Applications of the $\phi$–transform
Tauberian theorems. Part II

Jasson Vindas $^1$ (and Stevan Pilipović $^2$)

jvindas@cage.UGent.be

$^1$ Department of Mathematics, Ghent University
$^2$ Department of Mathematics and Informatics, University of Novi Sad

Generalized Functions in PDE, Geometry, Stochastics and Microlocal Analysis
(September 04, 2010, Novi Sad)
We will consider various applications of the $\phi-$transform of tempered distributions, defined as

$$F_{\phi}f(x, y) = \langle f(x + yt), \phi(t) \rangle, \quad (x, y) \in \mathbb{H}^{n+1},$$

where $f \in S'(\mathbb{R}^n)$, $\phi \in S(\mathbb{R}^n)$, and $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$. The essential assumption will be

$$\int_{\mathbb{R}^n} \phi(t) dt = 1.$$
We will consider various applications of the $\phi-$transform of tempered distributions, defined as

$$F_\phi f(x, y) = \langle f(x + yt), \phi(t) \rangle, \quad (x, y) \in \mathbb{H}^{n+1},$$

where $f \in S'(\mathbb{R}^n)$, $\phi \in S(\mathbb{R}^n)$, and $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$. The essential assumption will be

$$\int_{\mathbb{R}^n} \phi(t)dt = 1.$$
Outline

1. Summability of numerical series and Tauberian theorems
2. Boundary behavior of holomorphic functions
3. Stabilization in time for Cauchy problems
Weak-asymptotics (by dilation)

In the next definitions $L$ is a Karamata slowly varying function.

**Definition**

Let $f \in S'(\mathbb{R}^n)$.

- It has *weak-asymptotic behavior* at 0 (resp. at infinity) if
  \[ \exists g \in S'(\mathbb{R}^n) \text{ such that } \lim_{\varepsilon \to 0^+} \varepsilon^{-\alpha} L(\varepsilon) \frac{f(\varepsilon t)}{\varepsilon^{-\alpha} L(\varepsilon)} = g(t) \quad \text{(resp. } \lim_{\lambda \to \infty} \frac{f(\lambda t)}{\lambda^{-\alpha} L(\lambda)} = g(t)) \text{ in } S'(\mathbb{R}^n). \]

- It is *weak-asymptotically bounded* at 0 (resp. at infinity) if the net
  \[ \left\{ \frac{f(\varepsilon t)}{\varepsilon^{-\alpha} L(\varepsilon)} \right\}_{0 < \varepsilon < 1} \quad \text{(resp. } \left\{ \frac{f(\lambda t)}{\lambda^{-\alpha} L(\lambda)} \right\}_{1 < \lambda < \infty}) \]
  is weakly bounded in $S'(\mathbb{R}^n)$. 
We write:

- For weak-asymptotic behavior
  \[ f(\varepsilon t) \sim \varepsilon^\alpha L(\varepsilon)g(t) \quad \text{in } S'(\mathbb{R}^n) \text{ as } \varepsilon \to 0^+ \]
  \[ f(\lambda t) \sim \lambda^\alpha L(\lambda)g(t) \quad \text{in } S'(\mathbb{R}^n) \text{ as } \lambda \to \infty \]

- For weak-asymptotic boundedness
  \[ f(\varepsilon t) = O(\varepsilon^\alpha L(\varepsilon)) \quad \text{in } S'(\mathbb{R}^n) \text{ as } \varepsilon \to 0^+ \]
  \[ f(\lambda t) = O(\lambda^\alpha L(\lambda)) \quad \text{in } S'(\mathbb{R}^n) \text{ as } \lambda \to \infty \]
**Theorem**

A function $f$ has weak-asymptotic behavior if and only if

$$
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} F_\phi f(\varepsilon x, \varepsilon y) = F_{x,y}, \quad \text{for each } |x|^2 + y^2 = 1, \ y > 0, \ (1)
$$

(respectively,

$$
\lim_{\lambda \to \infty} \frac{1}{\lambda^\alpha L(\lambda)} F_\phi f(\lambda x, \lambda y) = F_{x,y},
$$

$$
\limsup_{\varepsilon \to 0^+} \sup_{|x|^2 + y^2 = 1, \ y > 0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} |F_\phi f(\varepsilon x, \varepsilon y)| < \infty, \quad \text{for some } k \in \mathbb{N}, \ (2)
$$

(respectively,

$$
\limsup_{\lambda \to \infty} \sup_{|x|^2 + y^2 = 1, \ y > 0} \frac{y^k}{\lambda^\alpha L(\lambda)} |F_\phi f(\lambda x, \lambda y)| < \infty).
$$

In such a case, $g$ is completely determined by $F_\phi g(x, y) = F_{x,y}$. 

