Generalized functions, analytic representations, and applications to generalized prime number theory

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Sato's hyperfunctions

A useful idea in analysis is to study functions of a real variable via analytic functions. One looks for representations

$$f(x) = F(x + i0) - F(x - i0) := \lim_{y \to 0^+} F(x + iy) - F(x - iy), \quad (1)$$

with suitable interpretation of the limit.

Let $\mathcal{O}(\Omega)$ be the space of analytic functions on $\Omega \subseteq \mathbb{C}$. In 1959 Sato introduced the so-called space of hyperfunctions

$$\mathcal{B} = \mathcal{B}(\mathbb{R}) := \mathcal{O}(\mathbb{C} \setminus \mathbb{R}) / \mathcal{O}(\mathbb{C}) .$$

So $\mathcal{B}$ contains all objects "of the form"

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Most spaces occurring in functional analysis are embedded into the space of Sato hyperfunctions.

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Analytic representation of distributions

Starting with the work of Köthe, many authors investigated the problem of representing distributions via analytic functions (Tillmann, Silva, ...). One has:

Theorem

Every distribution admits the representation

$$f(x) = \lim_{y\to 0^+} F(x+iy) - F(x-iy)\ , \ \text{in } \mathcal{D}', \tag{2}$$

where F is analytic except on $\mathbb{R}$ and satisfies: for every compact $[a,b]$ there are constants $K, k > 0$ such that

$$|F(x+iy)| \leq \frac{K}{|y|^k}\ , \quad x \in [a,b], 0 < |y| < 1\ . \tag{3}$$

Conversely, if $F \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ satisfies (3), then (2) exists.

Constructing analytic representations: Cauchy transform

Denoting as $\mathcal{O}_{\mathcal{D}'}(\mathbb{C})$ the space of analytic functions on $\mathbb{C} \setminus \mathbb{R}$ satisfying the bounds

$$|F(x+iy)| \leq \frac{K}{|y|^k} , \quad x \in [a,b], 0 < |y| < 1 .$$

we obtain $\mathcal{D}' \cong \mathcal{O}_{\mathcal{D}'}(\mathbb{C} \setminus \mathbb{R})/\mathcal{O}(\mathbb{C})$.

How to find F? The simplest way is via the Cauchy transform:

$$F(z) = \frac{1}{2\pi i}\left\langle f(t), \frac{1}{t-z}\right\rangle, \quad \Im m\, z \neq 0 .$$

The Cauchy transform is well-defined in various distribution spaces, e.g. if $f \in \mathcal{E}'$, namely a compactly supported distribution. Recall $f \in \mathcal{E}'$ is the dual of $\mathcal{E} = C^\infty$.

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Constructing analytic representations: Fourier-Laplace transform

If $f \in \mathcal{S}'$, we can use the Fourier-Laplace transform representation.
Decompose the Fourier transform $\hat{f} = \hat{f}_- + \hat{f}_+$, where $\hat{f}_-$ and $\hat{f}_-$ have supports in $(-\infty, 0]$ and $[0, \infty)$. Then

$$F(z) = \begin{cases} \frac{1}{2\pi}\langle \hat{f}_+(u), e^{izu}\rangle & \text{if } \Im m\, z > 0\ , \\ -\frac{1}{2\pi}\langle \hat{f}_-(u), e^{izu}\rangle & \text{if } \Im m\, z < 0\ . \end{cases}$$

In this case F satisfies the global bound

$$|F(x+iy)| \leq \frac{K(1+|x|+|y|)^m}{|y|^k}, \quad y \neq 0\ . \qquad (4)$$

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Hardy spaces

The classical Hardy space H^p, $1 \leq p \leq \infty$, is defined as the space of analytic functions on $\Im m\, z > 0$ such that

$$\sup_{0<y\leq 1} ||F(\cdot + iy)||_p < \infty. \tag{5}$$

A classical result tells us that for every $F \in H^p$,

$$f(x) := \lim_{y \to 0^+} F(x + iy)$$

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The space $\mathcal{D}'_{L^p}$

- $\mathcal{D}_{L^p} = \{\phi \in \mathcal{E} : \phi^{(n)} \in L^p, \forall n\}$.
- $\mathcal{D}'_{L^p}$ is the dual of $\mathcal{D}_{L^q}$ where $1/p + 1/q = 1$ (with a technical variant when $p = 1$).

