On the Jump Behavior of Distributions and Logarithmic Averages

Jasson Vindas
jvindas@math.lsu.edu

Department of Mathematics
Louisiana State University

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A theorem of Ferenc Lukács (1920)

Let $f \in L^1[-\pi, \pi]$ having Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Its conjugate series is defined as $\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)$

A classical theorem of F. Lukács states that if there is d such that

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h |f(x_0 + t) - f(x_0 - t) - d| \, \mathrm{d}t = 0$$

then

$$\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} (a_n \sin nx_0 - b_n \cos nx_0) = -\frac{d}{\pi}$$

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A recent generalizations
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F. Móricz has generalized this result by extending the notion of symmetric jump to the existence of

$$d = \lim_{h \to 0^+} \frac{1}{h} \int_0^h (f(x_0 + t) - f(x_0 - t))\, \mathrm{d}t$$

He gave the formula for jumps using logarithmic Abel-Poisson means of the conjugate series,

$$\lim_{r \to 1^-} \frac{1}{\log(1-r)} \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0) r^n = \frac{1}{\pi} d$$

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Outline

Distributional jump behavior

Definition

A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to have a distributional jump behavior at $x = x_0 \in \mathbb{R}$ if it satisfies

$$\lim_{\epsilon \to 0^+} f(x_0 + \epsilon x) = \gamma_- H(-x) + \gamma_+ H(x) \qquad \textit{in } \mathcal{D}'(\mathbb{R})$$

The jump is defined as the number $[f]_{x=x_0} = \gamma_+ - \gamma_-$

The meaning of the above limit is in the weak topology of $\mathcal{D}'(\mathbb{R})$, that is, for all $\phi \in \mathcal{D}(\mathbb{R})$

$$\lim_{\epsilon \to 0^+} \langle f(x_0 + \epsilon x), \phi(x) \rangle = \gamma_- \int_{-\infty}^{0} \phi(x)\,\mathrm{d}x + \gamma_+ \int_{0}^{\infty} \phi(x)\,\mathrm{d}x$$

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First order jump behavior

Example

- Let $f \in L^1_{\text{loc}}(\mathbb{R})$, we say that it has a first order jump behavior if for some constants $\gamma_\pm$

$$\lim_{h \to 0^\pm} \frac{1}{h} \int_0^h f(x_0 + t)\mathrm{d}t = \gamma_\pm$$

- The above definition still makes sense even if f is not locally (Lebesgue) integrable but just Denjoy locally integrable

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Jumps and logarithmic Cesàro means

distributional version of logarithmic-Cesàro means of conjugate series

Theorem

Let $f \in \mathcal{S}'(\mathbb{R})$ have the distributional jump behavior at $x = x_0$. Consider a decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$ where

$$\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0] \quad \textit{and} \quad \operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$$

Then, there exists $k \in \mathbb{N}$ such that $e^{ix_0 t}\hat{f}_\pm * t_\pm^k$ are continuous and

$$\left(e^{ix_0 t}\hat{f}_\pm(t) * t_\pm^k\right)(x) \sim \pm[f]_{x=x_0}\frac{|x|^k}{i}\log|x|, \quad \textit{as } |x| \to \infty,$$

in the ordinary sense

Jumps and angular behavior of harmonic conjugates

distributional version of Abel-Poisson means of conjugate series

A function U, harmonic on $\Im m\, z > 0$, is called a harmonic representation of f if in the weak topology of $\mathcal{D}'(\mathbb{R})$

$$\lim_{y \to 0^+} U(x + iy) = f(x)$$

Theorem

Let $f \in \mathcal{D}'(\mathbb{R})$ have jump behavior at $x = x_0$ and V be a harmonic conjugate to a harmonic representation of it. Then

$$V(z) \sim \frac{1}{\pi}[f]_{x=x_0} \log |z - x_0| \ , \quad z \to x_0$$

on angular regions of the form $\eta < \arg(z - x_0) < \pi - \eta$, $0 < \eta \leq \pi/2$

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Symmetric jump behavior

The symmetric jump of a distribution $f \in \mathcal{D}'(\mathbb{R})$ at a point $x = x_0$ is studied by means of the jump distribution at $x = x_0$ defined by

$$\psi_{x_0}(x) = f(x_0 + x) - f(x_0 - x)$$

Definition

A distribution f is said to have a symmetric jump behavior at $x = x_0$ if the jump distribution ψ_{x_0} has jump behavior at $x = 0$. In such a case, we define the symmetric jump of f at $x = x_0$ as the number $[f]_{x=x_0} = [\psi_{x_0}]_{x=0}/2$.

Since ψ_{x_0} is an odd distribution, it is easy to see that the jump behavior of ψ_{x_0} is of the form

$$\lim_{\epsilon \to 0^+} \psi_{x_0}(\epsilon x) = [f]_{x=x_0} \operatorname{sgn} x$$

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Abel-Poisson means of the conjugate series

Applications to Fourier series

Corollary

Let $f \in \mathcal{S}'(\mathbb{R})$ be a 2π-periodic distribution with Fourier series

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If f has a symmetric jump at $x = x_0$, then

$$\lim_{r \to 1^-} \frac{1}{\log(1-r)} \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0)\, r^n = \frac{1}{\pi} [f]_{x=x_0}$$

Logarithmic Riesz means of the conjugate series

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If f has a symmetric jump behavior at $x = x_0$, then there is a $k \in \mathbb{N}$ such that

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{0<n<x} (a_n \sin nx_0 - b_n \cos nx_0) \left(1 - \frac{n}{x}\right)^k = -\frac{1}{\pi} [f]_{x=x_0}$$