Applications of the $\phi-$transform Tauberian theorems. Part II

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Generalized Functions in PDE, Geometry, Stochastics and Microlocal Analysis
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We will consider various applications of the $\phi-$transform of tempered distributions, defined as

$$F_{\phi}f(x,y) = \langle f(x+yt), \phi(t) \rangle, \quad (x,y) \in \mathbb{H}^{n+1},$$

where $f \in \mathcal{S}'(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$, and $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$.

- The essential assumption will be

$$\int_{\mathbb{R}^n} \phi(t) \mathrm{d}t = 1.$$

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Outline

Weak-asymptotics (by dilation)

In the next definitions L is a Karamata slowly varying function.

Definition

Let $f \in \mathcal{S}'(\mathbb{R}^n)$.

- It has weak-asymptotic behavior at 0 (resp. at infinity) if $\exists g \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon \to 0^+} \frac{f(\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)} = g(t) \quad \left(\text{resp. } \lim_{\lambda \to \infty} \frac{f(\lambda t)}{\lambda^\alpha L(\lambda)} = g(t) \right) \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

- It is weak-asymptotically bounded at 0 (resp. at infinity) if the net

$$\left\{ \frac{f(\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)} \right\}_{0<\varepsilon<1} \quad \left(\text{resp. } \left\{ \frac{f(\lambda t)}{\lambda^\alpha L(\lambda)} \right\}_{1<\lambda<\infty} \right)$$

is weakly bounded in $\mathcal{S}'(\mathbb{R}^n)$.

Weak-asymptotics

Notation

We write:

- For weak-asymptotic behavior

$$f(\varepsilon t) \sim \varepsilon^{\alpha} L(\varepsilon) g(t) \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \ \text{ as } \varepsilon \to 0^+$$

$$f(\lambda t) \sim \lambda^{\alpha} L(\lambda) g(t) \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \ \text{ as } \lambda \to \infty$$

- For weak-asymptotic boundedness

$$f(\varepsilon t) = O(\varepsilon^{\alpha} L(\varepsilon)) \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \ \text{ as } \varepsilon \to 0^+$$

$$f(\lambda t) = O(\lambda^{\alpha} L(\lambda)) \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \ \text{ as } \lambda \to \infty$$

Tauberian theorems for the $\phi-$transform

Weak-asymptotic behavior

Theorem

f has weak-asymptotic behavior if and only if

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{\alpha}L(\varepsilon)}F_{\phi}f(\varepsilon x,\varepsilon y)=F_{x,y}\ ,\quad \text{for each } |x|^2+y^2=1,\ y>0,\quad (1)$$

$$\left(\text{resp. } \lim_{\lambda\to\infty}\frac{1}{\lambda^{\alpha}L(\lambda)}F_{\phi}f(\lambda x,\lambda y)=F_{x,y}\right)$$

$$\limsup_{\varepsilon\to 0^+}\sup_{|x|^2+y^2=1,\ y>0}\frac{y^k}{\varepsilon^{\alpha}L(\varepsilon)}\left|F_{\phi}f\left(\varepsilon x,\varepsilon y\right)\right|<\infty,\quad \text{for some } k\in\mathbb{N},\quad (2)$$

$$\left(\text{resp. } \limsup_{\lambda\to\infty}\sup_{|x|^2+y^2=1,\ y>0}\frac{y^k}{\lambda^{\alpha}L(\lambda)}\left|F_{\phi}f\left(\lambda x,\lambda y\right)\right|<\infty\right).$$

In such a case, g is completely determined by $F_{\phi}g(x,y)=F_{x,y}$.

