of series, *Proc. Amer. Math. Soc.* **120** (1994), 421–424.
[22] Vindas, J., and Estrada, R., Distributional Point Values and Convergence of Fourier Series and Integrals, *J. Fourier. Anal. Appl.* **13** (2007), 551–576.
[23] Walter, G., Pointwise convergence of distribution expansions, *Studia Math.* **26** (1966), 143–154.

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