*Jasson Vindas*  
Applications of the $\phi$-transform
Tauberian theorems for the $\phi-$transform

Weak-asymptotic boundedness

**Theorem**

The estimate

$$
\limsup_{\varepsilon \to 0^+} \sup_{|x|^2 + y^2 = 1, \; y > 0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} |F_\phi f(\varepsilon x, \varepsilon y)| < \infty, \text{ for some } k \in \mathbb{N},
$$

$$
\left( \text{resp. } \limsup_{\lambda \to \infty} \sup_{|x|^2 + y^2 = 1, \; y > 0} \frac{y^k}{\lambda^\alpha L(\lambda)} |F_\phi f(\lambda x, \lambda y)| < \infty \right)
$$

is necessary and sufficient for $f$ to be weak-asymptotically bounded, namely, as $\varepsilon \to 0^+$ (resp. $\lambda \to \infty$)

$$
f(\varepsilon t) = O(\varepsilon^\alpha L(\varepsilon)) \quad (\text{resp. } f(\lambda t) = O(\lambda^\alpha L(\lambda))) \quad \text{in } S'(\mathbb{R}^n).
$$
Let $\sum_{n=0}^{\infty} c_n$ be a numerical series. We are interested in **divergent** series. Let $\rho$ be a function. We write

$$\sum_{n=0}^{\infty} c_n = \beta \ (\rho)$$

if $\sum_{n=0}^{\infty} c_n \rho(\varepsilon n)$ is convergent for small $\varepsilon$ and

$$\lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \beta.$$ 

In such a case we say that the series is $\rho$-summable.
Let \( \sum_{n=0}^{\infty} c_n \) be a numerical series. We are interested in divergent series. Let \( \rho \) be a function. We write

\[
\sum_{n=0}^{\infty} c_n = \beta \quad (\rho)
\]

if \( \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) \) is convergent for small \( \varepsilon \) and

\[
\lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \beta.
\]

In such a case we say that the series is \( \rho \)-summable.
Let $\sum_{n=0}^{\infty} c_n$ be a numerical series. We are interested in divergent series. Let $\rho$ be a function. We write

$$\sum_{n=0}^{\infty} c_n = \beta \ (\rho)$$

if $\sum_{n=0}^{\infty} c_n \rho(\varepsilon n)$ is convergent for small $\varepsilon$ and

$$\lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \beta.$$ 

In such a case we say that the series is $\rho$-summable.
Let $\sum_{n=0}^{\infty} c_n$ be a numerical series. We are interested in **divergent** series. Let $\rho$ be a function. We write

$$\sum_{n=0}^{\infty} c_n = \beta \ (\rho)$$

if $\sum_{n=0}^{\infty} c_n \rho(\varepsilon n)$ is convergent for small $\varepsilon$ and

$$\lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \beta.$$ 

In such a case we say that the series is $\rho$-summable.
Examples of summability methods

- Cesàro summability if \( \rho(u) = (1 - u)^k \chi_{[0,1]}(u) \). Notation:
  \[
  \sum_{n=0}^{\infty} c_n = \beta \quad (C, k).
  \]
  Connected with Fourier series.

- If \( \rho(u) = e^{-u} \), we obtain Abel summability. Notation:
  \[
  \sum_{n=0}^{\infty} c_n = \beta \quad (A).
  \]
  Related to radial behavior of analytic functions.

- If \( \rho(u) = \frac{u}{e^u - 1} \), we obtain Lambert summability. One writes:
  \[
  \sum_{n=0}^{\infty} c_n = \beta \quad (L).
  \]
  Important in combinatorics and prime number theory.
Examples of summability methods

- Cesàro summability if $\rho(u) = (1 - u)^k \chi_{[0,1]}(u)$. Notation:

$$\sum_{n=0}^{\infty} c_n = \beta \quad (C, k).$$

Connected with Fourier series.

- If $\rho(u) = e^{-u}$, we obtain Abel summability. Notation:

$$\sum_{n=0}^{\infty} c_n = \beta \quad (A).$$

Related to radial behavior of analytic functions.

- If $\rho(u) = \frac{u}{e^u - 1}$, we obtain Lambert summability. One writes:

$$\sum_{n=0}^{\infty} c_n = \beta \quad (L).$$

Important in combinatorics and prime number theory.

Jasson Vindas

Applications of the $\phi$-transform
Examples of summability methods

- Cesàro summability if $\rho(u) = (1 - u)^k \chi_{[0,1]}(u)$. Notation:

  $$\sum_{n=0}^{\infty} c_n = \beta \quad (C, k).$$

  Connected with Fourier series.

