A generalized prime random approximation procedure and some of its applications

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$\boxed{1}$ Introduction: In this lecture I am going to discuss a generalized prime random method to approximate a class of right continuous unbounded non-decreasing functions in a certain precise fashion. This procedure was recently developed by F. Broucke and myself (work to appear in MATH. Z.) and improves upon pioneer work of Diamond, Montgomery, and Vorhauer. We'll review the latter work in the next section, here we'll just motivate some definitions.

We need to introduce the notion of Beurling generalized numbers. There are two approaches, I'll start with the one à la Knopfmacher, popularized through his book "Abstract analytic number theory" (1975).

An *arithmetical semigroup* is a infinite commutative semigroup G with unity e such that

(I) there is a countable set of indecomposables $Q \subseteq G$ such that each $g \in G$ admits a unique factorization

$$g = q_1^{\alpha_1} \cdots q_k^{\alpha_k}, \qquad \alpha_j \in \mathbb{N},$$

and

(II) G is provided with a "norm function" $|\cdot| : G \longrightarrow (0,\infty) = \mathbb{R}^+$, that is, a positive function satisfying

(i) $|g_1 \cdot g_2| = |g_1| \cdot |g_2|$

(ii) $|g| > 1, \ \forall g \neq e$.

(iii) $|\{g \in G : |g| \leq x\}|$ is finite for all $x > 0$.

We are interested in the relationship between

$$N(x) = \sum_{\substack{|g| \leq x \\ g \in G}} 1 \quad \text{and} \quad \pi(x) = \sum_{\substack{|q| \leq x \\ q \in Q}} 1 .$$

Example 1. Let K be an algebraic number field and let $\mathcal{O}_K$ its ring of algebraic integers. The set $\mathcal{O}_K$ of non-zero ideals of $\mathcal{O}_K$ is an arithmetical semigroup with $|I|$ the norm of I $(= \# \mathcal{O}_K/I)$.

Let us now introduce Beurling's view, the notion of g-primes from 1937. If we arrange $G = \{g_k : k \in \mathbb{N}\}$ and $Q = \{q_k : k \in \mathbb{N}\}$ in such way that $1 = |e| < |g_1| \le |g_2| \le \cdots$ ($e = g_0$)

and $|q_1| \le |q_2| \le |q_3| \le \cdots$, and then name $\rho_k = |q_k|$ and $n_k = |g_k|$, we obtain two sequences of positive real numbers such that

(1) $\mathcal{P} : 1 < p_1 \le p_2 \le p_3 \le \cdots$

(2) $\mathcal{N} : n_0 = 1 < n_1 \le n_2 \le \cdots$, each $n_k = p_{\nu_1}^{\alpha_1} \cdots p_{\nu_j}^{\alpha_j}$

for $\nu_1 < \cdots < \nu_j$ and $\alpha_i > 0$.

<u>Def.</u> A Beurling g-prime system is an <u>unbounded</u> sequence $\mathcal{P}$ as in (1). Its associated (multi-)set of generalized integers arises as in (2), where multiplicities are taken into account. We write

$$\pi_{\mathcal{P}}(x) = \sum_{p_k \le x} 1, \quad N_{\mathcal{P}}(x) = \sum_{n_k \le x} 1.$$

Other number-theoretic functions can be introduced:

$$\Pi_{\mathcal{P}}(x) = \pi_{\mathcal{P}}(x) + \frac{1}{2}\pi_{\mathcal{P}}(x^{\frac{1}{2}}) + \cdots = \sum_{p_k^\alpha \le x} \frac{1}{\alpha},$$

$$\zeta_{\mathcal{P}}(s) = \sum_{k=0}^{\infty} \frac{1}{n_k^s}, \quad \psi_{\mathcal{P}}(x) = \sum_{p_k^\alpha \le x} \log p_k, \quad M_{\mathcal{P}}(x) = \sum_{n_k \le x} \mu(n_k),$$

where $\mu(n_k)$ arises as

$$\sum_{k=1}^{\infty} \frac{\mu(n_k)}{n_k^s} = \frac{1}{\zeta(s)},$$

and so on...

$\boxed{2}$ The DMVZ random approximation method.