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Weak-asymptotic boundedness

Theorem

The estimate

$$\limsup_{\varepsilon\to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} |F_\phi f(\varepsilon x, \varepsilon y)| < \infty, \quad \textit{for some } k \in \mathbb{N},$$

$$\left(\textit{resp. } \limsup_{\lambda\to\infty} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\lambda^\alpha L(\lambda)} |F_\phi f(\lambda x, \lambda y)| < \infty \right)$$

is necessary and sufficient for f to be weak-asymptotically bounded, namely, as $\varepsilon \to 0^+$ (resp. $\lambda \to \infty$)

$$f(\varepsilon t) = O(\varepsilon^\alpha L(\varepsilon)) \quad (\textit{resp. } f(\lambda t) = O(\lambda^\alpha L(\lambda))\,) \quad \textit{in } \mathcal{S}'(\mathbb{R}^n).$$

Summability of series

Let $\sum_{n=0}^{\infty} c_n$ be a numerical series. We are interested in divergent series. Let ρ be a function. We write

$$\sum_{n=0}^{\infty} c_n = \beta \quad (\rho)$$

if $\sum_{n=0}^{\infty} c_n\, \rho(\varepsilon n)$ is convergent for small ε and

$$\lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n\, \rho(\varepsilon n) = \beta.$$

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Examples of summability methods

- Cesàro summability if $\rho(u) = (1-u)^k \chi_{[0,1]}(u)$. Notation:

$$\sum_{n=0}^{\infty} c_n = \beta \quad (\mathrm{C}, k).$$

  Connected with Fourier series.

- If $\rho(u) = e^{-u}$, we obtain Abel summability. Notation:

$$\sum_{n=0}^{\infty} c_n = \beta \quad (\mathrm{A}).$$

  Related to radial behavior of analytic functions.

- If $\rho(u) = \frac{u}{e^u - 1}$, we obtain Lambert summability. One writes:

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  Important in combinatorics and prime number theory.

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A typical Tauberian question

Our setting will be:

- $\rho \in \mathcal{S}(\mathbb{R})$, many important kernels are of this form
- $\{c_n\}_{n=0}^{\infty}$ has at most polynomial growth
- Observe $\sum_{n=0}^{\infty} c_n = \beta \Rightarrow \rho$-summability (Abelian theorem)
- Tauberian question: Is it possible to go back to convergence under an additional hypothesis?

A Tauberian theorem for series looks like

$$\rho\text{-summability \& Tauberian hypothesis} \Rightarrow \sum_{n=0}^{\infty} c_n = \beta$$

A typical Tauberian hypothesis is $c_n = O(1/n)$.

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The role of distributional point values

Point values of distributions were defined in Pilipović's lecture. We set $f(x)=\sum_{n=0}^{\infty}c_n e^{inx}$.

Lemma

$f(0)=\beta$, distributionally, if and only if $\sum_{n=0}^{\infty}c_n$ is ρ-summable to $\beta\rho(0)$, $\forall\rho\in\mathcal{S}(\mathbb{R})$.

Proof: Set $\varphi=\frac{1}{2\pi}\hat{\rho}\in\mathcal{S}(\mathbb{R})$, namely, $\rho(u)=\int_{-\infty}^{\infty}\varphi(t)e^{iut}\mathrm{d}t$. Then,

$$\sum_{n=0}^{\infty}c_n\rho(\varepsilon n)=\sum_{n=0}^{\infty}c_n\int_{-\infty}^{\infty}\varphi(t)e^{i\varepsilon nt}\mathrm{d}t=\langle f(\varepsilon t),\varphi(t)\rangle$$

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$$\sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \sum_{n=0}^{\infty} c_n \int_{-\infty}^{\infty} \varphi(t) e^{i\varepsilon n t} \mathrm{d}t = \langle f(\varepsilon t), \varphi(t) \rangle$$

The role of point values for convergence

Theorem

Suppose $f(0) = \beta$, distributionally. Then, the Tauberian condition $c_n = O(1/n)$ implies that $\sum_{n=0}^{\infty} c_n = \beta$.