- If $\rho(u) = e^{-u}$, we obtain Abel summability. Notation:

  $$\sum_{n=0}^{\infty} c_n = \beta \quad (A).$$

  Related to radial behavior of analytic functions.

- If $\rho(u) = \frac{u}{e^u - 1}$, we obtain Lambert summability. One writes:

  $$\sum_{n=0}^{\infty} c_n = \beta \quad (L).$$

  Important in combinatorics and prime number theory.
Examples of summability methods

- Cesàro summability if $\rho(u) = (1 - u)^k \chi_{[0,1]}(u)$. Notation:
  $$\sum_{n=0}^{\infty} c_n = \beta \quad (C, k).$$
  Connected with Fourier series.

- If $\rho(u) = e^{-u}$, we obtain Abel summability. Notation:
  $$\sum_{n=0}^{\infty} c_n = \beta \quad (A).$$
  Related to radial behavior of analytic functions.

- If $\rho(u) = \frac{u}{e^u - 1}$, we obtain Lambert summability. One writes:
  $$\sum_{n=0}^{\infty} c_n = \beta \quad (L).$$
  Important in combinatorics and prime number theory.
A typical Tauberian question

Our setting will be:

- $\rho \in S(\mathbb{R})$, many important kernels are of this form
- $\{c_n\}_{n=0}^{\infty}$ has at most polynomial growth
- Observe $\sum_{n=0}^{\infty} c_n = \beta \Rightarrow \rho$-summability (Abelian theorem)
- Tauberian question: Is it possible to go back to convergence under an additional hypothesis?

A Tauberian theorem for series looks like

$$\rho$\text{-summability} \& \text{Tauberian hypothesis} \Rightarrow \sum_{n=0}^{\infty} c_n = \beta$$

A typical Tauberian hypothesis is $c_n = O(1/n)$. 
A typical Tauberian question

Our setting will be:

- \( \rho \in S(\mathbb{R}) \), many important kernels are of this form
- \( \{c_n\}_{n=0}^{\infty} \) has at most polynomial growth
- Observe \( \sum_{n=0}^{\infty} c_n = \beta \Rightarrow \rho\text{-summability (Abelian theorem)} \)
- Tauberian question: Is it possible to go back to convergence under an additional hypothesis?

A Tauberian theorem for series looks like

\[
\rho\text{-summability } \& \text{ Tauberian hypothesis } \Rightarrow \sum_{n=0}^{\infty} c_n = \beta
\]

A typical Tauberian hypothesis is \( c_n = O(1/n) \).
A typical Tauberian question

Our setting will be:

- $\rho \in \mathcal{S}(\mathbb{R})$, many important kernels are of this form
- $\{c_n\}_{n=0}^{\infty}$ has at most polynomial growth
- Observe $\sum_{n=0}^{\infty} c_n = \beta \Rightarrow \rho$-summability (Abelian theorem)
- Tauberian question: Is it possible to go back to convergence under an additional hypothesis?

A Tauberian theorem for series looks like

$$\rho\text{-summability & Tauberian hypothesis} \Rightarrow \sum_{n=0}^{\infty} c_n = \beta$$

A typical Tauberian hypothesis is $c_n = O(1/n)$. 
Point values of distributions were defined in Pilipović’s lecture. We set \( f(x) = \sum_{n=0}^{\infty} c_n e^{inx} \).

**Lemma**

\( f(0) = \beta, \) distributionally, if and only if \( \sum_{n=0}^{\infty} c_n \) is \( \rho \)-summable to \( \beta \rho(0), \forall \rho \in S(\mathbb{R}) \).

**Proof:** Set \( \varphi = \frac{1}{2\pi} \hat{\rho} \in S(\mathbb{R}), \) namely, \( \rho(u) = \int_{-\infty}^{\infty} \varphi(t)e^{iut}dt \). Then,

\[
\sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \sum_{n=0}^{\infty} c_n \int_{-\infty}^{\infty} \varphi(t)e^{i\varepsilon nt}dt = \langle f(\varepsilon t), \varphi(t) \rangle
\]
Point values of distributions were defined in Pilipović’s lecture. We set \( f(x) = \sum_{n=0}^{\infty} c_n e^{inx} \).

**Lemma**

\( f(0) = \beta, \) distributionally, if and only if \( \sum_{n=0}^{\infty} c_n \) is \( \rho \)-summable to \( \beta \rho(0), \forall \rho \in S(\mathbb{R}) \).

