Structural Theorems for Quasiasymptotics of Distributions at the Origin

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Key words Slowly varying functions, quasiasymptotics of distributions, almost homogeneous functions.
MSC (2000) 46F 05, 46F 99, 41A 60.

An open question concerning the quasiasymptotic behavior of distributions at the origin is solved. The question is the following: Suppose that a tempered distribution has quasiasymptotic at the origin in $\mathcal{S}'(\mathbb{R})$, then the tempered distribution has quasiasymptotic in $\mathcal{D}'(\mathbb{R})$, does the converse implication hold? The second purpose of this article is to give complete structural theorems for quasiasymptotics at the origin. For this purpose, asymptotically homogeneous functions with respect to slowly varying functions are introduced and analyzed.

1 Introduction and Preliminaries.

The purpose of this introductory section is to fix the notation and state the problems to be considered in this article.

The Schwartz spaces of test functions and distributions on the real line $\mathbb{R}$ are denoted by $\mathcal{D}$ and $\mathcal{D}'$, respectively; the spaces of rapidly decreasing functions and its dual, the space of tempered distributions, are denoted by $\mathcal{S}$ and $\mathcal{S}'$. We refer the reader to [9] for the properties of these spaces. A real-valued measurable function L defined in some interval of the form $(0, A)$, $A > 0$, is said to be *slowly varying function at the origin* if L is positive near 0 and

$$\lim_{x\to 0^+} \frac{L(ax)}{L(x)} = 1, \tag{1}$$

for any $a > 0$. In the same way one defines slowly varying functions at infinity. Since we will only be dealing with slowly varying functions at the origin, we shall refer to them just as slowly varying functions, suppressing the indication that it is at the origin. We refer to [10] and [1] for properties of such functions.

The main subject of this article are the so-called quasiasymptotic behaviors of distributions at the origin [7, 8, 13, 2, 3, 5]. Let L be slowly varying. We say that $f \in \mathcal{D}'$ has *quasiasymptotic behavior at the origin (has quasiasymptotic at* 0*) in* $\mathcal{D}'$ *with respect to* $\epsilon^\alpha L(\epsilon)$, $\alpha \in \mathbb{R}$, if for some $g \in \mathcal{D}'$ and every $\phi \in \mathcal{D}$,

$$\lim_{\epsilon\to 0^+} \left\langle \frac{f(\epsilon x)}{\epsilon^\alpha L(\epsilon)}, \phi(x) \right\rangle = \langle g(x), \phi(x)\rangle \,. \tag{2}$$

If (2) holds, we also say that f has *quasiasymptotic of order* α *at the origin with respect to the slowly varying function* L. In [2] such generalized functions are called asymptotically homogeneous generalized functions. Note

The first author gratefully acknowledges support by the Louisiana State Board of Regents grant LEQSF(2005-2007)-ENH-TR-21.
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that in the usual definition is assumed $g \neq 0$, but we extend the definition by allowing g to be 0. This is because all the results of this article are applicable to this situation as well.

We also express (2) by using the notation

$$f(\epsilon x) = \epsilon^{\alpha} L(\epsilon) g(x) + o(\epsilon^{\alpha} L(\epsilon)),\ \epsilon \to 0^{+} \text{ in } \mathcal{D}', \tag{3}$$

which should always be interpreted in the weak topology of $\mathcal{D}'$, i.e, in the sense of (2). If (3) holds for $f \in \mathcal{D}'$ and $f = h$ in a neighborhood of 0, $h \in \mathcal{D}'$, then (3) holds for h as well. This means that the distribution f in (3) may be defined only in a neighborhood of zero and therefore the quasiasymptotic is a local property.

It is easy to prove [5, p. 161] that (3) forces g to be homogeneous with the degree of homogeneity α. Let H be the Heaviside function, i.e, the characteristic function of the interval $(0, \infty)$. Since we know explicitly all the homogeneous distributions on the real line [5, p. 72], either g has the form

$$g(x) = C_{-} x_{-}^{\alpha} + C_{+} x_{+}^{\alpha},\ \text{if } \alpha \notin \{-1, -2, -3, \dots\}, \tag{4}$$

for some constants C_{-} and C_{+}, or

$$g(x) = \gamma \delta^{(k-1)}(x) + \beta x^{-k},\ \text{if } \alpha = -k \in \{-1, -2, -3, \dots\}, \tag{5}$$

for some constants γ and β. It is well-known that the distributions on the right hand side of (4) are defined for $\alpha \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$, as well. In fact, $x_{+}^{\alpha} = x^{\alpha} H(x)$ and $x_{-}^{\alpha} = |x|^{\alpha} H(-x)$ if $\Re\alpha > -1$. For the other values they are defined by analytic continuation and they have simple poles at the negative integers [5, p. 65]. The first term on the right hand side of (5) is the $(k-1)$-derivative of the well-known delta distribution. The second term is defined as the value of $x_{-}^{\alpha} + x_{+}^{\alpha}$ at $\alpha = -k$ if k is even, and as the value of $x_{+}^{\alpha} - x_{-}^{\alpha}$ at $\alpha = -k$ if k is odd. These choices are well defined since they eliminate the corresponding poles [5, p. 66].

Suppose now that $f \in \mathcal{S}'$. If we replace the space $\mathcal{D}$ by $\mathcal{S}$ in (2), then we say that f has quasiasymptotic at 0 with respect to $\epsilon^{\alpha} L(\epsilon)$ in $\mathcal{S}'$. It is obvious that if a tempered distribution has quasiasymptotic at 0 in $\mathcal{S}'$, then it will have it in $\mathcal{D}'$. It was shown by one of the authors in [8] that if L is bounded near the origin and $\alpha < 0$, $\alpha \notin \{-1, -2, -3, \dots\}$, then the converse is true. In the same article (see Remark 2 in [8]), the author proposed an open question: does the converse hold without these restrictions on L and α? The main scope of this article is to give a positive answer to this question in the general case. We should mention that, besides the case already cited from [8], a positive answer to this question is known for Łojasiewicz point values, for instance a proof can be found in [4].

If $\alpha = 0$, $L = 1$ and g is a constant distribution, then (3) reduces to the well-known notion of distributional point values due to Łojasiewicz [6]. If $g(x)$ is a linear combination of $H(x)$ and $H(-x)$, then the notion corresponds to that of distributional jump behavior [11, 4]. It was shown by Łojasiewicz that the existence of the point value at 0 is equivalent to the existence of constants $\gamma \in \mathbb{R}$, $m \in \mathbb{N}$ and a continuous function in a neighborhood $U \subset \mathbb{R}$ of 0, F, such that

$$F^{(m)} = f \ \text{ in } \ U \ \text{ and } \ F(x) = \gamma x^{m} + o(x^{m}),\ x \to 0.$$

Such types of characterizations are called structural theorems. A structural theorem for quasiasymptotics of order $\alpha > -1$ at the origin has been given in [8, Theorem 1] under the assumption that L is bounded. The second goal of this article is to give structural theorems for quasiasymptotics of all orders. It is remarked that these structural theorems hold without any restriction on L.

The key points for our structural theorems are the notions of asymptotically and associate asymptotically homogeneous functions. Our analysis of such functions is interesting in itself as a contribution to the asymptotic analysis, in general.

The plan for this article is as follows. In Section 2, we make some technical remarks about slowly varying functions while in Section 3 we introduce and study a new class of functions having interesting asymptotic properties. We call these functions *asymptotically homogeneous functions with respect to a slowly varying function*; they are the main tool in our analysis. In Section 4, we give structural theorems for quasiasymptotics where the order is not a negative integer. We also reduce the study of quasiasymptotics with negative integer orders to the case of order -1. Section 5 is dedicated to the structural theorem for the negative integer case. Finally, we show in Section 6 that if a tempered distribution has quasiasymptotic at 0 in $\mathcal{D}'$, then it will have quasiasymptotic at 0 in $\mathcal{S}'$, solving the open question posted in [8].