The space $\mathcal{D}'_{L^2}$ is easy to understand: $f \in \mathcal{D}'_{L^2}$ iff $\exists k$ such that

$$\int_{-\infty}^{\infty} |\hat{f}(u)|^2 (1 + |u|)^k < \infty .$$

Theorem

A function $F(z)$, analytic in $\Im m\, z > 0$, has boundary values

$$f(x) = \lim_{y \to 0^+} F(x + iy) \quad \text{in } \mathcal{D}'_{L^p}$$

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Boundary values: Summary

Let $F(z)$ be analytic on the half-plane $\Im m\, z > 0$. Then

$$|F(x+iy)| \leq \frac{K}{y^k} \text{ (locally) } \implies F(x+i0) \in \mathcal{D}' .$$

$$|F(x+iy)| \leq \frac{K(1+|x|+y)^m}{y^k} \implies F(x+i0) \in \mathcal{S}' .$$

$$||F(\,\cdot\,+iy)||_p \leq \frac{K}{y^k} \implies F(x+i0) \in \mathcal{D}'_{L^p} .$$

The class of real analytic functions over $\mathbb{R}$ is characterized by

$$\sup_{x\in[a,b]} |f^{(p)}(x)| \leq h^p p!, \quad \text{for some } h = h_{a,b} \, .$$

Replace $p!$ by a sequence $\{M_p\}_{p=0}^{\infty}$ satisfying (convexity): $M_p^2 \leq M_{p-1}M_{p+1}$. Define $\mathcal{E}^{\{M_p\}} \subset \mathcal{E}(= C^{\infty})$ as those functions such that

$$\sup_{x\in[a,b]} |f^{(p)}(x)| \leq h^p M_p, \quad \text{for some } h = h_{a,b} \, .$$

Example. $M_p = (p!)^s$ gives rise to the Gevrey classes.

Hadamard problem (1912): The class is called quasi-analytic if $\mathcal{E}^{\{M_p\}} \cap \mathcal{D} = \{0\}$. Find conditions over M_p for quasi-analyticity.

Theorem (Denjoy-Carleman, 1921, 1926)

The class $\mathcal{E}^{\{M_p\}}$ is quasi-analytic iff

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The sequence

We will work with the following conditions on $\{M_p\}_{p=0}^{\infty}$:

(M.1) $M_p^2 \leq M_{p-1}M_{p+1}$ (logarithmic convexity)

(M.2) $M_p \leq AH^p M_q M_{p-q}$ for $0 \leq q \leq p$ (stability under ultradifferential operators)

(M.3') $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$ (non-quasianalyticity)

The following two functions are useful:

$$M(\rho) = \sup_{p\in\mathbb{N}} \log\left(\frac{\rho^p M_0}{M_p}\right) \qquad \text{(the associated function)}.$$

$$M^*(\rho) = \sup_{p\in\mathbb{N}} \log\left(\frac{\rho^p M_0 p!}{M_p}\right).$$

The nonquasi-analyticity condition $(M.3')$ is equivalent to:

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Ultradistribution spaces

$$\mathcal{S}^{M_p}_{h} = \left\{ \phi \in \mathcal{S} : \sup_{x,\alpha,\beta} \frac{(1+|x|)^{\alpha}|\phi^{(\beta)}(x)|}{h^{\alpha+\beta}M_{\alpha}M_{\beta}} < \infty \right\},$$

$$\mathcal{D}^{M_p}_{h,L^q} = \left\{ \phi \in \mathcal{D}_{L^q} : \sup_{\alpha} \frac{||\phi^{(\alpha)}||_q}{h^{\alpha}M_{\alpha}} < \infty \right\},$$

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Beurling-type spaces:

$$\mathcal{S}^{(M_p)} = \bigcap_{h>0} \mathcal{S}^{(M_p)}_{h}, \quad \mathcal{D}^{(M_p)}_{L^q} = \bigcap_{h>0} \mathcal{D}^{(M_p)}_{h}$$

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Roumieu-type spaces: Replace intersections by unions, resulting spaces: $\mathcal{S}^{\{M_p\}}$, $\mathcal{D}^{\{M_p\}}_{L^q}$, $\mathcal{D}^{\{M_p\}}$.

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Boundary values in ultradistribution spaces

Let $F(z)$ be analytic on $\Im m\, z > 0$. Assume $(M.1)$, $(M.2)$, $(M.3)$.