**Proof:** Set \( \varphi = \frac{1}{2\pi} \hat{\rho} \in S(\mathbb{R}), \) namely, \( \rho(u) = \int_{-\infty}^{\infty} \varphi(t)e^{iut} dt \). Then,

\[
\sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \sum_{n=0}^{\infty} c_n \int_{-\infty}^{\infty} \varphi(t)e^{i\varepsilon nt} dt = \langle f(\varepsilon t), \varphi(t) \rangle
\]
The role of point values for convergence

**Theorem**

Suppose \( f(0) = \beta \), distributionally. Then, the Tauberian condition \( c_n = O(1/n) \) implies that \( \sum_{n=0}^{\infty} c_n = \beta \).

**Proof:** From the last lemma \( \lim_{\epsilon \to 0^+} \sum_{n=0}^{\infty} c_n \rho(\epsilon n) = \beta \), if \( \rho(0) = 1 \).

If we were able to replace \( \rho(u) = \chi_{[0,1]}(u) \), we would have

\[
\lim_{\epsilon \to 0^+} \sum_{0 \leq n \leq \frac{1}{\epsilon}} c_n = \beta.
\]

Fix arbitrary \( \sigma > 1 \). Choose \( 0 \leq \rho < 1 \) such that \( \text{supp} \rho \subseteq [0, \sigma] \) and \( \rho(u) = 1 \) for \( u \in [0, 1] \). Then

\[
\limsup_{\epsilon \to 0^+} \left| \sum_{0 \leq n \leq \frac{1}{\epsilon}} c_n - \beta \right| \leq \limsup_{\epsilon \to 0^+} \left| \sum_{1 < \epsilon n \leq \sigma} c_n \rho(\epsilon n) \right| < (\sigma - 1) O(1).
\]

Since \( \sigma \) was arbitrary, we conclude the convergence of the series.
The role of point values for convergence

**Theorem**

Suppose $f(0) = \beta$, distributionally. Then, the Tauberian condition $c_n = O(1/n)$ implies that $\sum_{n=0}^{\infty} c_n = \beta$.

**Proof:** From the last lemma $\lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \beta$, if $\rho(0) = 1$. If we were able to replace $\rho(u) = \chi_{[0,1]}(u)$, we would have

$$\lim_{\varepsilon \to 0^+} \sum_{0 \leq n \leq \frac{1}{\varepsilon}} c_n = \beta.$$ 

Fix arbitrary $\sigma > 1$. Choose $0 \leq \rho < 1$ such that $\text{supp } \rho \subseteq [0, \sigma]$ and $\rho(u) = 1$ for $u \in [0, 1]$. Then

$$\limsup_{\varepsilon \to 0^+} \left| \sum_{0 \leq n \leq \frac{1}{\varepsilon}} c_n - \beta \right| \leq \limsup_{\varepsilon \to 0^+} \left| \sum_{1 < \varepsilon n \leq \sigma} c_n \rho(\varepsilon n) \right| < (\sigma - 1)O(1).$$

Since $\sigma$ was arbitrary, we conclude the convergence of the series.

Jasson Vindas

Applications of the $\phi$-transform
The role of point values for convergence

Theorem

Suppose \( f(0) = \beta \), distributionally. Then, the Tauberian condition \( c_n = O(1/n) \) implies that \( \sum_{n=0}^{\infty} c_n = \beta \).

Proof: From the last lemma \( \lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \beta \), if \( \rho(0) = 1 \).

If we were able to replace \( \rho(u) = \chi_{[0,1]}(u) \), we would have

\[
\lim_{\varepsilon \to 0^+} \sum_{0 \leq n \leq \frac{1}{\varepsilon}} c_n = \beta.
\]

Fix arbitrary \( \sigma > 1 \). Choose \( 0 \leq \rho < 1 \) such that \( \text{supp} \rho \subseteq [0, \sigma] \) and \( \rho(u) = 1 \) for \( u \in [0, 1] \). Then

\[
\limsup_{\varepsilon \to 0^+} \left| \sum_{0 \leq n \leq \frac{1}{\varepsilon}} c_n - \beta \right| \leq \limsup_{\varepsilon \to 0^+} \left| \sum_{1 < \varepsilon n \leq \sigma} c_n \rho(\varepsilon n) \right| < (\sigma - 1)O(1).
\]

Since \( \sigma \) was arbitrary, we conclude the convergence of the series.
Suppose that
\[ \lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \beta \]

Set \( \varphi = \frac{1}{2\pi} \hat{\rho} \in S(\mathbb{R}) \)

Then,
\[ \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \langle f(\varepsilon t), \phi(t) \rangle = F_{\phi} f(0, \varepsilon) \]

Therefore, \( \rho \)-summability is equivalent to the boundary limit
\[ \lim_{\varepsilon \to 0^+} F_{\phi} f(0, \varepsilon) = \beta. \]
The connection with the $\phi$–transform

