Complex Tauberian theorems for Laplace transforms

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Universidad Nacional de Costa Rica (September 11, 2019)

Complex Tauberian theorems have been strikingly useful tools in diverse areas such as:

- Analytic number theory.
- Spectral theory for (pseudo-)differential operators.
- Last three decades: operator theory and semigroups.

We will discuss some developments in complex Tauberians for Laplace transforms. We will be concerned with two groups of statements:

- Wiener-Ikehara theorems.
- Ingham-Karamata theorems.

Main questions:

1. Relax boundary requirements to a minimum.
2. Mild Tauberian hypotheses (one-sided conditions).
3. Optimal Tauberian constants: sharp versions.
4. Best possible error terms.

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The classical Wiener-Ikehara theorem

Theorem (Wiener-Ikehara, Laplace transforms)

Let S be a non-decreasing function (Tauberian hypothesis) such that $\mathcal{L}\{S;z\} = \int_{0^-}^{\infty} e^{-zt} S(t)\,\mathrm{d}t$ converges for $\Re e\, z > 1$. If

$$\mathcal{L}\{S;z\} - \frac{A}{z-1}$$

has analytic continuation through $\Re e\, z = 1$, then $S(x) \sim Ae^{x}$.

Theorem (Wiener-Ikehara, version for Dirichlet series)

Let $a_n \geq 0$. Suppose $\sum_{n=1}^{\infty} a_n n^{-z}$ converges for $\Re e\, z > 1$. If

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From the Wiener-Ikehara theorem to the PNT:

The Prime Number Theorem (PNT) asserts that

$$\pi(x) = \sum_{p \leq x} 1 \sim \frac{x}{\log x}$$

- PNT is equivalent to $\psi(x) = \sum_{p^{\alpha} \leq x} \log p = \sum_{n \leq x} \Lambda(n) \sim x$.
- $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ has analytic continuation to $\mathbb{C}$ except for a simple pole with residue 1 at $z = 1$.
- Logarithmic differentiation of $\zeta(z) = \prod_{p}(1 - p^{-z})^{-1}$ leads to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)}, \quad \Re e\, z > 1.$$

- $(z-1)\zeta(z)$ has no zeros on $\Re e\, z = 1$, so

$$-\frac{d}{dz}(\log((z-1)\zeta(z))) = -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$$

is analytic in a region containing $\Re e\, z \geq 1$. The rest follows from the Wiener-Ikehara theorem.

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Remarks on the Wiener-Ikehara theorem

- Historically, the Wiener-Ikehara theorem improved a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(z) \ll |z|^N$ on

$$G(z) = \mathcal{L}\{S; z\} - \frac{A}{z-1}$$

- The hypothesis $G(z)$ has analytic continuation to $\Re e\, z = 1$ can be significantly relaxed to "good boundary behavior":
  1. G(z) has continuous extension to $\Re e\, z = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{x \to 1^+} G(x + iy) \in L^1(I)$ for every finite interval I.
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Pseudofunctions and pseudomeasures

Pseudofunctions and pseudomeasures are notions that naturally arise in harmonic analysis.

- Pseudomeasures: $PM(\mathbb{R}) = \{g : \widehat{g} \in L^{\infty}(\mathbb{R})\}$
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in PM(\mathbb{R}) : \lim_{|x|\to\infty} \widehat{g}(x) = 0\}$

Given an open set $U \subseteq \mathbb{R}$, we define the local spaces:

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- Every Radon measure is a local pseudomeasure.

Let G be analytic on $\Re e\, z > \alpha$ and $U \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $\alpha + iU$ if it has distributional boundary values there, i.e.

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Extension of the Korevaar-Wiener-Ikehara theorem

We call a function S log-linearly slowly decreasing if for each $\varepsilon > 0$ there exists $\delta > 0$

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Theorem (Debruyne and V., 2016)

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The Fatou-Riesz theorem

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Séries trigonométriques et séries de Taylor,

Fatou proved the following theorem on analytic continuation of power series.

Theorem (Fatou-Riesz theorem)

Suppose that $F(z) = \sum_{n=0}^{\infty} c_n z^n$ *converges for* $|z| < 1$ *and* $\lim_{n\to\infty} c_n = 0$ *(this is the Tauberian condition). If* $F(z)$ *has analytic continuation to a neigborhood of* $z = 1$*, then* $\sum_{n=0}^{\infty} c_n$ *converges and*

$$\sum_{n=0}^{\infty} c_n = F(1).$$

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The Ingham-Karamata theorem for Laplace transforms

In 1935 Ingham and Karamata obtained a Fatou-Riesz type Tauberian theorem for Laplace transforms. The result makes use of *slow decrease*.

A function τ is called slowly decreasing if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h) - \tau(x)) > -\varepsilon.$$

that is, $\tau(x+h) - \tau(x) > -\varepsilon$ for $x > X_\varepsilon$ and $0 \le h < \delta_\varepsilon$.

Theorem (Ingham and Karamata, independently)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing (Tauberian hypothesis). Suppose its Laplace transform

$$\mathcal{L}\{\tau; z\} = \int_0^\infty \tau(t)e^{-zt}\mathrm{d}t$$

converges on $\Re e\, z > 0$ and has L^1_{loc}-boundary behavior on $\Re e\, z = 0$, then $\lim_{x\to\infty} \tau(x) = 0$.

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Developments in the 1980s: Newman's contour integration method

In 1980 Newman gave a simple contour integration proof of the next Tauberian theorem.

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Let a_n be bounded (Tauberian hypothesis). If $F(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$ has analytic continuation beyond $\Re e\, z = 1$, then

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- It is contained in the Ingham-Karamata theorem; however, Newman's proof method is simple and very attractive.
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Newman's Tauberian theorem from above provides a relatively simple way to prove the PNT.