- Beurling case:

$$(\forall A > 0)(\exists \lambda)(\exists K)\left(|F(x+iy)| \leq K e^{M^*\left(\frac{\lambda}{y}\right)}, y < 1, |x| \leq A\right) \implies F(x+i0) \in \mathcal{D}'^{(M_p)}.$$

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- Roumieu case: If one replaces $(\exists \lambda)$ by $(\forall \lambda)$, one obtains boundary values in the ultradistribution spaces $\mathcal{D}'^{\{M_p\}}$, $\mathcal{S}'^{\{M_p\}}$, and $\mathcal{D}'^{\{M_p\}}_{L^q}$.

Boundary values in ultradistribution spaces

Let $F(z)$ be analytic on $\Im m\, z > 0$. Assume $(M.1)$, $(M.2)$, $(M.3)$.

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The prime number theorem

The prime number theorem (PNT) states that

$$\pi(x) \sim \frac{x}{\log x} , \quad x \to \infty ,$$

where

$$\pi(x) = \sum_{\substack{p \leq x \\ p \text{ prime}}} 1 .$$

We will consider generalizations of the PNT for Beurling's generalized numbers

Beurling's problem

In 1937, Beurling raised and studied the following question.

- Let $1 < p_1 \leq p_2, \ldots$ be a non-decreasing sequence tending to infinity (generalized primes).
- Arrange all possible products of the p_j in a non-decreasing sequence $1 < n_1 \leq n_2, \ldots,$ where every n_k is repeated as many times as represented by $p_{\nu_1}^{\alpha_1} p_{\nu_2}^{\alpha_2} \ldots p_{\nu_m}^{\alpha_m}$ with $\nu_j < \nu_{j+1}$ (generalized numbers).
- Denote $N(x) = \sum_{n_k \leq x} 1$ and $\pi(x) = \sum_{p_k \leq x} 1$.

Beurling's problem: Find conditions over N which ensure the validity of the PNT, i.e.,

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Beurling's PNT

Beurling studied the problem in connection with the asymptotics

$$N(x) \sim ax\ .$$

Conditions on the reminder in $N(x) = ax + R(x)$ are *needed*.

Theorem (Beurling, 1937)

if

$$N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right),$$

where $a > 0$ *and* $\gamma > 3/2$*, then the PNT holds.*

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Beurling's condition is sharp, namely, the PNT does not necessarily hold if $\gamma = 3/2$*.*

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The L^2-conjecture: Kahane's PNT

In 1969, Bateman and Diamond conjectured that

$$\int_1^\infty \left| \frac{(N(x) - ax)\log x}{x} \right|^2 \frac{\mathrm{d}x}{x} < \infty$$

would suffice for the PNT. The above L^2-condition extends that of Beurling.

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An average condition for the PNT

Schlage-Puchta and I recently showed.

Theorem (2012, extending Beurling)

Suppose there exist constants $a > 0$ *and* $\gamma > 3/2$ *such that*

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Then the prime number theorem still holds.

The hypothesis means that there exists some $m \in \mathbb{N}$ such that:

$$\int_{0}^{x} \frac{N(t) - at}{t}\left(1 - \frac{t}{x}\right)^{m} \mathrm{d}t = O\left(\frac{x}{\log^{\gamma} x}\right) .$$

Technique: Distributional methods in the analysis of boundary behavior of zeta functions.

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Comparison of conditions for the PNT

The three conditions

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$$E(u) = \frac{(N(e^{u}) - ae^{u})u}{e^{u}} .$$

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In all three cases, this error function satisfies the membership condition:

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The newest general PNT

Theorem (2013, extending all earlier results)

Suppose that $E \in \mathcal{D}'^{(M_p)}_{L^2}$, where the sequence satisfies (M.1) and (M.2) and the associated function M satisfies:

$$\int_1^{\infty} \frac{M(x)}{x^3}\,\mathrm{d}x < \infty\,. \tag{6}$$

Then the prime number theorem holds.

Example. If $M_p = (p!)^s$ with $1/2 < s$, then (6) holds because

$$Ax^{1/s} \le M(x) \le Bx^{1/s}.$$

Remark. The condition (6) implies the bound

$$M(x) = o(x^2/\log x).$$

Is this growth condition sharp for the PNT? I conjecture so.

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