Landau showed the following PNT:

<u>Theorem</u> (Landau, 1907) Suppose that

(1) $N_{\mathcal{P}}(x) = \rho x + O(x^{\theta})$,

for some $\rho > 0$ and $0 \leq \theta < 1$. Then, $\exists c > 0$ such that

(2) $\psi_{\mathcal{P}}(x) = x + O(x \exp(-c\sqrt{\log x}))$

In a breakthrough paper, Diamond, Montgomery and Vorhauer showed that Landau's PNT cannot be improved:

<u>Theorem</u> (Diamond, Montgomery, Vorhauer, 2006).
Let $\frac{1}{2} < \theta < 1$. There is a Beurling number system $\mathcal{P}$ such (1) holds for some $\rho > 0$ and such that for some $a > 0$:

$$\psi_{\mathcal{P}}(x) = x + O(x \exp(-a\sqrt{\log x})),$$

$$\psi_{\mathcal{P}}(x) = x + \Omega(x \exp(-a\sqrt{\log x})), \text{ and}$$

$\zeta_{\mathcal{P}}(s)$ has infinitely many zeros on the curve

$$\sigma = 1 - \frac{a^2}{2\log t} \quad (s = \sigma + it). ///$$

There are two key steps implemented by them in the proof of this theorem. The first one is an old idea that goes back to Beurling, namely, one first attempts to construct a continuous analog of a Beurling number system having the desired properties. This is already challenging and we won't say anything how it is done. In general, a non-necessarily discrete number system is a pair of right continuous non-decreasing functions Π and N linked by the relation

$$\zeta(s) = \int_{1^-}^{\infty} x^{-s}\, dN(x) = \exp\left(\int_{1^-}^{\infty} x^{-s}\, d\Pi(x) \right)$$

The second step is a discretization of the continuous number system.

For the latter, they used a probabilistic argument to generate the following generalized prime random approximation result. Since Zhang (2017) did some contributions to improve the method, we shall refer to it as the DMVZ method.

Theorem (Diamond, Montgomery, Vorhauer, 2006; Zhang 2007). Let $f \in L^1_{loc}[1,\infty)$ be a positive function such that

(3) $$f(u) \ll \frac{1}{\log u} \quad \text{and} \quad \int_1^\infty f(u)\,du = \infty.$$

Then, there is a Beurling prime number system $\mathcal{P}$: $1 < p_1 < p_2 < \cdots < p_k \to \infty$ such that

(4) $$\left| \sum_{p_k \le x} p_k^{-it} - \int_1^x u^{-it} f(u)\,du \right| \ll \sqrt{x} + \sqrt{\frac{x \log(|t|+1)}{\log(x+1)}}.$$

3 The new g-prime random approximation method

If we write $F(x) = \int_1^x f(u)\,du$ in the DMVZ method, then the first inequality of (3) implies that F satisfies a Chebyshev type inequality, while $t=0$ in (4) leads to

(5) $$\pi_{\mathcal{P}}(x) - F(x) \ll \sqrt{x}.$$

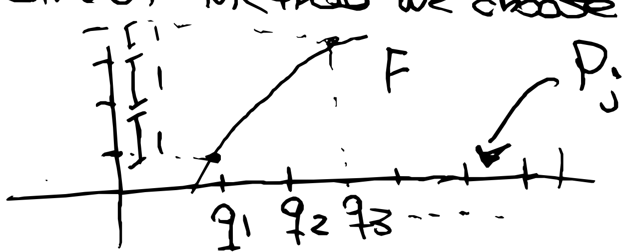
For various applications, (5) is not strong enough. Our improvement to the DMVZ method is precisely at this level:

<u>Theorem</u> (Broucke, V., 2024). Let F be a non-decreasing right continuous function on $[1,\infty)$ such that

$$F(x) \ll \frac{x}{\log x},$$

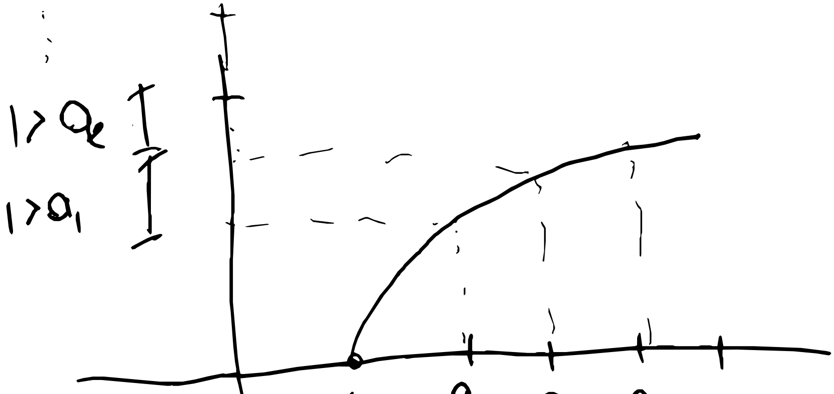
$F(1)=0$ and $F(x) \longrightarrow \infty$. Then, there is a set of generalized primes $\mathcal{P}$: $1 < p_1 \leq p_2 \leq \cdots$ such that (4) holds and $|\pi_{\mathcal{P}}(x) - F(x)| \leq 2$. If F is continuous, we can actually choose it such that $|\pi_{\mathcal{P}}(x) - F(x)| \leq 1$.