2 Remarks on Slowly varying functions

Along the proofs in the next sections, there are some arguments about slowly varying functions which repeat over and over again, so in order to avoid repetitions, we choose to have them here. Throughout this section L will be slowly varying. Our first obvious observation is that only the behavior of L near 0 plays a role in (2), and so we may put to L any behavior we want in intervals of the form $[A,\infty)$. Moreover, if $\tilde{L}$ is any measurable function which satisfies

$$\lim_{x\to 0^+}\frac{\tilde{L}(x)}{L(x)}=1,$$

we may replace L by $\tilde{L}$ in any statement about quasiasymptotics without losing generality in the statement. One of the basic results in the theory of slowly varying functions is a representation formula (see first two pages of [10]). Furthermore, the representation formula completely characterizes all the slowly varying functions; L is slowly varying at the origin if and only if there exist measurable functions u and w defined on some interval $(0,B]$, u being bounded and having a finite limit at 0 and w being continuous in $[0,B]$ with $w(0)=0$, such that

$$L(x)=exp\left(u(x)+\int_x^B\frac{w(t)}{t}dt\right),\ \ x\in(0,B].$$

This formula is important because it enables us to obtain some estimates on L. Since we are looking for suitable modifications of L, our first observation is that we can always assume that L is defined in the whole $(0,\infty)$ and L is always positive. This is shown by extending u and w to $(0,\infty)$ in any way we want. Moreover, given any fixed $\sigma>0$, by reducing B and then modifying u and w, we can assume, when it is convenient, that $B=1$, u is bounded in $(0,\infty)$ and $|w(x)|<\sigma$, $x\in(0,\infty)$. In particular this implies

$$\tilde{M}\min\left\{x^{-\sigma},x^{\sigma}\right\}<\frac{L(\epsilon x)}{L(\epsilon)}<M\max\left\{x^{-\sigma},x^{\sigma}\right\},\ x,\epsilon\in(0,\infty),$$

for some positive constants M and $\tilde{M}$. Under the assumption of the last estimate we can use Lebesgue's dominated convergence theorem in

$$\int_0^\infty\left(\frac{L\left(\epsilon x\right)}{L(\epsilon)}-1\right)\phi(x)dx,$$

for $\phi\in\mathcal{S}$, to deduce that

$$L(\epsilon x)H(x)=L(\epsilon)H(x)+o\left(L(\epsilon)\right),\ \epsilon\to 0^+,\ \text{in } \mathcal{S}'. \tag{6}$$

The reader should keep in mind (6) since from now on it will be implicitly used without any further reference, specially for differentiating asymptotic expressions in the future sections. We may also impose more conditions on w to obtain more reasonable assumptions on L. For example, the assumption $t^{-1}w(t)\in L^1[1,\infty)$ implies

$$\tilde{M}<L(x)<M,\ x>1,$$

for some positive constants $\tilde{M}$ and M. We finally comment a well-known fact [10], [1]: As soon as (1) holds for each $a>0$, it automatically holds uniformly for a in compact subsets of $(0,\infty)$.

3 Asymptotically Homogeneous Function

In this section the concept of asymptotically homogeneous functions with respect to a slowly varying function is introduced and studied.

Our analysis in the next sections is based on the properties of the parametric coefficients resulting after performing several integrations of the quasiasymptotic. Here we take a comprehensive approach, we single out the defining asymptotic properties of such coefficients and proceed to obtain their behaviors at 0. This idea has been previously applied in [5, p. 365] in the context of summability of Fourier series of distributions; more recently, in [11] and [12] to obtain interesting relations between Łojasiewicz point values and quasiasymptotics of order -1 at ∞, where a structural theorem is obtained by means of a pointwise Fourier inversion formula for Łojasiewicz point values. It is remarkable that such results strongly depend on asymptotic properties of functions which are called *asymptotically homogeneous functions of degree 0 at* ∞ [5]. They arise as the main coefficient of integration of the quasiasymptotic of order -1 with respect to $L=1$. We will generalize these ideas by extending the concept of asymptotically homogeneous functions. Our main motivation is given in the following proposition.

Proposition 3.1 *Let $f\in\mathcal{D}'$ have quasiasymptotic behavior at the origin*

$$f\left(\epsilon x\right)=L(\epsilon)g(\epsilon x)+o\left(\epsilon^{\alpha}L(\epsilon)\right),\ \epsilon\to 0^{+},\ \text{in } \mathcal{D}', \tag{7}$$

where L is a slowly varying function and g is a homogeneous distribution of order $\alpha\in\mathbb{R}$. Let $k\in\mathbb{N}$. Suppose that g admits a primitive of order k, $G_k\in\mathcal{D}'$, $G_k^{(k)}=g$, which is homogeneous of degree $k+\alpha$. Then, for any given F_k, a k-primitive of f in $\mathcal{D}'$, i.e, $F_k^{(k)}=f$, there exist functions $c_0,\dots,c_{k-1}$, continuous in $(0,\infty)$ such that

$$F_k\left(\epsilon x\right)=L(\epsilon)G_k(\epsilon x)+\sum_{j=0}^{k-1}c_j(\epsilon)\frac{(\epsilon x)^{k-1-j}}{(k-1-j)!}+o\left(\epsilon^{\alpha+k}L(\epsilon)\right),\ \epsilon\to 0^{+},\ \text{in } \mathcal{D}'. \tag{8}$$

Furthermore, for each $a>0$, they satisfy the following asymptotic property, $j\in\{0,\dots,k-1\}$,

$$c_j(a\epsilon)=c_j(\epsilon)+o\left(\epsilon^{\alpha+j+1}L(\epsilon)\right),\ \epsilon\to 0^{+}. \tag{9}$$

P r o o f. Recall, any $\phi\in\mathcal{D}$ is of the form

$$\phi=C_\phi\phi_0+\theta',\ \text{where } C_\phi=\int_{-\infty}^{\infty}\phi(t)dt,\ \theta\in\mathcal{D} \tag{10}$$

and $\phi_0\in\mathcal{D}$ is chosen so that $\int_{-\infty}^{\infty}\phi_0(t)dt=1$.

The evaluations of primitives F_1 of f and G_1 of g on ϕ are given by

$$\langle F_1,\phi\rangle=C_\phi\langle F_1,\phi_0\rangle-\langle f,\theta\rangle\ \text{ and }\ \langle G_1,\phi\rangle=C_\phi\langle G_1,\phi_0\rangle-\langle g,\theta\rangle.$$

This implies

$$\left\langle\frac{F_1(\epsilon x)}{\epsilon^{\alpha+1}L(\epsilon)},\phi(x)\right\rangle=C_\phi\left\langle\frac{F_1(\epsilon x)}{\epsilon^{\alpha+1}L(\epsilon)},\phi_0(x)\right\rangle-\left\langle\frac{f(\epsilon x)}{\epsilon^{\alpha}L(\epsilon)},\theta(x)\right\rangle, \tag{11}$$

and

$$\left\langle\frac{G_1(\epsilon x)}{\epsilon^{\alpha+1}L(\epsilon)},\phi(x)\right\rangle=C_\phi\left\langle\frac{G_1(\epsilon x)}{\epsilon^{\alpha+1}L(\epsilon)},\phi_0(x)\right\rangle-\left\langle\frac{g(\epsilon x)}{\epsilon^{\alpha}L(\epsilon)},\theta(x)\right\rangle. \tag{12}$$

With $c_0(\epsilon)=\langle(F_1-G_1)(\epsilon x),\phi_0(x)\rangle,\ \epsilon\in(0,\infty)$, from (7), it follows

$$F_1\left(\epsilon x\right)=L(\epsilon)G_1(\epsilon x)+c_0(\epsilon)+o\left(\epsilon^{\alpha+1}L(\epsilon)\right),\ \epsilon\to 0^{+},\ \text{in } \mathcal{D}'. \tag{13}$$

So relation (8) follows from (13) and (7), by induction.