- Suppose that
  \[
  \lim_{{\varepsilon \to 0^+}} \sum_{{n=0}}^{\infty} c_n \rho(\varepsilon n) = \beta
  \]

- Set $\varphi = \frac{1}{2\pi} \hat{\rho} \in S(\mathbb{R})$

- Then,
  \[
  \sum_{{n=0}}^{\infty} c_n \rho(\varepsilon n) = \langle f(\varepsilon t), \phi(t) \rangle = F_\phi f(0, \varepsilon)
  \]

- Therefore, $\rho$–summability is equivalent to the boundary limit
  \[
  \lim_{{\varepsilon \to 0^+}} F_\phi f(0, \varepsilon) = \beta.
  \]
The connection with the $\phi-$transform

- Suppose that
  \[
  \lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \beta
  \]

- Set $\phi = \frac{1}{2\pi} \hat{\rho} \in S(\mathbb{R})$

- Then,
  \[
  \sum_{n=0}^{\infty} c_n \rho(\varepsilon n) = \langle f(\varepsilon t), \phi(t) \rangle = F_\phi f(0, \varepsilon)
  \]

- Therefore, $\rho$-summability is equivalent to the boundary limit
  \[
  \lim_{\varepsilon \to 0^+} F_\phi f(0, \varepsilon) = \beta.
  \]
Theorem

\( f(0) = \beta, \text{ distributionally, if and only if} \)

1. there exist \( k \in \mathbb{N} \) and \( M > 0 \) such that

\[
|F_\phi f(\varepsilon x, \varepsilon y)| \leq \frac{M}{y^k}, \quad |x|^2 + y^2 = 1, \quad 0 < \varepsilon < 1.
\]

2. and, for each \( |x|^2 + y^2 = 1, \)

\[
\lim_{\varepsilon \to 0^+} F_\phi f(\varepsilon x, \varepsilon y) = \beta.
\]
Application: Littlewood’s Tauberian theorem

Theorem (Littlewood, 1912)

Suppose that

$$\lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n e^{-\varepsilon n} = \beta.$$ 

The Tauberian condition $c_n = O(1/n)$ implies $\sum_{n=0}^{\infty} c_n = \beta$.

Proof: Choose $\phi$ in such a way

$$F(z) := \sum_{n=0}^{\infty} c_n e^{-y_n+ix} = F_\phi f(x, y) =, \, z = x + iy, \, y > 0.$$ 

We check the conditions of the previous theorem, i.e.,
Proof of Littlewood’s theorem (continuation):

We check the two conditions of the previous theorem, i.e.,

- The estimate: for $|x|^2 + y^2 = 1$, and $0 < \varepsilon \leq 1$:

$$|F(\varepsilon x + i\varepsilon y)| < |F(\varepsilon x + i\varepsilon y) - F(i\varepsilon y)| + F(i\varepsilon y) \leq O(1) + \sum_{n=0}^{\infty} c_n e^{-\varepsilon y_n} e^{i\varepsilon x_n}$$

$$\leq O(1) + O(1) \sum_{n=0}^{\infty} \frac{e^{-\varepsilon y_n}}{n} |e^{i\varepsilon x_n} - 1| < O(1) + O(1) \varepsilon \sum_{n=0}^{\infty} e^{-\varepsilon y_n}$$

$$= \frac{O(1)}{y}$$

- Since $F$ is analytic, bounded on cones with vertex at the origin, and has a radial limit, we must have that it has $\beta$ as non-tangential boundary value, namely, for each $z \in \mathbb{H}^2$,

$$\lim_{\varepsilon \to 0^+} F(\varepsilon z) = \beta.$$
Proof of Littlewood's theorem (continuation):

We check the two conditions of the previous theorem, i.e.,

- The estimate: for $|x|^2 + y^2 = 1$, and $0 < \varepsilon \leq 1$:

$$|F(\varepsilon x + i\varepsilon y)| < |F(\varepsilon x + i\varepsilon y) - F(i\varepsilon y)| + F(i\varepsilon y) \leq O(1) + \left| \sum_{n=0}^{\infty} c_n e^{-\varepsilon yn} e^{i\varepsilon xn} \right|$$

$$\leq O(1) + O(1) \sum_{n=0}^{\infty} \frac{e^{-\varepsilon yn}}{n} \left| e^{i\varepsilon xn} - 1 \right| < O(1) + O(1) \varepsilon \sum_{n=0}^{\infty} e^{-\varepsilon yn}$$

$$= \frac{O(1)}{y}$$

- Since $F$ is analytic, bounded on cones with vertex at the origin, and has a radial limit, we must have that it has $\beta$ as non-tangential boundary value, namely, for each $z \in \mathbb{H}^2$,

$$\lim_{\varepsilon \to 0^+} F(\varepsilon z) = \beta.$$
Proof of Littlewood’s theorem (continuation):
We check the two conditions of the previous theorem, i.e.,