Proof: From the last lemma $\lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \beta$, if $\rho(0) = 1$.
If we were able to replace $\rho(u) = \chi_{[0,1]}(u)$, we would have

$$\lim_{\varepsilon \to 0^+} \sum_{0 \leq n \leq \frac{1}{\varepsilon}} c_n = \beta.$$

Fix arbitrary $\sigma > 1$. Choose $0 \leq \rho < 1$ such that $\operatorname{supp} \rho \subseteq [0, \sigma]$ and $\rho(u) = 1$ for $u \in [0, 1]$. Then

$$\limsup_{\varepsilon \to 0^+} \left| \sum_{0 \leq n \leq \frac{1}{\varepsilon}} c_n - \beta \right| \leq \limsup_{\varepsilon \to 0^+} \left| \sum_{1 < \varepsilon n \leq \sigma} c_n \rho(\varepsilon n) \right| < (\sigma - 1) O(1).$$

Since σ was arbitrary, we conclude the convergence of the series.

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The connection with the $\phi-$transform

- Suppose that

$$\lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \beta$$

- Set $\varphi = \frac{1}{2\pi}\hat{\rho} \in \mathcal{S}(\mathbb{R})$
- Then,

$$\sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \langle f(\varepsilon t), \phi(t) \rangle = F_\phi f(0, \varepsilon)$$

- Therefore, ρ-summability is equivalent to the boundary limit

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Tauberian theorem for distributional point values

Theorem

$f(0) = \beta$, *distributionally, if and only if*

1. *there exist* $k \in \mathbb{N}$ *and* $M > 0$ *such that*

$$|F_\phi f(\varepsilon x, \varepsilon y)| \leq \frac{M}{y^k}, \ |x|^2 + y^2 = 1, \ 0 < \varepsilon < 1.$$

2. *and, for each* $|x|^2 + y^2 = 1$,

$$\lim_{\varepsilon \to 0^+} F_\phi f(\varepsilon x, \varepsilon y) = \beta.$$

Application: Littlewood's Tauberian theorem

Theorem (Littlewood, 1912)

Suppose that

$$\lim_{\varepsilon\to 0^+}\sum_{n=0}^{\infty} c_n e^{-\varepsilon n}=\beta.$$

The Tauberian condition $c_n=O(1/n)$ implies $\sum_{n=0}^{\infty} c_n=\beta$.

Proof: Choose ϕ in such a way

$$F(z):=\sum_{n=0}^{\infty} c_n e^{-yn+ix}=F_{\phi}f(x,y)=,\ z=x+iy,\ y>0.$$

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Proof of Littlewood's theorem (continuation):

We check the two conditions of the previous theorem, i.e.,

- The estimate: for $|x|^2 + y^2 = 1$, and $0 < \varepsilon \leq 1$:

$$|F(\varepsilon x + i\varepsilon y)| < |F(\varepsilon x + i\varepsilon y) - F(i\varepsilon y)| + F(i\varepsilon y) \leq O(1) + \left|\sum_{n=0}^{\infty} c_n e^{-\varepsilon y n} e^{i\varepsilon x n}\right|$$

$$\leq O(1) + O(1)\sum_{n=0}^{\infty} \frac{e^{-\varepsilon y n}}{n}\left|e^{i\varepsilon x n} - 1\right| < O(1) + O(1)\varepsilon \sum_{n=0}^{\infty} e^{-\varepsilon y n}$$

$$= \frac{O(1)}{y}$$

- Since F is analytic, bounded on cones with vertex at the origin, and has a radial limit, we must have that it has β as non-tangential boundary value, namely, for each $z \in \mathbb{H}^2$,

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Laplace transform

Let Γ be a closed convex acute cone with vertex at the origin. Acute means that the conjugate cone

$$\Gamma^* = \{\xi \in \mathbb{R}^n : \ \xi \cdot u \geq 0, \forall u \in \Gamma\} \text{ has non-empty interior.}$$

Set

$$\mathcal{S}'_\Gamma = \left\{h \in \mathcal{S}'(\mathbb{R}^n) : \ \operatorname{supp} h \subseteq \Gamma\right\}$$

$$C_\Gamma = \operatorname{int} \Gamma^* \text{ and } T^{C_\Gamma} = \mathbb{R}^n + iC_\Gamma.$$

Given $h \in \mathcal{S}'_\Gamma$, its Laplace transform is defined as

$$\mathcal{L}\{h; z\} = \left\langle h(u), e^{iz\cdot u}\right\rangle, \quad z \in T^{C_\Gamma};$$

it is a holomorphic function on the tube domain T^{C_Γ}.