- One works here with the Möbius

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors,} \\ 0 & \text{otherwise.} \end{cases}$$

- Property: μ is the Dirichlet convolution inverse of 1. So,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} = \frac{1}{\zeta(z)} \quad (\zeta \text{ is the Riemann zeta function})$$

- Applying the previous theorem, $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = \frac{1}{\zeta(1)} = 0.$
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Newman's contour integration method was adapted to a variety of Tauberian problems in numerous articles.

Its importance was recognized by the semigroup community. Here is a sample (extending a result of Korevaar and Zagier):

Theorem (Arendt and Batty, 1988)

Let $\rho \in L^{\infty}(\mathbb{R})$ (Tauberian hypothesis) vanish on $(-\infty, 0)$. Suppose that $\mathcal{L}\{\rho; z\}$ has analytic continuation at every point of the complement of iE where $E \subset \mathbb{R}$ is a closed null set. If $0 \notin iE$ and

$$\sup_{t \in E} \sup_{x>0} \left| \int_0^x e^{-itu} \rho(u) \mathrm{d}u \right| < \infty,$$

then the (improper) integral of ρ converges to $b = \mathcal{L}\{\rho; 0\}$, that is,

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If $E = \emptyset$, the result is due to Korevaar and Zagier (independently), who also obtained it via Newman's contour integration technique.

In this case, the result is contained in the Ingham-Karamata theorem:

- Set $\tau(x) = \int_0^x \rho(u)\mathrm{d}u - b \Rightarrow \mathcal{L}\{\tau; z\} = \dfrac{\mathcal{L}\{\rho; z\} - b}{z}$ with $b = \mathcal{L}\{\rho; 0\}$.
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The Arendt-Batty Tauberian theorem readily extends to functions with values on a Banach space. Here is a sample application of the vector-valued version:

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Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a reflexive Banach space X. Denote the spectrum of its infinitesimal generator A as $\sigma(A)$. If $\sigma(A) \cap i\mathbb{R}$ is countable and no eigenvalue of A lies on the imaginary axis, then

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In recent times, Tauberian methods have been revisited to study rates of converge that can be a applied to PDE, e.g. decay estimates for damped wave equations.

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Input from operator theory

In the same spirit, Katznelson and Tzafriri established earlier the following important result in asymptotic operator theory:

Theorem (Katznelson and Tzafriri, 1986)

Let T be a power-bounded operator on a Banach space ($\sup_{n\in\mathbb{N}} \|T^n\| < \infty$). Then,

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if and only if $\sigma(T) \cap \partial\mathbb{D} \subseteq \{1\}$.

- This can be deduced from a power series Tauberian theorem that preceded the Arendt-Batty theorem.
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Extension of the Ingham-Karamata theorem

Theorem (Debruyne and V., 2019)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing, vanish on $(-\infty, 0)$, and have convergent Laplace transform on $\Re e\, z > 0$. Suppose there is a closed null set $E \subset \mathbb{R}$ such that:

(I) *$\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) *$0 \notin E$.*

Then

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Conversely, (1) implies that $\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on the whole imaginary axis.

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Some tools involved in the proof

Our proof is based on:

- Boundedness theorems for Laplace transforms with local pseudo-measure behavior.
- Characterizations of local pseudofunctions through behavior outside exceptional sets.

Other results

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has "good boundary behavior" on the line segment $1 + i(-\lambda, \lambda)$.

One can show that there are $c_\lambda, C_\lambda > 0$ such that

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The inequalities hold with

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These constants are best possible.

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$\mathrm{Lip}(I; M)$ denotes the class of Lipschitz continuous functions on I with Lipschitz constant M.

Known result: Suppose that

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There is an absolute contant $\mathfrak{C} > 0$ such that

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$\mathfrak{C} = 6$, Ingham (1935)

$\mathfrak{C} = 2$, Korevaar, Zagier, and other people ...

Problem: Find the optimal value of $\mathfrak{C}$.

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and the value of $\pi/2$ in this inequality cannot be improved.

Combining this with the Graham-Vaaler sharp Wiener-Ikehara theorem, one can consider 'Lipschitz continuous functions only from below'. We obtained the sharp inequality

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$$\Psi(\delta) = \limsup_{x\to\infty} \sup_{h\in[0,\delta]} |\tau(x+h) - \tau(x)|.$$

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Let $\tau \in L^{\infty}(\mathbb{R})$. Suppose that $\hat{\tau} \in \mathrm{PF}_{loc}(-\lambda, \lambda)$ (in particular if continuous there). Then,

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For some applications of these results in analytic number theory, see:

- On PNT equivalences for Beurling numbers, Monatsh. Math. 184 (2017), 401–424.
- On Diamond's L^1 criterion for asymptotic density of Beurling generalized integers, Michigan Math. J. 68 (2019), 211–223.

Important book references on complex Tauberians

- W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, 2011.
- J. Korevaar, Tauberian theory. A century of developments, 2004.

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- On PNT equivalences for Beurling numbers, Monatsh. Math. 184 (2017), 401–424.
- On Diamond's L^1 criterion for asymptotic density of Beurling generalized integers, Michigan Math. J. 68 (2019), 211–223.

Important book references on complex Tauberians

- W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, 2011.
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Some references

The talk is based on recent collaborative works with G. Debruyne:

- Generalization of the Wiener-Ikehara theorem, Illinois J. Math. 60 (2016), 613–624.
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- Complex Tauberian theorems for Laplace transforms with local pseudofunction boundary behavior, J. Anal. Math. 138 (2019), 799–833.

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