- The estimate: for $|x|^2 + y^2 = 1$, and $0 < \varepsilon \leq 1$:

$$ |F(\varepsilon x + i\varepsilon y)| < |F(\varepsilon x + i\varepsilon y) - F(i\varepsilon y)| + F(i\varepsilon y) \leq O(1) + \left| \sum_{n=0}^{\infty} c_n e^{-\varepsilon y n} e^{i\varepsilon x n} \right| $$

$$ \leq O(1) + O(1) \sum_{n=0}^{\infty} \frac{e^{-\varepsilon y n}}{n} |e^{i\varepsilon x n} - 1| < O(1) + O(1) \varepsilon \sum_{n=0}^{\infty} e^{-\varepsilon y n} $$

$$ = \frac{O(1)}{y} $$

- Since $F$ is analytic, bounded on cones with vertex at the origin, and has a radial limit, we must have that it has $\beta$ as non-tangential boundary value, namely, for each $z \in \mathbb{H}^2$,

$$ \lim_{\varepsilon \to 0^+} F(\varepsilon z) = \beta. $$
Let $\Gamma$ be a closed convex acute cone with vertex at the origin. Acute means that the conjugate cone

$$\Gamma^* = \{ \xi \in \mathbb{R}^n : \xi \cdot u \geq 0, \forall u \in \Gamma \}$$

has non-empty interior.

Set

$$S'_\Gamma = \{ h \in S'(\mathbb{R}^n) : \text{supp } h \subseteq \Gamma \}$$

$$C_\Gamma = \text{int } \Gamma^* \text{ and } T^{C_\Gamma} = \mathbb{R}^n + iC_\Gamma.$$

Given $h \in S'_\Gamma$, its Laplace transform is defined as

$$\mathcal{L} \{ h; z \} = \left\langle h(u), e^{iz \cdot u} \right\rangle, \quad z \in T^{C_\Gamma};$$

it is a holomorphic function on the tube domain $T^{C_\Gamma}$. 
Let $\Gamma$ be a closed convex acute cone with vertex at the origin. Acute means that the conjugate cone

$$\Gamma^* = \{\xi \in \mathbb{R}^n : \xi \cdot u \geq 0, \forall u \in \Gamma\}$$

has non-empty interior.

Set

$$S'_{\Gamma} = \{h \in S'((\mathbb{R}^n) : \text{supp } h \subseteq \Gamma\}$$

$$C_{\Gamma} = \text{int }\Gamma^* \text{ and } T^{C_{\Gamma}} = \mathbb{R}^n + iC_{\Gamma}.$$

Given $h \in S'_{\Gamma}$, its Laplace transform is defined as

$$\mathcal{L}\{h; z\} = \left\langle h(u), e^{iz \cdot u}\right\rangle, \quad z \in T^{C_{\Gamma}};$$

it is a holomorphic function on the tube domain $T^{C_{\Gamma}}$. 
Laplace transforms as $\phi-$transforms

We may express the Laplace transform as a $\phi-$transform if we fix a direction in $C_\Gamma$.

- Fix $\omega \in C_\Gamma$
- Choose $\eta_\omega \in S(\mathbb{R}^n)$ such that $\eta_\omega(u) = e^{-\omega \cdot u}$, $\forall u \in \Gamma$
- Set

$$\phi_\omega = 1/(2\pi)^n \hat{\eta}_\omega \text{ and } \hat{f} = (2\pi)^n h$$

Then,

$$\mathcal{L}\{h; x + i\sigma\omega\} = F_{\phi_\omega}f(x, \sigma), \quad x \in \mathbb{R}^n, \sigma \in \mathbb{R}_+.$$
Laplace transforms as $\phi-\text{transforms}$

We may express the Laplace transform as a $\phi-$transform if we fix a direction in $C_{\Gamma}$.