Thus we shall concentrate in showing (9). We set $F_m=F_k^{(k-m)}$ and $G_m=G_k^{(k-m)}, m\in\{1,\dots,k\}$. By differentiating relation (8) $(k-m)$-times, it follows

$$F_m(\epsilon x)=L(\epsilon)G_m(\epsilon x)+\sum_{j=0}^{m-1}c_j(\epsilon)\frac{(\epsilon x)^{m-1-j}}{(m-1-j)!}+o\left(\epsilon^{\alpha+m}L(\epsilon)\right),\ \epsilon\to 0^{+},\ \text{in } \mathcal{D}'. \tag{14}$$

Choose $\phi \in \mathcal{D}$ such that $\int_{-\infty}^{\infty} \phi(x) x^j dx = 0$ for $j = 1, \ldots, m-1$, and $\int_{-\infty}^{\infty} \phi(x) dx = 1$, then evaluating (14) at ϕ, it follows

$$c_{m-1}(a\epsilon) + L(a\epsilon) \left\langle G_m(a\epsilon x), \phi(x) \right\rangle + o\left(\epsilon^{\alpha+m} L(\epsilon)\right)$$

$$= \left\langle F_m(a\epsilon x), \phi(x) \right\rangle = \frac{1}{a} \left\langle F_m(\epsilon x), \phi\left(\frac{x}{a}\right) \right\rangle$$

$$= c_{m-1}(\epsilon) + L(\epsilon) \left\langle G_m(a\epsilon x), \phi(x) \right\rangle + o\left(\epsilon^{\alpha+m} L(\epsilon)\right),\ \epsilon \to 0^+$$

and so, with $j = m-1 \in \{0, \ldots, k-1\}$, for each $a > 0$,

$$c_j(a\epsilon) = c_j(\epsilon) + o\left(\epsilon^{\alpha+j+1} L(\epsilon)\right),\ \epsilon \to 0^+.$$

□

Remark 3.2 If supp $f \subset [0, \infty)$, then we can choose ϕ_0 in (10) to be supported by $(-\infty, 0]$ and take in (11) and (12) $\langle F_1, \phi_0 \rangle = 0$ and $\langle G_1, \phi_0 \rangle = 0$. We have supp $F_1 \subset [0, \infty)$ and supp $G_1 \subset [0, \infty)$. So we can prove by induction that $c_j = 0, j = 0, ..., k-1$ in (9) and that F_k and G_k are supported by $[0, \infty)$. We will come back to this in Remark 5.5.

In the rest of this section, we study continuous functions having the same asymptotic properties at the origin as the coefficients of integration of (7), i.e., functions defined in some interval of the form $(0, A)$, $0 < A$, having the behavior

$$c(ax) = c(x) + o\left(x^\alpha L(x)\right),\ x \to 0^+,$$

for each $a > 0$. Consider $b(x) = c(x) x^{-\alpha}, x \in (0, A)$. Then, for each $a > 0$,

$$b(ax) = a^{-\alpha} b(x) + o(L(x)),\ x \to 0^+.$$

Definition 3.3 A function b is said to be *asymptotically homogeneous of degree α at 0 with respect to the slowly varying function L*, if it is continuous and defined in some interval $(0, A)$, $A > 0$, and for each $a > 0$,

$$b(ax) = a^\alpha b(x) + o(L(x)),\ x \to 0^+.$$

We shall study these functions in details. We first study asymptotically homogeneous functions of positive degree.

Theorem 3.4 *Suppose that b is asymptotically homogeneous of degree $\alpha > 0$ with respect to the slowly varying function L. Then*

$$b(x) = o(L(x)),\ x \to 0^+.$$

Proof. Let $0 < \eta$. We keep $\eta < 2^\alpha - 1$. Let $x_0 > 0$ such that

$$\left| b\left(\frac{x}{2}\right) - 2^{-\alpha} b(x) \right| \le \eta L(x) \ \text{ and } \ |L(2x) - L(x)| \le \eta L(x),\ 0 < x < x_0. \tag{15}$$

Let $M = \max\left\{ \frac{|b(x)|}{L(x)} : \frac{1}{2} x_0 \le x \le x_0 \right\}$ and $x \in [x_0/2, x_0]$. From (15) it follows

$$\left| \frac{b(x/2^n)}{L(x/2^n)} \right| \le 2^{-\alpha n} \frac{|b(x)|}{L(x/2^n)} + \eta \sum_{j=0}^{n-1} 2^{-\alpha(n-1-j)} \frac{L(x/2^j)}{L(x/2^n)}.$$

Thus, with $t = x/2^n$, and $t \in [x_0/2^{n+1}, x_0/2^n]$,

$$\left| \frac{b(t)}{L(t)} \right| \le 2^{-n\alpha} M \frac{L(2^n t)}{L(t)} + \eta \sum_{j=0}^{n-1} 2^{-j\alpha} \frac{L(2^{j+1} t)}{L(t)}.$$

By this and

$$L(2^{j+1}t)/L(2^j t) \leq (1+\eta),\ j=0,\dots,n-1,$$

we have that if $t \in \left[2^{-(n+1)}x_0, 2^{-n}x_0\right]$, then

$$\left|\frac{b(t)}{L(t)}\right| \leq M\left(\frac{1+\eta}{2^\alpha}\right)^n + \eta(1+\eta)\sum_{j=0}^{\infty}\left(\frac{1+\eta}{2^\alpha}\right)^j = M\left(\frac{1+\eta}{2^\alpha}\right)^n + \eta(1+\eta)\frac{2^\alpha}{2^\alpha-1-\eta}.$$

Let us prove that for every $\varepsilon > 0$ there exists a positive δ such that $|b(t)/L(t)| < \varepsilon,\ t \in (0,\delta)$. First, we have to take so small η such that

$$\eta(1+\eta)\frac{2^\alpha}{2^\alpha-1-\eta} < \frac{\varepsilon}{2}$$

and $n_0 \in \mathbb{N}$ such that

$$M\left(\frac{1+\eta}{2^\alpha}\right)^n < \frac{\varepsilon}{2},\ \ n \geq n_0.$$

This implies that we have to take $\delta = x_0/2^{n_0}$. This completes the proof. □

In order to obtain further progress, we need the following lemma.

Lemma 3.5 *Let b be an asymptotically homogeneous function of degree α at the origin with respect to L defined in $(0,A)$. Then, the relation*

$$b(ax) = a^\alpha b(x) + o(L(x)),\ x \to 0^+,$$

holds uniformly for a in compact subsets of $(0,A)$.

P r o o f. We rather work with $f(x) = e^{\alpha x} b\left(e^{-x}\right)$ and $g(x) = L\left(e^{-x}\right)$, we may assume that $A=1$ and hence that f and g are defined in $[0,\infty)$. By using a linear transformation between an arbitrary compact subinterval of $[0,\infty)$ and $[0,1]$, it is enough to show that

$$f(h+x) - f(x) = o(e^{\alpha x}g(x)),\ x \to \infty, \tag{16}$$

uniformly for $h \in [0,1]$. Suppose that (16) is false. Then, there exist $0 < \varepsilon < 1$, a sequence $\{h_m\}_{m=1}^{\infty} \in [0,1]^{\mathbb{N}}$ and an increasing sequence of real numbers $\{x_m\}_{m=1}^{\infty}$, $x_m \to \infty,\ m \to \infty$, such that

$$\left|f\left(h_m + x_m\right) - f\left(x_m\right)\right| \geq \varepsilon e^{\alpha x_m} g\left(x_m\right),\ m \in \mathbb{N}. \tag{17}$$

Define, for $n \in \mathbb{N}$,

$$A_n = \left\{h \in [0,2] : \left|f\left(h + x_m\right) - f\left(x_m\right)\right| < \frac{\varepsilon}{3} e^{\alpha x_m} g\left(x_m\right), m \geq n\right\},$$

$$B_n = \left\{h \in [0,2] : \left|f\left(h + x_m + h_m\right) - f\left(h_m + x_m\right)\right| < \frac{\varepsilon}{3} e^{\alpha x_m} g\left(x_m + h_m\right), m \geq n\right\}.$$

Note that

$$[0,2] = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n,$$

so we can select N such that $\mu(A_n)$, $\mu(B_n) > \frac{3}{2}$ (here $\mu(\cdot)$ stands for Lebesgue measure), for all $n \geq N$. For each $n \in \mathbb{N}$, put $C_n = \{h_n\} + B_n$. Then, we have $\mu(C_n) > \frac{3}{2}, n \geq N$, and $C_n, A_n \subseteq [0,3]$. It follows that $A_n \bigcap C_n \neq \emptyset, n > N$. For each $n \geq N$, select $u_n \in A_n \bigcap C_n$. In particular, we have $u_n - h_n \in B_n$, and hence,

$$\left|f\left(u_n + x_n\right) - f\left(x_n\right)\right| < \frac{\varepsilon}{3} e^{\alpha x_n} g\left(x_n\right),$$

$$|f(u_n + x_n) - f(x_n + h_n)| < \frac{\varepsilon}{3} e^{\alpha x_n} g(x_n + h_n)$$

which implies that for all $n \geq N$,

$$|f(x_n + h_n) - f(x_n)| < \frac{\varepsilon}{3} e^{\alpha x_n} (g(x_n) + g(x_n + h_n)) .$$

Using that $g(x+h) - g(x) = o(g(x))$, $x \to \infty$, uniformly for h on compact subsets of $(0, \infty)$, we have that for all n sufficiently large, $g(x_n + h_n) \leq 2g(x_n)$, which implies that for n big enough

$$|f(x_n + h_n) - f(x_n)| < \varepsilon e^{\alpha x_n} g(x_n) ,$$

in contradiction to (17), Therefore, (16) must hold uniformly for $h \in [0, 1]$. □

The next theorem explores the asymptotic behavior of asymptotically homogeneous functions of negative degree.

