A quick distributional way to the prime number theorem

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AMS-SIAM Special Session on Asymptotic Methods in Analysis with Applications
January 6 , 2009

The prime number theorem

The aim of this talk is to give a purely distributional proof of the Prime Number Theorem (PNT), that is,

$$\pi(x) \sim \frac{x}{\log x} , \quad x \to \infty ,$$

where

$$\pi(x) = \sum_{p \text{ prime}, \ p<x} 1 .$$

The word distributional refers to Schwartz distributions.

The tecniques

The proof is based on:

- Chebyshev elementary estimate
- The non-vanishing of the Riemann zeta function on $\Re e\, z = 1$
- Arguments from generalized asymptotics
  - $S-$asymptotics
  - Quasiasymptotics

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  - $S-$asymptotics
  - Quasiasymptotics

Outline

Notation
from distribution theory

- $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ denote the spaces of smooth compactly supported functions and smooth rapidly decreasing functions
- $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ the spaces of distributions and tempered distributions
- The Fourier transform in $\mathcal{S}(\mathbb{R})$ is defined as

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} e^{ixt}\phi(t)\mathrm{d}t$$

- The evaluation of f at a test function ϕ is denoted by

$$\langle f(x), \phi(x)\rangle$$

Quasiasymptotics
Generalized asymptotics

The idea is to study the weak asymptotic behavior of the dilates of f. So we look for asymptotic representations

$$f(\lambda x) \sim \rho(\lambda) g(x) .$$

Definition

We say that $f \in \mathcal{D}'(\mathbb{R})$ has quasiasymptotic behavior at ∞ in $\mathcal{D}'(\mathbb{R})$ with respect to ρ if for some $g \in \mathcal{D}'(\mathbb{R})$ and each $\phi \in \mathcal{D}(\mathbb{R})$,

$$\lim_{\lambda \to \infty} \left\langle \frac{f(\lambda x)}{\rho(\lambda)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle .$$

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Quasiasymptotics
Generalized asymptotics

We will study in connection to the PNT a particular case of quasiasymptotics, namely, a limit of the form

$$\lim_{\lambda\to\infty} f(\lambda x) = \beta H(x) , \quad \text{in } \mathcal{D}'(\mathbb{R}) , \tag{1}$$

where $H(x)$ is the Heaviside function.

- (1) should be always interpreted in the weak topology of $\mathcal{D}'(\mathbb{R})$, i.e.,

$$\lim_{\lambda\to\infty} \langle f(\lambda x), \phi(x)\rangle = \beta \int_0^\infty \phi(x)\mathrm{d}x , \ \ \forall\, \phi \in \mathcal{D}(\mathbb{R}) . \tag{2}$$

- We may also talk about (1) in other spaces of distributions; for instance in $\mathcal{D}'(0,\infty)$

$S-$asymptotics

Generalized asymptotics

Let $f \in \mathcal{D}'(\mathbb{R})$ and $\beta \in \mathbb{R}$ a relation of the form

$$\lim_{h\to\infty} f(x+h) = \beta \, , \quad \text{in } \mathcal{D}'(\mathbb{R}) \, ,$$

means that the limit is taken in the weak topology of $\mathcal{D}'(\mathbb{R})$, that is, for each $\phi \in \mathcal{D}(\mathbb{R})$ the following limit holds,

$$\lim_{h\to\infty} \langle f(x+h), \phi(x)\rangle = \beta \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x \, . \qquad (3)$$

- The above relation is an example of the so-called $S-$asymptotics of generalized functions, i.e.,

$$f(x+h) \sim \rho(h)g(x) \, , \quad \text{in } \mathcal{D}'(\mathbb{R}) \, .$$

- $\lim_{h\to\infty} f(x+h) = \beta$ in $\mathcal{S}'(\mathbb{R})$ means that $f \in \mathcal{S}'(\mathbb{R})$ and ϕ can be taken from $\mathcal{S}(\mathbb{R})$ in (3)