- Fix $\omega \in C_{\Gamma}$
- Choose $\eta_\omega \in S(\mathbb{R}^n)$ such that $\eta_\omega(u) = e^{-\omega \cdot u}$, $\forall u \in \Gamma$
- Set $\phi_\omega = 1/(2\pi)^n \hat{\eta}_\omega$ and $\hat{f} = (2\pi)^n h$

Then,

$$\mathcal{L} \{ h; x + i\sigma \omega \} = F_{\phi_\omega} f(x, \sigma), \quad x \in \mathbb{R}^n, \ \sigma \in \mathbb{R}_+.$$
Corollary

Let $h \in S'_\Gamma$. Then, an estimate (for some $\omega \in C_\Gamma$, $k \in \mathbb{N}$)

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2 + \sigma^2 = 1} \frac{\sigma^k \varepsilon^{n+\alpha}}{L(1/\varepsilon)} \left| \mathcal{L} \{ h; \varepsilon (x + i\sigma \omega) \} \right| < \infty,$$

and the existence of an open subcone $C' \subset C_\Gamma$ such that

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon^{\alpha+n}}{L(1/\varepsilon)} \mathcal{L} \{ h; i\varepsilon \xi \} = G(i\xi), \quad \text{for each } \xi \in C',$$

are necessary and sufficient for

$$h(\lambda u) \sim \lambda^\alpha L(\lambda)g(u) \quad \text{as } \lambda \to \infty \text{ in } S'(\mathbb{R}^n), \quad \text{for some } g \in S'_\Gamma.$$

In such a case $G(z) = \mathcal{L} \{ g; z \}$, $z \in T^{C_\Gamma}$. 
Boundary behavior of holomorphic functions

The last corollary may be reformulated in terms of boundary behavior of holomorphic functions.

- Let $F(z)$ be holomorphic on the tube domain $T^{C_{\Gamma}}$.
- Suppose $F$ admits a boundary distribution
  \[ f(x) = F(x + i0^+) \in S'(\mathbb{R}^n) \]
  Assume $F$ satisfies a "tempered growth condition" (i.e., it belongs to the Vladimirov algebra).

Then $f$ has weak-asymptotic behavior at 0 if and only if

- $F(i\varepsilon \xi)$ has the same kind of asymptotics for $\xi \in C' \subseteq C_{\Gamma}$.
- There is a direction $\omega \in C_{\Gamma}$ such that $F(\varepsilon x + i\varepsilon \sigma \omega)$ satisfies a certain estimate (in fact, the same as in the Tauberians for the $\phi-$transform!)
The last corollary may be reformulated in terms of boundary behavior of holomorphic functions.

- Let $F(z)$ be holomorphic on the tube domain $T^{C_\Gamma}$
- Suppose $F$ admits a boundary distribution
  \[ f(x) = F(x + i0^+) \in S'(\mathbb{R}^n) \]

- Assume $F$ satisfies a "tempered growth condition" (i.e., it belongs to the Vladimirov algebra).

Then $f$ has weak-asymptotic behavior at 0 if and only if

- $F(i\varepsilon \xi)$ has the same kind of asymptotics for $\xi \in C' \subseteq C_\Gamma$
- There is a direction $\omega \in C_\Gamma$ such that $F(\varepsilon x + i\varepsilon \sigma \omega)$ satisfies a certain estimate (in fact, the same as in the Tauberians for the $\phi$–transform!)
A Generalized Cauchy problem

We will consider the Cauchy problem

\[
\frac{\partial}{\partial t} U(x, t) = P \left( \frac{\partial}{\partial x} \right) U(x, t),
\]

\[
\lim_{t \to 0^+} U(x, t) = f(x) \text{ in } S'(\mathbb{R}^n).
\]

- $\Gamma \subseteq \mathbb{R}^n$ is a closed convex cone with vertex at the origin. Possible situation: $\Gamma = \mathbb{R}^n$.
- $P$ is a homogeneous polynomial of degree $d$. Assume:

\[
\Re P(iu) < 0, \quad u \in \Gamma, \ u \neq 0.
\]

- $f \in S'(\mathbb{R}^n)$. Assume $\text{supp} \hat{f} \subseteq \Gamma$. 

Jasson Vindas

Applications of the $\phi$-transform
Asymptotic stabilization in time for solutions

We ask for conditions which ensure the existence of a function $T : (A, \infty) \to \mathbb{R}_+$ such that the following limit exists

$$\lim_{t \to \infty} \frac{U(x, t)}{T(t)} = \ell,$$

uniformly for $x$ in compacts of $\mathbb{R}^n$. 
If $U$ is required to have slow growth over $\mathbb{H}^{n+1}$, i.e.,

$$\sup_{(x,t) \in \mathbb{H}^{n+1}} |U(x, t)| \left( t + \frac{1}{t} \right)^{-k_1} (1 + |x|)^{-k_2} < \infty,$$

for some $k_1, k_2 \in \mathbb{N}$, then the Cauchy problem has a unique solution. Moreover,

$$U(x, t) = \frac{1}{(2\pi)^n} \left< f(u), e^{ix \cdot u} e^{P(it^{1/d}u)} \right>.$$
If $U$ is required to have slow growth over $\mathbb{H}^{n+1}$, i.e.,