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Laplace transforms as $\phi-$transforms

We may express the Laplace transform as a $\phi-$transform if we fix a direction in C_Γ.

- Fix $\omega \in C_\Gamma$
- Choose $\eta_\omega \in \mathcal{S}(\mathbb{R}^n)$ such that $\eta_\omega(u) = e^{-\omega \cdot u}$, $\forall u \in \Gamma$
- Set

$$\phi_\omega = 1/(2\pi)^n \hat{\eta}_\omega \text{ and } \hat{f} = (2\pi)^n h$$

Then,

$$\mathcal{L}\{h; x + i\sigma\omega\} = F_{\phi_\omega} f(x, \sigma), \quad x \in \mathbb{R}^n,\ \sigma \in \mathbb{R}_+.$$

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Tauberian theorem for Laplace Transforms

Corollary

Let $h \in \mathcal{S}'_\Gamma$. Then, an estimate (for some $\omega \in C_\Gamma$, $k \in \mathbb{N}$)

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2+\sigma^2=1} \frac{\sigma^k \varepsilon^{n+\alpha}}{L(1/\varepsilon)} \left|\mathcal{L}\left\{h; \varepsilon\left(x+i\sigma\omega\right)\right\}\right| < \infty, \tag{3}$$

and the existence of an open subcone $C' \subset C_\Gamma$ such that

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon^{\alpha+n}}{L(1/\varepsilon)} \mathcal{L}\left\{h; i\varepsilon\xi\right\} = G(i\xi), \quad \text{for each } \xi \in C', \tag{4}$$

are necessary and sufficient for

$$h(\lambda u) \sim \lambda^\alpha L(\lambda) g(u) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}^n), \quad \text{for some } g \in \mathcal{S}'_\Gamma.$$

In such a case $G(z) = \mathcal{L}\{g; z\}$, $z \in T^{C_\Gamma}$.

Boundary behavior of holomorphic functions

The last corollary may be reformulated in terms of boundary behavior of holomorphic functions.

- Let $F(z)$ be holomorphic on the tube domain T^{C_Γ}
- Suppose F admits a boundary distribution

$$f(x) = F(x + i0^+) \in \mathcal{S}'(\mathbb{R}^n)$$

- Assume F satisfies a "tempered growth condition" (i.e., it belogs to the Vladimirov algebra).

Then f has weak-asymptotic behavior at 0 if and only if

- $F(i\varepsilon\xi)$ has the same kind of asymptotics for $\xi \in C' \subseteq C_\Gamma$
- There is a direction $\omega \in C_\Gamma$ such that $F(\varepsilon x + i\varepsilon\sigma\omega)$ satisfies a certain estimate (in fact, the same as in the Tauberians for the $\phi-$transform!)

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- Let $F(z)$ be holomorphic on the tube domain T^{C_Γ}
- Suppose F admits a boundary distribution

$$f(x) = F(x + i0^+) \in \mathcal{S}'(\mathbb{R}^n)$$

- Assume F satisfies a "tempered growth condition" (i.e., it belogs to the Vladimirov algebra).

Then f has weak-asymptotic behavior at 0 if and only if

- $F(i\varepsilon\xi)$ has the same kind of asymptotics for $\xi \in C' \subseteq C_\Gamma$
- There is a direction $\omega \in C_\Gamma$ such that $F(\varepsilon x + i\varepsilon\sigma\omega)$ satisfies a certain estimate (in fact, the same as in the Tauberians for the $\phi-$transform!)