Theorem 3.6 *Suppose that b is asymptotically homogeneous of degree $-\alpha < 0$ at 0 with respect to the slowly varying function L. Then, there exists a number γ such that,*

$$b(x) = \frac{\gamma}{x^\alpha} + o(L(x)) , \ x \to 0^+. \tag{18}$$

In particular, we have that for each $\sigma > 0$,

$$b(x) = \frac{\gamma}{x^\alpha} + o\left(\frac{1}{x^\sigma}\right) , \ x \to 0^+.$$

P r o o f. Again, we rather work with $f(x) = e^{-\alpha x} b(e^{-x})$ and $g(x) = L(e^{-x})$. Then f satisfies

$$f(h + x) - f(x) = o\left(e^{-\alpha x} g(x)\right) , \ x \to \infty,$$

uniformly for $h \in [0, 1]$. Given $\varepsilon > 0$, we can find $x_0 > 0$ such that for all $x > x_0$ and $h \in [0, 1]$,

$$|f(x + h) - f(x)| \leq \varepsilon e^{-\alpha x} g(x) \ \text{ and } \ |g(h + x) - g(x)| \leq \left(e^{\frac{\alpha}{2}} - 1\right) g(x).$$

So we have that

$$\begin{aligned}
&|f(h + n + x) - f(x)| \\
&\leq |f(h + n + x) - f(n + x)| + |f(n + x) - f(x)| \\
&\leq \varepsilon e^{-\alpha(n+x)} g(n + x) + \sum_{j=0}^{n-1} |f(j + 1 + x) - f(j + x)| \\
&\leq \varepsilon e^{-\alpha x} \sum_{j=0}^{n} e^{-\alpha j} g(j + x) \\
&\leq \varepsilon e^{-\alpha x} g(x) \frac{e^{\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}} - 1},
\end{aligned}$$

where the last estimate follows from $g(x + j) \leq g(x) e^{\alpha j/2}$. Since $g(x) = o\left(e^{\alpha x}\right)$ as $x \to \infty$, it shows that there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{x \to \infty} f(x) = \gamma.$$

Moreover, the estimate shows that

$$f(x) = \gamma + o\left(e^{-\alpha x} g(x)\right) , \ x \to \infty,$$

thus, changing the variables back, we have obtained,

$$b(x) = \frac{\gamma}{x^\alpha} + o(L(x)) , \ x \to 0^+.$$

□

We remark that (18) trivially implies that b is asymptotically homogeneous of degree $-\alpha$ with respect to L.

We now focus our attention to asymptotically homogeneous function of degree 0. Using the properties of slowly varying functions we can roughly estimate their growth at approaching 0. For instance, we easily obtain the following estimate.

Proposition 3.7 *Let b be asymptotically homogeneous of degree 0 at the origin with respect to the slowly varying function L. If $\sigma > 0$ then,*

$$b(x) = o\left(\frac{1}{x^{\sigma}}\right),\ x \to 0^{+}.$$

In particular, $b(x)\left(L(x)\right)^{-1}$ is integrable near the origin.

P r o o f. We know that $L(x) = o\left(x^{-\sigma}\right)$, $x \to 0^{+}$. Then for each $a > 0$, $b(ax) = b(x) + o\left(x^{-\sigma}\right)$, $x \to 0^{+}$ and this implies that $x^{\sigma} b(x)$ is asymptotically homogeneous of degree σ with respect to the constant function 1. From Theorem 3.4, it follows that $b(x) = o\left(x^{-\sigma}\right)$, $x \to 0^{+}$. □

The next theorem will be very important in the next section.

Theorem 3.8 *Let b be asymptotically homogeneous of degree zero at the origin with respect to the slowly varying function L. Suppose that b is defined in $(0, A]$. Then*

$$b(\epsilon x)H(x) = b(\epsilon)H(x) + o(L(\epsilon))\ as\ \epsilon \to 0^{+},\ \text{in } \mathcal{D}', \tag{19}$$

where H is the Heaviside function.

P r o o f. Since for any $\phi \in \mathcal{D}$ there exists $\epsilon_{\phi} < 1$ such that

$$\langle b(\epsilon x), \phi(x)\rangle = \int_{0}^{\frac{1}{\epsilon}} b(\epsilon x)\phi(x)dx,\ \epsilon < \epsilon_{\phi},$$

we can assume that $A = 1$. Our aim is to show that for some $\epsilon_0 < 1$,

$$\frac{b(\epsilon x) - b(\epsilon)}{L(\epsilon)},\ x \in (0, \infty),\ \epsilon < \epsilon_0$$

is dominated by an integrable function in $(0, 1)$ for the use of the Lebesgue theorem. For this goal, we assume that L satisfies the following estimate,

$$\frac{L(\epsilon x)}{L(\epsilon)} \leq M \max\left\{x^{-\frac{1}{2}}, x^{\frac{1}{2}}\right\},\ x, \epsilon \in (0, \infty). \tag{20}$$

By Lemma 3.5, there exists $0 < \epsilon_0 < 1$ such that

$$|b(\epsilon x) - b(\epsilon)| < L(\epsilon),\ x \in [1/2, 2],\ \epsilon < \epsilon_0.$$

We keep $\epsilon < \epsilon_0$ and $x \in \left[2^{-n-1}, 2^{-n}\right]$. Then

$$|b(\epsilon x) - b(\epsilon)| \leq |b(2\epsilon x) - b((2x\epsilon)/2)| + |b(2\epsilon x) - b(\epsilon)|$$

$$\leq L(2\epsilon x) + |b(2\epsilon x) - b(\epsilon)| \leq \sum_{i=1}^{n} L\left(2^{i}\epsilon x\right) + L(\epsilon)$$

$$\leq \sum_{i=1}^{n} (2^{i}x)^{-1/2} L(\epsilon) + L(\epsilon).$$

It follows from (20) that if $\epsilon < \epsilon_0$ and $x \leq 1$, then

$$\left|\frac{b(\epsilon x) - b(\epsilon)}{L(\epsilon)}\right| \leq M_1 x^{-\frac{1}{2}} + 1,$$

where $M_1 = M(\sqrt{2}+1)$. Similarly, one gets the estimate, for $\epsilon < \epsilon_0$ and $x \geq 1$,

$$\left| \frac{b(\epsilon x) - b(\epsilon)}{L(\epsilon)} \right| \leq M_1 x^{\frac{1}{2}} + 1,$$

Therefore we can apply Lebesgue's dominated convergence theorem to deduce (19). □

Remark 3.9 We want to remark that in Definition 3.3 we may replace the word continuous by measurable and Theorem 3.4, Lemma 3.5, Theorem 3.6 and Proposition 3.7 would be still true. Indeed, the only place where we used the continuity was in the proof of Theorem 3.4 to deduce that the function was locally bounded in some smaller interval of the form $(0, B]$, however, Lemma 3.5 implies that if one only assumes measurability, then the function satisfies this condition. Note that Theorem 3.8 is also true if we replace the continuity of b by the hypothesis locally bounded in $(0, A]$.

4 Structural Theorems for some cases

In this section we apply the results of Section 3 to obtain the structural theorems for the quasiasymptotics in several cases. We begin with the following proposition.

