On the Jump Behavior of Distributions and Logarithmic Averages

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A theorem of Ferenc Lukács (1920)

Let $f \in L^1[-\pi, \pi]$ having Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Its conjugate series is defined as

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)$$

A classical theorem of F. Lukács states that if there is $d$ such that

$$\lim_{h \to 0^+} \frac{1}{h} \int_{0}^{h} |f(x_0 + t) - f(x_0 - t) - d| \, dt = 0$$

then

$$\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} (a_n \sin nx_0 - b_n \cos nx_0) = -\frac{d}{\pi}$$
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F. Móricz has generalized this result by extending the notion of symmetric jump to the existence of

$$d = \lim_{h \to 0^+} \frac{1}{h} \int_0^h (f(x_0 + t) - f(x_0 - t)) \, dt$$

He gave the formula for jumps using logarithmic Abel-Poisson means of the conjugate series,

$$\lim_{r \to 1^-} \frac{1}{\log(1 - r)} \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0) r^n = \frac{1}{\pi} d$$
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Outline

1. Distributional jump behavior
   - Example: Jump behavior of the first order

2. Jump behavior and logarithmic averages
   - Jumps and logarithmic averages in the Cesàro sense
   - Jumps and angular boundary behavior of harmonic conjugates

3. Applications to Fourier series of distributions
   - Symmetric jump behavior
   - Logarithmic Abel-Poisson means of the conjugate series
   - Logarithmic Cesàro-Riesz means of the conjugate series
Distributional jump behavior

Definition

A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to have a \textit{distributional jump behavior} at $x = x_0 \in \mathbb{R}$ if it satisfies

$$\lim_{\epsilon \to 0^+} f(x_0 + \epsilon x) = \gamma_- H(-x) + \gamma_+ H(x) \quad \text{in} \quad \mathcal{D}'(\mathbb{R})$$

The jump is defined as the number $[f]_{x=x_0} = \gamma_+ - \gamma_-$

The meaning of the above limit is in the \textit{weak topology} of $\mathcal{D}'(\mathbb{R})$, that is, for all $\phi \in \mathcal{D}(\mathbb{R})$

$$\lim_{\epsilon \to 0^+} \langle f(x_0 + \epsilon x), \phi(x) \rangle = \gamma_- \int_{-\infty}^{0} \phi(x) \, dx + \gamma_+ \int_{0}^{\infty} \phi(x) \, dx$$
### Definition

A distribution \( f \in \mathcal{D}'(\mathbb{R}) \) is said to have a **distributional jump behavior** at \( x = x_0 \in \mathbb{R} \) if it satisfies

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Let $f \in L^1_{\text{loc}}(\mathbb{R})$, we say that it has a first order jump behavior if for some constants $\gamma_{\pm}$

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\lim_{h \to 0^\pm} \frac{1}{h} \int_{0}^{h} f(x_0 + t)dt = \gamma_{\pm}
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The above definition still makes sense even if $f$ is not locally (Lebesgue) integrable but just Denjoy locally integrable.
First order jump behavior

Example

Let \( f \in L^1_{\text{loc}}(\mathbb{R}) \), we say that it has a first order jump behavior if for some constants \( \gamma_{\pm} \)

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The above definition still makes sense even if \( f \) is not locally (Lebesgue) integrable but just Denjoy locally integrable.
Let $f \in S'(\mathbb{R})$ have the distributional jump behavior at $x = x_0$. Consider a decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$ where

$$\text{supp } \hat{f}_- \subseteq (-\infty, 0] \quad \text{and} \quad \text{supp } \hat{f}_+ \subseteq [0, \infty)$$

Then, there exists $k \in \mathbb{N}$ such that $e^{ix_0 t} \hat{f}_\pm * t_\pm^k$ are continuous and

$$\left( e^{ix_0 t} \hat{f}_\pm(t) * t_\pm^k \right)(x) \sim \pm [f]_{x=x_0} \frac{|x|^k}{i} \log |x|, \quad \text{as } |x| \to \infty,$$

in the ordinary sense.
A function $U$, harmonic on $\mathbb{R}^+$, is called a harmonic representation of $f$ if in the weak topology of $\mathcal{D}'(\mathbb{R})$

$$\lim_{y\to 0^+} U(x + iy) = f(x)$$

**Theorem**

Let $f \in \mathcal{D}'(\mathbb{R})$ have jump behavior at $x = x_0$ and $V$ be a harmonic conjugate to a harmonic representation of it. Then

$$V(z) \sim \frac{1}{\pi} [f]_{x=x_0} \log |z - x_0|, \quad z \to x_0$$

on angular regions of the form $\eta < \arg(z - x_0) < \pi - \eta$, $0 < \eta \leq \pi/2$
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Symmetric jump behavior

The symmetric jump of a distribution $f \in D'(\mathbb{R})$ at a point $x = x_0$ is studied by means of the jump distribution at $x = x_0$ defined by

$$\psi_{x_0}(x) = f(x_0 + x) - f(x_0 - x)$$

**Definition**

A distribution $f$ is said to have a symmetric jump behavior at $x = x_0$ if the jump distribution $\psi_{x_0}$ has jump behavior at $x = 0$. In such a case, we define the symmetric jump of $f$ at $x = x_0$ as the number $[f]_{x=x_0} = [\psi_{x_0}]_{x=0}/2$.

Since $\psi_{x_0}$ is an odd distribution, it is easy to see that the jump behavior of $\psi_{x_0}$ is of the form

$$\lim_{\epsilon \to 0^+} \psi_{x_0}(\epsilon x) = [f]_{x=x_0} \, \text{sgn} x$$
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Corollary

Let \( f \in S'(\mathbb{R}) \) be a \( 2\pi \)-periodic distribution with Fourier series

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\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
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If \( f \) has a symmetric jump at \( x = x_0 \), then

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\lim_{r \to 1^-} \frac{1}{\log(1 - r)} \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0) r^n = \frac{1}{\pi} [f]_{x=x_0}
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If \( f \) has a symmetric jump behavior at \( x = x_0 \), then there is a \( k \in \mathbb{N} \) such that

\[
\lim_{x \to \infty} \frac{1}{\log x} \sum_{0<n<x} (a_n \sin nx_0 - b_n \cos nx_0) \left(1 - \frac{n}{x}\right)^k = -\frac{1}{\pi} [f]_{x=x_0}
\]