A Generalized Cauchy problem

We will consider the Cauchy problem

$$\frac{\partial}{\partial t}U(x,t) = P\left(\frac{\partial}{\partial x}\right)U(x,t),$$

$$\lim_{t\to 0^+} U(x,t) = f(x) \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

- $\Gamma \subseteq \mathbb{R}^n$ is a closed convex cone with vertex at the origin. Possible situation: $\Gamma = \mathbb{R}^n$.
- P is a homogeneous polynomial of degree d. Assume:

$$\Re e\, P(iu) < 0, \quad u \in \Gamma,\ u \neq 0.$$

- $f \in \mathcal{S}'(\mathbb{R}^n)$. Assume $\operatorname{supp} \hat{f} \subseteq \Gamma$.

Asymptotic stabilization in time for solutions

We ask for conditions which ensure the existence of a function $T:(A,\infty)\to\mathbb{R}_+$ such that the following limit exists

$$\lim_{t\to\infty}\frac{U(x,t)}{T(t)}=\ell,$$

uniformly for x in compacts of $\mathbb{R}^n$.

Generalized Cauchy problem
Solution

If U is required to have slow growth over $\mathbb{H}^{n+1}$, i.e.,

$$\sup_{(x,t)\in\mathbb{H}^{n+1}} |U(x,t)| \left(t+\frac{1}{t}\right)^{-k_1} (1+|x|)^{-k_2} < \infty, \quad \text{for some } k_1, k_2 \in \mathbb{N},$$

then the Cauchy problem has a unique solution. Moreover,

$$U(x,t) = \frac{1}{(2\pi)^n} \left\langle f(u), e^{ix\cdot u} e^{P\left(it^{1/d}u\right)} \right\rangle .$$

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Relation with the $\phi-$transform

Choose a test function $\eta \in \mathcal{S}(\mathbb{R}^n)$ with the property

$$\eta(u) = e^{P(iu)}, \text{ for } u \in \Gamma;$$

setting $\phi(\xi) = (2\pi)^{-n}\hat{\eta}(\xi)$, we express U as a $\phi-$transform,

$$U(x,t) = \left\langle f(\xi), \frac{1}{t^{n/d}}\phi\left(\frac{\xi - x}{t^{1/d}}\right)\right\rangle = F_{\phi}f(x,y), \quad \text{with } y = t^{1/d},$$

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Stabilization along d-curves

We say U *stabilizes along d-curves* (at infinity), relative to $\lambda^{\alpha}L(\lambda)$, if the following two conditions hold:

1. there exist the limits
$$\lim_{\lambda\to\infty}\frac{U(\lambda x,\lambda^{d}t)}{\lambda^{\alpha}L(\lambda)}=U_0(x,t),\quad (x,t)\in\mathbb{H}^{n+1};$$
2. there are constants $C\in\mathbb{R}_+$ and $k\in\mathbb{N}$ such that
$$\left|\frac{U(\lambda x,\lambda^{d}t)}{\lambda^{\alpha}L(\lambda)}\right|\leq\frac{M}{t^{k}},\quad |x|^2+t^2,\ t>0.$$

Stabilization in time for Cauchy problems

Theorem

The solution U to the Cauchy problem stabilizes along d-curves if and only if f has weak-asymptotic behavior at infinity, relative to $\lambda^{\alpha}L(\lambda)$.

Corollary

If U stabilizes along d-curves, relative to $\lambda^{\alpha}L(\lambda)$, then U stabilizes in time with respect to $T(t) = t^{\alpha/d}L(t^{1/d})$. That is, there is a constant ℓ such that

$$\lim_{t\to\infty}\frac{U(x,t)}{T(t)} = \ell,$$

uniformly for x in compacts of $\mathbb{R}^n$.

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Example: The heat equation

We immediately recover a result of Drozhzhinov and Zavialov for the heat equation.

Let U be the solution to the Cauchy problem (here actually $\Gamma = \mathbb{R}^n$)

$$\frac{\partial}{\partial t} U = \Delta_x U,$$

$$\lim_{t \to 0^+} U(x,t) = f(x) \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

If U stabilizes along parabolas (i.e., d=2), then it stabilizes in time.

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