Proposition 4.1 *Let $f \in \mathcal{D}'$ and let L be a slowly varying function. Suppose that*

$$f(\epsilon x) = C_- L(\epsilon) \frac{(\epsilon x)_-^\alpha}{\Gamma(\alpha+1)} + C_+ L(\epsilon) \frac{(\epsilon x)_+^\alpha}{\Gamma(\alpha+1)} + o\left(\epsilon^\alpha L(\epsilon)\right),\ \epsilon \to 0^+,\ \text{in } \mathcal{D}',$$

where $\alpha \notin \{-1, -2, \dots\}$. Given $k \in \mathbb{N}$ and F_k, a k-primitive of f, there exist $\gamma_0, \dots, \gamma_{k-1}$ such that in the sense of convergence in $\mathcal{D}'$,

$$F_k(\epsilon x) = \sum_{j=0}^{k-1} \gamma_j \frac{(\epsilon x)^j}{j!} + C_- L(\epsilon) \frac{(\epsilon x)_-^{\alpha+k}}{\Gamma(\alpha+k+1)} + C_+ L(\epsilon) \frac{(\epsilon x)_+^{\alpha+k}}{\Gamma(\alpha+k+1)} + o\left(\epsilon^{\alpha+k} L(\epsilon)\right) \tag{21}$$

as $\epsilon \to 0^+$.

P r o o f. Proposition 3.1 implies that there are continuous functions on $(0, \infty)$, $c_0, \dots, c_{k-1}$, such that

$$F_k(\epsilon x) = C_- L(\epsilon) \frac{(\epsilon x)_-^{\alpha+k}}{\Gamma(\alpha+k+1)} + C_+ L(\epsilon) \frac{(\epsilon x)_+^{\alpha+k}}{\Gamma(\alpha+k+1)} + \sum_{j=0}^{k-1} c_{k-1-j}(\epsilon) \frac{(\epsilon x)^j}{j!} + o\left(\epsilon^{\alpha+k} L(\epsilon)\right)$$

as $\epsilon \to 0^+$, in $\mathcal{D}'$, and for each $a > 0$,

$$c_{k-1-j}(a\epsilon) = c_{k-1-j}(\epsilon) + o\left(\epsilon^{k-j+\alpha} L(\epsilon)\right),\ \epsilon \to 0^+.$$

For those j's such that $k + \alpha < j$, we can apply Theorem 3.4 to $\epsilon^{j-k-\alpha} c_{k-1-j}(\epsilon)$, to obtain that

$$\epsilon^j c_{k-1-j}(\epsilon) = o\left(\epsilon^{\alpha+k} L(\epsilon)\right),\ \epsilon \to 0^+.$$

So we set $\gamma_j = 0$ for $k + \alpha < j$. For the rest of j's, $j < k + \alpha$, we apply Theorem 3.6 to find γ_j such that $c_{k-1-j}(\epsilon) = \gamma_j + o(\epsilon^{k-j+\alpha} L(\epsilon))$ as $\epsilon \to 0^+$. □

Proposition 4.1 enables us to characterize the quasiasymptotics in the cases where the order is not a negative integer.

Theorem 4.2 *Let $f \in \mathcal{D}'$ have quasiasymptotic behavior at* 0 *in $\mathcal{D}'$,*

$$f(\epsilon x) = C_- L(\epsilon) \frac{(\epsilon x)_-^\alpha}{\Gamma(\alpha+1)} + C_+ L(\epsilon) \frac{(\epsilon x)_+^\alpha}{\Gamma(\alpha+1)} + o\left(\epsilon^\alpha L(\epsilon)\right),\ \epsilon \to 0^+. \tag{22}$$

If $\alpha \notin \{-1,-2,\dots\}$, then there exist an integer m, a m-primitive F of f, i.e. $F^{(m)} = f$, such that F is continuous in $[-1,1]$ and

$$\lim_{x\to\pm 0} \frac{\Gamma(\alpha+m+1)F(x)}{|x|^{\alpha+m} L(|x|)} = C_{\pm}. \tag{23}$$

Conversely, if these conditions hold, then (by differentiation) (22) follows.

P r o o f. It follows from the definition of convergence in $\mathcal{D}'$ that there is $m \in \mathbb{N}$ such that any m-primitive of f is continuous in $[-1,1]$ and (21) holds uniformly for $x \in [-1,1]$. Pick a specific m-primitive of f, F_m, then from Proposition 4.1, and the argument just discussed, there is a polynomial p of degree at most $m-1$, such that

$$F_m(\epsilon x) = p(\epsilon x) + C_- L(\epsilon)\frac{(\epsilon x)_-^{\alpha+m}}{\Gamma(\alpha+m+1)} + C_+ L(\epsilon)\frac{(\epsilon x)_+^{\alpha+m}}{\Gamma(\alpha+m+1)} + o\left(\epsilon^{\alpha+m}L(\epsilon)\right)$$

as $\epsilon \to 0^+$, uniformly for $x \in [-1,1]$. Then setting $F = F_m - p$, $x = 1, -1$ and replacing ϵ by x, relation (23) follows at once. □

We now turn our attention to the case $\alpha \in \{-1,-2,\dots\}$. Since $\epsilon p_{k-1}(\epsilon x)/L(\epsilon) \to 0,\ \epsilon \to 0^+$, we have the following proposition.

Proposition 4.3 *Let $f \in \mathcal{D}'$ have quasiasymptotic behavior at 0,*

$$f(\epsilon x) = L(\epsilon)g(\epsilon x) + o\left(\epsilon^{-k}L(\epsilon)\right),\ \epsilon \to 0^+,\ \text{in } \mathcal{D}', \tag{24}$$

where $k \in \{-2,-3,\dots\}$ and g be a homogeneous distribution of degree $-k$. Let G be a homogeneous distribution of degree -1 such that $G^{(k-1)} = g$. Then for any $(k-1)$-primitive of f, F_{k-1}, we have that

$$F_{k-1}(\epsilon x) = L(\epsilon)G(\epsilon x) + o\left(\epsilon^{-1}L(\epsilon)\right),\ \epsilon \to 0^+,\ \text{in } \mathcal{D}'. \tag{25}$$

Conversely, relation (25) implies (24).

P r o o f. It follows directly from Proposition 3.1 and Theorem 3.4. □

Proposition 4.3 reduces our study to the case of quasiasymptotics of order -1 for which we shall proceed to study a particular case; we postpone the general case until the next section.

Proposition 4.4 *Let $f \in \mathcal{D}'$ have quasiasymptotic behavior at 0 in $\mathcal{D}'$,*

$$f(\epsilon x) = \gamma L(\epsilon)\delta(\epsilon x) + o\left(\epsilon^{-1}L(\epsilon)\right),\ \epsilon \to 0^+. \tag{26}$$

For each $k \in \mathbb{N}$, let F_k be a k-primitive of f. Then, there exists c, an asymptotically homogeneous function of degree 0 with respect to L, such that for each k, there is a polynomial p_{k-1} of degree at most $k-1$ such that

$$F_{k+1}(\epsilon x) = p_{k-1}(\epsilon x) + \gamma L(\epsilon)\frac{(\epsilon x)^k}{2k!}\,\mathrm{sgn}x + c(\epsilon)\frac{(\epsilon x)^k}{k!} + o\left(\epsilon^k L(\epsilon)\right),\ \epsilon \to 0^+,\ \text{in } \mathcal{D}'. \tag{27}$$

Moreover, there exists $n_0 \in \mathbb{N}$ such that for $k \geq n_0$, any F_{k+1} is continuous in $[-1,1]$ and (27) holds uniformly for $x \in [-1,1]$. In particular, by taking $x = 1, -1$, one has for $k \geq n_0$

$$F_{k+1}(x) = p_{k-1}(x) + \gamma L(|x|)\frac{x^k}{2k!}\,\mathrm{sgn}x + c\left(|x|\right)\frac{x^k}{k!} + o\left(|x|^k L(|x|)\right),\ x \to 0. \tag{28}$$

Conversely, if (28) holds, then (27) is satisfied in $\mathcal{D}'$, and (by differentiation) the behavior (26) follows.