$$\sup_{(x,t)\in\mathbb{H}^{n+1}} |U(x,t)| \left( t + \frac{1}{t} \right)^{-k_1} (1 + |x|)^{-k_2} < \infty,$$

for some $k_1, k_2 \in \mathbb{N}$, then the Cauchy problem has a unique solution. Moreover,

$$U(x, t) = \frac{1}{(2\pi)^n} \left\langle f(u), e^{ix \cdot u} e^{P(it^{1/d} u)} \right\rangle.$$
Choose a test function $\eta \in S(\mathbb{R}^n)$ with the property

$$\eta(u) = e^{P(iu)}, \text{ for } u \in \Gamma;$$

setting $\phi(\xi) = (2\pi)^{-n} \hat{\eta}(\xi)$, we express $U$ as a $\phi$–transform,

$$U(x, t) = \left\langle f(\xi), \frac{1}{t^{n/d}} \phi \left( \frac{\xi - x}{t^{1/d}} \right) \right\rangle = F_\phi f(x, y), \text{ with } y = t^{1/d},$$
Choose a test function $\eta \in S(\mathbb{R}^n)$ with the property

$$\eta(u) = e^{P(iu)}, \text{ for } u \in \Gamma;$$

setting $\phi(\xi) = (2\pi)^{-n} \hat{\eta}(\xi)$, we express $U$ as a $\phi$–transform,

$$U(x, t) = \left\langle f(\xi), \frac{1}{t^{n/d}} \phi \left( \frac{\xi - x}{t^{1/d}} \right) \right\rangle = F_\phi f(x, y), \text{ with } y = t^{1/d},$$
Stabilization along $d$-curves

We say $U$ stabilizes along $d$-curves (at infinity), relative to $\lambda^\alpha L(\lambda)$, if the following two conditions hold:

1. There exist the limits

$$\lim_{\lambda \to \infty} \frac{U(\lambda x, \lambda^d t)}{\lambda^\alpha L(\lambda)} = U_0(x, t), \quad (x, t) \in \mathbb{H}^{n+1};$$

2. There are constants $C \in \mathbb{R}_+$ and $k \in \mathbb{N}$ such that

$$\left| \frac{U(\lambda x, \lambda^d t)}{\lambda^\alpha L(\lambda)} \right| \leq \frac{M}{t^k}, \quad |x|^2 + t^2, \quad t > 0.$$
The solution $U$ to the Cauchy problem stabilizes along $d$-curves if and only if $f$ has weak-asymptotic behavior at infinity, relative to $\lambda^\alpha L(\lambda)$.

**Corollary**

If $U$ stabilizes along $d$-curves, relative to $\lambda^\alpha L(\lambda)$, then $U$ stabilizes in time with respect to $T(t) = t^{\alpha/d} L(t^{1/d})$. That is, there is a constant $\ell$ such that

$$\lim_{t \to \infty} \frac{U(x, t)}{T(t)} = \ell,$$

uniformly for $x$ in compacts of $\mathbb{R}^n$. 

Jasson Vindas

Applications of the $\phi$-transform
Theorem

The solution $U$ to the Cauchy problem stabilizes along $d$-curves if and only if $f$ has weak-asymptotic behavior at infinity, relative to $\lambda^\alpha L(\lambda)$.

Corollary

If $U$ stabilizes along $d$-curves, relative to $\lambda^\alpha L(\lambda)$, then $U$ stabilizes in time with respect to $T(t) = t^{\alpha/d} L(t^{1/d})$. That is, there is a constant $\ell$ such that

$$\lim_{t \to \infty} \frac{U(x, t)}{T(t)} = \ell,$$

uniformly for $x$ in compacts of $\mathbb{R}^n$. 
Example: The heat equation

We immediately recover a result of Drozhzhinov and Zavialov for the heat equation.

Let $U$ be the solution to the Cauchy problem (here actually $\Gamma = \mathbb{R}^n$)

$$\frac{\partial}{\partial t} U = \Delta_x U,$$

$$\lim_{t \to 0^+} U(x, t) = f(x) \text{ in } S'(\mathbb{R}^n).$$

If $U$ stabilizes along parabolas (i.e., $d=2$), then it stabilizes in time.
Example: The heat equation

We immediately recover a result of Drozhzhinov and Zavialov for the heat equation.
Let $U$ be the solution to the Cauchy problem (here actually $\Gamma = \mathbb{R}^n$)

$$\frac{\partial}{\partial t} U = \Delta x U,$$

$$\lim_{t \to 0^+} U(x, t) = f(x) \quad \text{in} \ S'(\mathbb{R}^n).$$

If $U$ stabilizes along parabolas (i.e., $d=2$), then it stabilizes in time.