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Riemann zeta function
Properties

Consider the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} , \ \Re e \, z > 1 .$$

Properties

- $\zeta(z) - \dfrac{1}{z-1}$ admits an analytic continuation to a neighborhood of $\Re e \, z = 1$
- $\zeta(1 + ix)$, $x \neq 0$, is free of zeros

Chebyshev function

We denote by Λ the von Mangoldt function defined on the natural numbers as

$$\Lambda(n) = \begin{cases} 0\,, & \text{if } n = 1\,, \\ \log p\,, & \text{if } n = p^m \text{ with } p \text{ prime and } m > 0\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

and by ψ the Chebyshev function

$$\psi(x) = \sum_{p^m < x} \log p = \sum_{n < x} \Lambda(n)\,.$$

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Chebyshev's elementary estimate

It is very well known since the time of Chebyshev that

- The PNT is equivalent to the statement

$$\psi(x) \sim x \tag{4}$$

- Chebyshev's elementary estimate: $\exists M > 0$ such that $\psi(x) < Mx$

Our approach to the PNT will be to show (4). The proof is based on finding the (quasi-) asymptotic behavior of $\psi'(x)$; observe that

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The distribution $v(x)$

We shall study the (S-)asymptotic properties of the distribution

$$v(x) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \delta(x - \log n) \ .$$

clearly $v \in \mathcal{S}'(\mathbb{R})$. Let us take the Fourier-Laplace transform of v, that is, for $\Im m\, z > 0$

$$\hat{v}(z) = \left\langle v(t), e^{izt} \right\rangle = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1-iz}} = -\frac{\zeta'(1-iz)}{\zeta(1-iz)} \ ,$$

a formula that Riemann obtained by logarithmic differentiation of the Euler product $\zeta(z) = \prod_p 1/(1-p^{-z})$. Then,

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Properties of $v(x)$ to be used

It follows from the properties of ζ that the distributional boundary value of $\hat{v}(z) - \frac{i}{z}$ is a function, i.e.,

- $\hat{v}(x) - \frac{i}{(x + i0)} \in L^1_{\mathrm{loc}}(\mathbb{R})$

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The plan
Steps

1. To show that

$$\lim_{h\to\infty} v(x+h) = 1\ , \quad \text{in } \mathcal{S}'(\mathbb{R})$$

2. To show that

$$\lim_{\lambda\to\infty} \psi'(\lambda x) = \lim_{\lambda\to\infty} \sum_{n=1}^{\infty} \Lambda(n)\delta(\lambda x - n) = H(x)\ , \quad \text{in } \mathcal{D}'(0,\infty)$$

3. Final step, Step 2 is used to conclude

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$\lim_{h\to\infty} v(x+h) = 1$ in $\mathcal{S}'(\mathbb{R})$
Step 1

- First, $v(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$, as $h \to \infty$

Proof.

Set $g(x) = e^{-x}\psi(e^{x})$, by Chebyshev estimate $g(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$. Next, $g'(x+h) = O(1)$, but $g'(x) = -g(x) + e^{-x}\sum \Lambda(n)\delta(x - \log n) = -g(x) + v(x)$. □

- Second, $\lim_{h\to\infty} \langle v(x+h), \phi(x)\rangle = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x$, for ϕ in a dense subspace of $\mathcal{S}(\mathbb{R})$

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Step 1 (continuation)

Proof.

Let $\phi = \widehat{\phi_1}$ with supp ϕ_1 compact.

$$\langle v(x+h), \phi(x)\rangle = \int_{-h}^{\infty} \phi(x)\mathrm{d}x + \left\langle v(x+h) - H(x+h)\, , \widehat{\phi_1}(x)\right\rangle$$

$$= \int_{-h}^{\infty} \phi(x)\mathrm{d}x + \left\langle \hat{v}(x) - \frac{i}{(x+i0)}\, , e^{-ihx}\phi_1(x)\right\rangle$$

$$= \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x + o(1), \quad h \to \infty$$

□

- Banach-Steinhaus theorem immediately gives the result

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$\lim_{\lambda\to\infty} \psi'(\lambda x) = H(x)$, in $\mathcal{D}'(0,\infty)$

Step 2

Proof.