P r o o f. Relation (27) follows from Proposition 3.1 and Theorem 3.6. The existence of n_0 follows from the definition of convergence in $\mathcal{D}'$. The converse is shown by applying Theorem 3.8 and differentiating $(k+1)$-times. □

Proposition 4.4 can be considered as a structural theorem. We shall give the general version in the next section, we shall present an alternative reformulation as well.

5 Quasiasymptotic of negative integer order.

The first part of this section will be dedicated to the quasiasymptotic

$$f(\epsilon x)=\gamma\epsilon^{-1}L(\epsilon)\delta(x)+\beta\epsilon^{-1}L(\epsilon)x^{-1}+o\left(\epsilon^{-1}L(\epsilon)\right),\ \epsilon\to 0^{+},\ \text{in }\mathcal{D}'. \tag{29}$$

For each $k\in\mathbb{N}$, select a k-primitive of f, say F_k, such that $F_k'=F_{k-1}$. We shall study, as we have been doing, the coefficients of the integration of (29). We should introduce some notation that will be needed. In the following for all $k\in\mathbb{N}$ we denote by l_k the primitive of $\log|x|$ with the property that $l_k(0)=0$ and $l_k'=l_{k-1}$. We have an explicit formula for them:

$$l_k(x)=\frac{x^k}{k!}\log|x|-\frac{x^k}{k!}\sum_{j=1}^{k}\frac{1}{j},\ x\in\mathbb{R},$$

which can be easily verified by direct differentiation. They satisfy

$$l_k(ax)=a^k l_k(x)+\frac{(ax)^k}{k!}\log a,\ a>0. \tag{30}$$

We now proceed to integrate (29) once, so we obtain

$$F_1(\epsilon x)=c_0(\epsilon)+\frac{\gamma}{2}L(\epsilon)\operatorname{sgn}x+\beta L(\epsilon)\log|x|+o(L(\epsilon)),\ \epsilon\to 0^{+},\ \text{in }\mathcal{D}'. \tag{31}$$

Now, using the standard trick of evaluating in $\phi\in\mathcal{D}$ with the property $\int_{-\infty}^{\infty}\phi(x)dx=1$, we obtain that

$$\begin{aligned}
&c_0(\epsilon a)+\frac{\gamma}{2}L(\epsilon)\int_{-\infty}^{\infty}\operatorname{sgn}x\phi(x)dx+\beta L(\epsilon a)\int_{-\infty}^{\infty}\log|x|\,\phi(x)dx+o(L(\epsilon))\\
&=\langle F_1(\epsilon a x),\phi(x)\rangle=\frac{1}{a}\left\langle F_1(\epsilon x),\phi\left(\frac{x}{a}\right)\right\rangle\\
&=c_0(\epsilon)+\frac{\gamma}{2}L(\epsilon)\int_{-\infty}^{\infty}\operatorname{sgn}x\phi(x)dx+\beta L(\epsilon)\int_{-\infty}^{\infty}\log|ax|\,\phi(x)dx+o(L(\epsilon)),\ \epsilon\to 0^{+},
\end{aligned}$$

for each $a>0$. So, we see that c_0 satisfies that for each $a>0$,

$$c_0(\epsilon a)=c_0(\epsilon)+\beta\log a L(\epsilon)+o(L(\epsilon)),\ \epsilon\to 0^{+}. \tag{32}$$

We call a continuous function defined in some interval of the form $(0,A)$, $A>0$, and satisfying (32), for some β, *associate asymptotically homogeneous of degree* 0 *with respect to* L. Using the same method of Lemma 3.5 and Theorem 3.8, one shows the following important lemma.

Lemma 5.1 *Assume that* b *is a continuous function defined in an interval of the form* $(0,A]$ *and that it satisfies (32). Then (32) holds uniformly on compact subsets of* $(0,A]$. *Additionally, we have*

$$b(|\epsilon x|)=b(\epsilon)+\beta L(\epsilon)\log|x|+o(L(\epsilon)),\ \epsilon\to 0^{+},\ \text{in }\ \mathcal{D}'.$$

Further integration of (31) gives,

$$F_{k+1}(\epsilon x)=\sum_{j=0}^{k}c_j(\epsilon)\frac{(\epsilon x)^{k-j}}{(k-j)!}+\gamma L(\epsilon)\operatorname{sgn}x\frac{(\epsilon x)^k}{2k!}+\beta L(\epsilon)\epsilon^k l^k(x)+o\left(\epsilon^k L(\epsilon)\right)$$

as $\epsilon\to 0^{+}$, in $\mathcal{D}'$; where the c_j's, for $j>0$, satisfy to be continuous on $(0,\infty)$ and for each $a>0$,

$$c_j(a\epsilon)=c_j(\epsilon)+o\left(\epsilon^j L(\epsilon)\right),\epsilon\to 0^{+}. \tag{33}$$

The proof of this assertion can be given as in Proposition 3.1. We have to choose $\phi\in\mathcal{D}$ so that, $\int_{-\infty}^{\infty}\phi(x)x^j dx=0$ for $j=1,\dots,k$ and $\int_{-\infty}^{\infty}\phi(x)dx=1$, then to evaluate $F_{k+1}(a\epsilon x)$ at ϕ taking into account (30). With this, one obtains (33).

So using the results of Section 3 and Lemma 5.1, one obtains the structural theorem for quasiasymptotics of order -1.

Theorem 5.2 *Let $f \in \mathcal{D}'$ have quasiasymptotic at 0 of the form (29). For each $k \in \mathbb{N}$, choose an arbitrary k-primitive F_k of f. Then there exists an associate asymptotically homogeneous function, c, satisfying (32) such that for any k, there is a polynomial p_{k-1} of degree at most $k-1$ for which F_{k+1} satisfies,*

$$F_{k+1}(\epsilon x) = p_{k-1}(\epsilon x) + c(\epsilon)\frac{(\epsilon x)^k}{k!} + \gamma L(\epsilon)\frac{(\epsilon x)^k}{2k!}\,\mathrm{sgn}x + \beta L(\epsilon)\epsilon^k l_k(x) + o\left(\epsilon^k L(\epsilon)\right) \tag{34}$$

as $\epsilon \to 0^+$, in the sense of convergence in $\mathcal{D}'$. Moreover, there exists $n_0 \in \mathbb{N}$, such that for all $k \geq n_0$, F_{k+1} is continuous in $[-1,1]$ and (34) holds uniformly for $x \in [-1,1]$. In particular for $k \geq n_0$, one has,

$$F_{k+1}(x) = p_{k-1}(x) + c\left(|x|\right)\frac{x^k}{k!} + \gamma\frac{x^k}{2k!}L\left(|x|\right)\,\mathrm{sgn}x - \beta L\left(|x|\right)\frac{x^k}{k!}\sum_{j=1}^{k}\frac{1}{j} + o\left(|x|^k L\left(|x|\right)\right) \tag{35}$$

as $x \to 0$, in the ordinary sense. Conversely, it follows from Lemma 5.1 that relation (35) implies (34), and (by differentiation) (29) follows.

Theorem 5.2 is a structural theorem, but we shall give a version free of c.

Theorem 5.3 *Let $f \in \mathcal{D}'$. Then f has quasiasymptotic at 0 of the form (29) if and only if there exists a $(k+1)$-primitive F of f, continuous on $[-1,1]$, such that for each $a > 0$,*

$$\lim_{x \to 0^+}\frac{k!\left(a^{-k}F(ax) - (-1)^k F(-x)\right)}{x^k L(x)} = \gamma + \beta\log a. \tag{36}$$

P r o o f. The limit (36) follows from (34) and (35) by direct computation. For the converse, rewrite (36) as

$$a^{-k}F(ax) - (-1)^k F(-x) = (\gamma + \beta\log a)\frac{x^k}{k!}L(x) + o\left(x^k L(x)\right),\ x \to 0^+,$$

for each $a > 0$. Set

$$c(x) = k!x^{-k}F(x) - \left(\frac{\gamma}{2} - \beta\sum_{j=1}^{k}\frac{1}{j}\right)L(x),\ x \in (0,1).$$

By setting $a = 1$ in (36), one sees that for $x < 0$,

$$F(x) = c\left(|x|\right)\frac{x^k}{k!} + \gamma L\left(|x|\right)\frac{x^k}{2k!}\,\mathrm{sgn}x - \beta L\left(|x|\right)\frac{x^k}{k!}\sum_{j=1}^{k}\frac{1}{j} + o\left(|x|^k L\left(|x|\right)\right),\ x \to 0.$$

Since

$$a^{-k}F(ax) - F(x) = \beta\log a\frac{x^k}{k!}L(x) + o\left(x^k L(x)\right),\ x \to 0^+,$$

it is clear that for each $a > 0$,

$$c(ax) = c(x) + \beta\log a L(x) + o(L(x)),\ x \to 0^+.$$

□

It is remarkable that, initially, no uniform condition on a is assumed in (36). However, the proof of Theorem 5.3 forces this relation to hold uniformly for a in compact subsets.