The proofs of both approximations results are probabilistic, using a classical inequality of Kolmogorov to control the size of the sum of independent random variables in combination with a Borel-Cantelli argument. The random variables here give us how the random generalized primes are selected. The main difference in both procedure is how these generalized random primes are distributed. Assume that F is continuous for simplicity. In our method we choose:

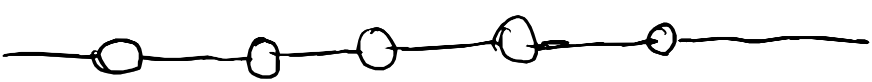


$p_j \in (q_{j-1}, q_j]$ are independent random variables distributed according to $dF|_{(q_{j-1}, q_j]}$.

In the DMVZ method this is done by choosing:



Then one chooses Bernoulli independent random variables X_i with $P(X_i = 1) = a_i$. Then the random primes come from $q_i X_i$



So, if we additionally choose $a = a_1 = a_2 = a_3 = \cdots$, this amounts to perform a Bernoulli trial to find the generalized primes.

For details how the desired results are obtained, we refer to the corresponding papers.

$\boxed{4}$ Application: Balazard's problem for general Dirichlet series.

M. Balazard posed the question

Question 1: Does there exist $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ with exactly one zero in its half-plane of convergence?

If RH (Riemann hypothesis) is true,

$$(6) \qquad \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

would provide an answer. But of course we'd like to find an unconditional example. We have not answered question 1 as it stands, but we obtained an answer in the case of general Dirichlet series.

Theorem (Broucke, V., 2024) There is a Beurling number system $\mathcal{P}$ with associated g-integers $n_0 = 1 < n_1 < \cdots$ such that

$$\sum_{k=1}^{\infty} \frac{\mu(n_k)}{n_k^s}$$

has abscissa of convergence $\frac{1}{2}$ and only vanishes at $s=1$ in $\{s : \operatorname{Re} s > \frac{1}{2}\}$. ///

The number system we constructed is related to a question recently posed by Hilberdink and Neamah (2020). We say that $\mathcal{P}$ is an $[\alpha, \beta, \gamma]$-system if

$$\Pi_{\mathcal{P}}(x) = Li(x) + O(x^{\alpha+\varepsilon})$$

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta+\varepsilon}), \quad \rho > 0,$$

$$M(x) = O(x^{\gamma+\varepsilon})$$

hold for any $\varepsilon > 0$, but for no $\varepsilon < 0$. They showed

$$\Theta = \max\{\alpha, \beta, \gamma\} \geq \frac{1}{2}, \text{ and at least}$$

two of the three numbers must equal Θ. If $\Theta < 1$, they call it a well-behaved system. Under RH, the classical integers are well-behaved. But are there unconditional examples of well-behaved systems? Furthermore, they asked:

<u>Question 2</u>: Are there $[0, \beta, \beta]$-systems with $\beta < 1$?

<u>Theorem</u> (Broucke, V., 2024) There is a Beurling prime number system $\mathcal{P}$ of type $[0, \frac{1}{2}, \frac{1}{2}]$.

Our theorem solves both Question 1 and Question 2. The idea of the proof is as follows. We normalize the logarithmic integral as

$$Li(x) = \int_1^x \frac{\left(1-\frac{1}{u}\right)du}{\log u},$$

and apply the random approximation method from Section 3 to $F(x) = li(x)$, where

$$Li(x) = \sum_{k=1}^{+\infty} \frac{1}{k} li(x^{\frac{1}{k}}), \text{ i.e., } li(x) = \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} Li(x^{\frac{1}{\nu}}),$$

with μ the classical Möbius function of the rational integers. This produces $\mathcal{P}$ with $li(x) = \Pi_{\mathcal{P}}(x) + O(1)$. Using also $\Pi_{\mathcal{P}}(x) = \sum_{k=1}^{+\infty} \frac{1}{k} \pi(x^{\frac{1}{k}})$, one sees that

$$(7) \quad \Pi_{\mathcal{P}}(x) = Li(x) + O(\log\log x).$$

The bound (4) is strong enough to lead (via contour integration)

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\frac{1}{2}+\varepsilon}) \text{ and } M_{\mathcal{P}}(x) = O(x^{\frac{1}{2}+\varepsilon}),$$

$\forall \varepsilon > 0$, so the system is of type $[0, \beta, \gamma]$ with $\beta, \gamma \leq \frac{1}{2}$. But the theorem of Hilberdink and Neamah quoted above yields $\beta = \gamma = \frac{1}{2}$. ///

Remark 1. The DMVZ method would only yield

$$\pi_p(x) = \mathrm{li}(x) + O(\sqrt{x}),$$

which not enough to solve Questions 1 and 2. The bound (7) is crucial for our argument.

Remark 2. Further applications of our method have been obtained in a number of articles by Frederik Broucke. //