Step 2 implies that $e^{x+h}v(x+h) \sim e^{x+h}$, in $\mathcal{D}'(\mathbb{R})$, explicitely,

$$\sum_{n=1}^{\infty} \Lambda(n)\phi(\log n - h) \sim e^{h} \int_{-\infty}^{\infty} e^{x}\phi(x)\mathrm{d}x , \quad \forall \phi \in \mathcal{D}(\mathbb{R})$$

Changing variable in the last integral and writing $\lambda = e^{h}$,

$$\langle \psi'(\lambda x), \phi_1(x)\rangle = \frac{1}{\lambda}\sum_{n=1}^{\infty} \Lambda(n)\phi_1\left(\frac{n}{\lambda}\right) \sim \int_{0}^{\infty} \phi_1(x)\mathrm{d}x , \qquad (5)$$

where $\phi_1(x) = \phi(\log x)$. Thus, (5) holds $\forall \phi_1 \in \mathcal{D}(0,\infty)$. □

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Final Step: $\psi(x) \sim x$

Proof

Formally,

$$\frac{1}{\lambda}\sum_{n<\lambda}\Lambda(n) = \left\langle \psi'(\lambda x), \chi_{[0,1)}(x) \right\rangle \ .$$

We approximate $\chi_{[0,1)}$ by elements of $\mathcal{D}(0,\infty)$.

- Let ε be an arbitrary small positive number
- Choose ϕ_1 and ϕ_2 with the properties:
  - $0 \leq \phi_1, \phi_2 \leq 1$
  - $\operatorname{supp} \phi_1 \subseteq (0,1]$, $\phi_1(x) = 1$ on $[\varepsilon, 1-\varepsilon]$
  - $\operatorname{supp} \phi_2 \subseteq (0,1+\varepsilon]$, and $\phi_2(x) = 1$ on $[\varepsilon, 1]$

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Final Step: $\psi(x) \sim x$

Proof (continuation)

- Evaluating at ϕ_2 and using Chebyshev's estimate:

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda}\sum_{n<\lambda}\Lambda(n) \leq \limsup_{\lambda\to\infty}\left(\frac{1}{\lambda}\sum_{n<\varepsilon\lambda}\Lambda(n) + \frac{1}{\lambda}\sum_{n=1}^{\infty}\Lambda(n)\phi_2\left(\frac{n}{\lambda}\right)\right)$$

$$\leq M\varepsilon + \lim_{\lambda\to\infty}\left\langle \psi'(\lambda x), \phi_2(x)\right\rangle$$

$$= M\varepsilon + \int_0^{1+\varepsilon}\phi_2(x)\mathrm{d}x \leq 1+\varepsilon(M+1)$$

- Likewise, $1-2\varepsilon \leq \liminf_{\lambda\to\infty}\frac{1}{\lambda}\sum_{n<\lambda}\Lambda(n)$
- Therefore, $\psi(\lambda) = \sum_{n<\lambda}\Lambda(n) \sim \lambda$

Final Step: $\psi(x) \sim x$

Proof (continuation)

- Evaluating at ϕ_2 and using Chebyshev's estimate:

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda}\sum_{n<\lambda}\Lambda(n) \leq \limsup_{\lambda\to\infty}\left(\frac{1}{\lambda}\sum_{n<\varepsilon\lambda}\Lambda(n) + \frac{1}{\lambda}\sum_{n=1}^{\infty}\Lambda(n)\phi_2\left(\frac{n}{\lambda}\right)\right)$$

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