We are now ready to state the structural theorem for negative integer orders which now follows trivially from Proposition 4.3, Theorem 5.2 and Theorem 5.3.

Theorem 5.4 *Let $f \in \mathcal{D}'$ and k be a positive integer. Then f has the quasiasymptotic behavior at the origin,*

$$f(\epsilon x) = \gamma \epsilon^{-k} L(\epsilon) \delta^{(k-1)}(x) + (-1)^{k-1}(k-1)! \beta \epsilon^{-k} L(\epsilon) x^{-k} + o\left(\epsilon^{-k} L(\epsilon)\right), \ \epsilon \to 0^+, \ \text{in } \mathcal{D}',$$

if and only if there exists $m \in \mathbb{N}$, $m \geq k$, a function c defined on $(0, \infty)$, such that it is an associate asymptotically homogeneous function of degree 0 at the origin with respect to L, satisfying

$$c(a\epsilon) = c(\epsilon) + \beta \log a L(\epsilon) + o(L(\epsilon)), \ \epsilon \to 0^+,$$

for each $a > 0$, and a m-primitive of f, F, which is continuous in $[-1, 1]$ and satisfies

$$F(x) = c\left(|x|\right) \frac{x^{m-k}}{(m-k)!} + \gamma L\left(|x|\right) \frac{x^{m-k}}{2(m-k)!} \operatorname{sgn} x - \beta L\left(|x|\right) \frac{x^{m-k}}{(m-k)!} \sum_{j=1}^{m-k} \frac{1}{j} + o\left(|x|^{m-k} L\left(|x|\right)\right)$$

as $x \to 0$, in the ordinary sense. The last property is equivalent to

$$\lim_{x \to 0^+} \frac{(m-k)!\left(a^{k-m} F(ax) - (-1)^{m-k} F(-x)\right)}{x^{m-k} L(x)} = \gamma + \beta \log a,$$

for each $a > 0$.

We end this section with the following two remarks, they deal with the simplest case of quasiasymptotic behavior at the origin.

Remark 5.5 In this remark, we analyze the case when supp $f \subseteq [0, \infty)$. The reader should compare this remark with Theorem 2 in [13], Chapter 1, Section 3.4. In such case if f has quasiasymptotic at the origin of order $\alpha \in \mathbb{R}$, then by considerations on the support, one easily sees that it should be of the form

$$f(\epsilon x) = C \epsilon^{\alpha} L(\epsilon) \frac{x_+^{\alpha}}{\Gamma(\alpha+1)} + o\left(\epsilon^{\alpha} L(\epsilon)\right) \ \text{ as } \ \epsilon \to 0^+ \ \text{ in } \mathcal{D}'. \tag{37}$$

Moreover, again by considerations on the supports, see Remark 3.2, or simply by working in $\mathcal{D}'[0, \infty)$ as it is done in [13] for the case of quasiasymptotics at ∞ in $\mathcal{S}'_+$, the structural theorem can be stated in a very simple form, even in the case $\alpha \in \{-1, -2, \dots\}$. We have that f satisfies (37) if and only if there is $n \in \mathbb{N}$, $n > -\alpha$, such that the n-primitive of f with support in $[0, \infty)$, denoted by $f^{(-n)}$, is continuous in a neighborhood of the origin and satisfies

$$\lim_{x \to 0^+} \frac{\Gamma(\alpha+n+1) f^{(-n)}(x)}{x^{\alpha+n} L(x)} = C. \tag{38}$$

Remark 5.6 In analogy with the case of quasiasymptotics at ∞, we have that if supp f, supp $g \subseteq [0, \infty)$, and f and g have quasiasymptotic at the origin with respect to $\epsilon^{\alpha} L_1(\epsilon)$ and $\epsilon^{\nu} L_2(\epsilon)$, respectively, then $f * g$ has quasiasymptotic at the origin with respect to $\epsilon^{\alpha+\nu+1} L_1(\epsilon) L_2(\epsilon)$. In [13], this assertion at ∞ is shown by means of their Tauberian theorem; see Lemma 1, Chapter 4, Section 11.1 in [13]. We give an argument for proving this claim at the origin based on Remark 5.5. The proof is very simple. Consider $f \otimes g \in \mathcal{D}'(\mathbb{R}^2)$. Then by (38) there exist $n > -\alpha$, $m > -\nu$, C_1 and C_2 such

$$\lim_{x \to 0^+} \frac{\Gamma(\alpha+n+1) f^{(-n)}(x)}{x^{\alpha+n} L_1(x)} = C_1$$

and

$$\lim_{y \to 0^+} \frac{\Gamma(\nu+m+1) g^{(-m)}(y)}{y^{\nu+m} L_2(y)} = C_2,$$

hence for each $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$\langle f \otimes g(\epsilon x, \epsilon y), \phi(x, y) \rangle$$

$$= \frac{(-1)^{n+m}}{\epsilon^{n+m}} \int\int f^{(-n)}(\epsilon x) g^{(-m)}(\epsilon y) \frac{\partial^{n+m}\phi}{\partial x^n \partial y^m}(x,y) dx dy$$

$$= \epsilon^{\alpha+\nu} L_1(\epsilon) L_2(\epsilon) \left\langle \frac{C_1 x_+^\alpha}{\Gamma(\alpha+1)} \otimes \frac{C_2 y_+^\alpha}{\Gamma(\nu+1)}, \phi(x,y) \right\rangle + o(\epsilon^{\alpha+\nu} L_1(\epsilon) L_2(\epsilon)), \ \epsilon \to 0^+.$$

It follows then from the definition of convolution and the last relation that

$$f * g(\epsilon x) = C_1 C_2 L_1(\epsilon) L_2(\epsilon) \frac{(\epsilon x)_+^{\alpha+\nu+1}}{\Gamma(\alpha+\nu+2)} + o(\epsilon^{\alpha+\nu+1} L_1(\epsilon) L_2(\epsilon)), \ \epsilon \to 0^+, \text{ in } \mathcal{D}'.$$

6 Quasiasymptotics at the origin of Tempered Distributions.

We conclude this article by solving the open problem concerning quasiasymptotics of tempered distributions indicated in the introduction. The solution is the content of Theorem 6.1. The proof adapts to our context some arguments of R. Estrada given in [4] (see proofs of Theorem 1 and Theorem 2 there).

Theorem 6.1 *Let $f \in \mathcal{S}'$. If f has quasiasymptotic at 0 in $\mathcal{D}'$, then f has quasiasymptotic at 0 in $\mathcal{S}'$.*

Proof. Let α be the order of the quasiasymptotic. We shall divide the proof into three cases:
$\alpha \notin \{-1,-2,-3,\dots\}$,
$\alpha = -1$,
$\alpha = -2,-3,\dots$.
Suppose its order is $\alpha \notin \{-1,-2,-3,\dots\}$ and

$$f(\epsilon x) = C_- L(\epsilon) \frac{(\epsilon x)_-^\alpha}{\Gamma(\alpha+1)} + C_+ L(\epsilon) \frac{(\epsilon x)_+^\alpha}{\Gamma(\alpha+1)} + o(\epsilon^\alpha L(\epsilon)), \ \ \epsilon \to 0^+, \text{ in } \mathcal{D}'.$$

Then, there are real numbers m and λ such that $m \in \mathbb{N}$, $m > -\alpha$, $\lambda > m+\alpha$, and a continuous m-primitive F of f such that

$$F(x) = \frac{|x|^{m+\alpha}}{\Gamma(m+\alpha+1)} L(|x|)(C_- H(-x) + C_+ H(x)) + o\left(|x|^{m+\alpha} L(|x|)\right), \ x \to 0,$$

and

$$F(x) = O\left(|x|^\lambda\right), \ |x| \to \infty. \tag{39}$$

We make the usual assumptions over L. Assume that L is positive, defined in $(0,\infty)$ and there exists $M_1 > 0$ such that

$$\frac{L(\epsilon x)}{L(\epsilon)} \le M_1 \max\left\{x^{-\frac{1}{2}}, x^{\frac{1}{2}}\right\}, \ \ \epsilon, x \in (0,\infty). \tag{40}$$

Let $\phi \in \mathcal{S}$, then we can decompose $\phi = \phi_1 + \phi_2 + \phi_3$, where supp $\phi_1 \subseteq (-\infty, 1]$, supp ϕ_2 is compact and supp $\phi_3 \subseteq [1,\infty)$. Observe that since $\phi_2 \in \mathcal{D}$ we have that

$$\langle f(\epsilon x), \phi_2(x) \rangle = C_- \epsilon^\alpha L(\epsilon) \left\langle \frac{x_-^\alpha}{\Gamma(\alpha+1)}, \phi_2(x) \right\rangle \tag{41}$$

$$+ C_+ \epsilon^\alpha L(\epsilon) \left\langle \frac{x_+^\alpha}{\Gamma(\alpha+1)}, \phi_2(x) \right\rangle + o(\epsilon^\alpha L(\epsilon)), \ \epsilon \to 0^+.$$

If we want to show (41) for ϕ, it is enough to show it for ϕ_3 placed instead of ϕ_2 in the relation because by symmetry it would follow for ϕ_1 and hence for ϕ. Set

$$G(x) = \frac{F(x)}{x^{\alpha+m} L(x)}, \ x > 0.$$

Then

$$\lim_{x\to 0^+} G(x) = \frac{C_+}{\Gamma(\alpha+m+1)}, \tag{42}$$

On combining (39), (40) and (42), we find a constant $M_2 > 0$ such that

$$|G(x)| < M_2(1 + x^{\lambda+\frac{1}{2}-m-\alpha}),\ x > 0. \tag{43}$$

Relation (43) together with (40) show that for $\epsilon \leq 1$,

$$\left| G(\epsilon x)\frac{L(\epsilon x)}{L(\epsilon)} x^{\alpha+m}\phi_3^{(m)}(x) \right| \leq 2M_1M_2x^{\lambda+1}\left|\phi_3^{(m)}(x)\right| H(x-1).$$

The right hand side of the last estimate belongs to $L^1(\mathbb{R})$ and thus we can use the Lebesgue dominated convergence theorem to obtain,

$$\begin{aligned}
&\lim_{\epsilon\to 0^+} \frac{1}{\epsilon^\alpha L(\epsilon)} \left\langle f(\epsilon x), \phi_3(x)\right\rangle \\
&= \lim_{\epsilon\to 0^+} (-1)^m \int_0^\infty G(\epsilon x)\frac{L(\epsilon x)}{L(\epsilon)} x^{\alpha+m}\phi_3^{(m)}(x)dx \\
&= (-1)^m \frac{C_+}{\Gamma(\alpha+m+1)} \int_0^\infty x^{\alpha+m}\phi_3^{(m)}(x)dx \\
&= C_+ \left\langle \frac{x_+^\alpha}{\Gamma(\alpha+1)}, \phi_3(x)\right\rangle .
\end{aligned}$$

This shows the result in the case $\alpha \notin \{-1,-2,-3,\dots\}$.

We now aboard the case $\alpha = -1$. Assume that

$$f(\epsilon x) = \gamma\epsilon^{-1}L(\epsilon)\delta(x) + \beta\epsilon^{-1}L(\epsilon)x^{-1} + o\left(\epsilon^{-1}L(\epsilon)\right),\ \epsilon\to 0^+,\ \text{in } \mathcal{D}'.$$

As in the last case, it suffices to assume that $\phi \in \mathcal{S}$, supp $\phi \subseteq [1,\infty)$ and show that

$$\lim_{\epsilon\to 0^+} \frac{\epsilon}{L(\epsilon)} \left\langle f(\epsilon x), \phi(x)\right\rangle = \beta\int_1^\infty \frac{\phi(x)}{x}dx.$$

We may proceed as in the previous case to apply the structural theorem, but we rather reduce it to the previous situation. So, set $g(x) = xf(x)$, then

$$g(\epsilon x) = \beta L(\epsilon) + o(L(\epsilon)),\ \epsilon\to 0^+,\ \text{in } \mathcal{D}'. \tag{44}$$

But $g \in \mathcal{S}'$, then since the order of the quasiasymptotic is 0, first case implies that (44) is valid in $\mathcal{S}'$. Therefore

$$\lim_{\epsilon\to 0^+} \frac{\epsilon}{L(\epsilon)} \left\langle f(\epsilon x), \phi(x)\right\rangle = \lim_{\epsilon\to 0^+} \frac{1}{L(\epsilon)} \left\langle g(\epsilon x), \frac{\phi(x)}{x}\right\rangle = \beta\int_1^\infty \frac{\phi(x)}{x}dx.$$

This shows the case $\alpha = -1$.

It remains to show the theorem when $\alpha \in \{-2,-3,\dots\}$. Suppose the order is $-k$, $k \in \{2,3,\dots\}$. It is easy to see that any primitive of order $(k-1)$ of f has quasiasymptotic of order -1 at the origin with respect to L (in fact this is the content of Proposition 4.3). The $(k-1)$-primitives of f are in $\mathcal{S}'$, so we can apply the case $\alpha = -1$ to them, and then by differentiation it follows that f has quasiasymptotic at the origin in $\mathcal{S}'$.

This completes the proof of Theorem 6.1. □

References

[1] N. H. Bingham, C. M. Goldie, J. L. Teugels, Regular variation. Encyclopedia of Mathematics and its Applications, (27 Cambridge University Press, Cambridge, 1989).

[2] Yu. N. Drozhzhinov, B. I. Zav'yalov, Asymptotically homogeneous generalized functions in a spherical representation and some applications, (Russian) Dokl. Akad. Nauk, 405 (2005), 18–21.

[3] Yu. N. Drozhzhinov, B. I. Zav'yalov, Tauberian theorems for generalized functions with values in Banach spaces, (Russian) Izv. Ross. Akad. Nauk Ser. Mat., 66 (2002), 47–118; translation in Izv. Math., 66 (2002), 701–769.

[4] R. Estrada, A Distributional Version of the Ferenc-Lucács Theorem, Sarajevo J. Math., 13 (2005), 75–92.

[5] R. Estrada, R. Kanwal, A distributional approach to Asymptotics: Theory and Applications, (Second Edition, Birkhäuser, Boston, 2002).

[6] S. Łojasiewicz, Sur la valeur et la limite d'une distribution en un point, Studia Math., 16 (1957), 1–36.

[7] S. Pilipović, Some properties of the quasiasymptotic of Schwartz distributions, Part II-Quasiasymptotic at 0, Publ. Inst. Math. (Beograd), 57 (1988), 131–135.

[8] S. Pilipović, On the Behavior of a Distribution at the Origin, Math. Nachr., 141 (1989), 27–32.

[9] L. Schwartz, Théorie des Distributions, I, II, (Hermann, Paris, 1950).

[10] E. Seneta, Regularly Varying Functions, (Springer, Berlin, 1976).

[11] J. Vindas, R. Estrada, On Distributional Point Values and Convergence of Fourier Series and Integrals, J. Fourier. Anal. Appl., 13 (2007), 551–576.

[12] J. Vindas, R. Estrada, On the Order of Summability of the Fourier Inversion Formula, preprint.

[13] V. S. Vladimirov, Yu. N. Drozhzinov, B. I. Zav'yalov, Tauberian Theorems for Generalized Functions, (Kluver Academic Publishers, Maine, 1988).

[14] V. S. Vladimirov, Yu. N. Drozhzhinov, B. I. Zav'yalov, Tauberian theorems for generalized functions in a scale of regularly varying functions and functionals, (Russian) dedicated to Jovan Karamata, Publ. Inst. Math. (Beograd), (85) (2002), 123–132.