Structural theorems for quasiasymptotics of Schwartz distributions

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Summary

The aim of this talk is to communicate new structural theorems for quasiasymptotics of Schwartz distributions.

- Review of the definition of quasiasymptotics at infinity and at the origin and the known properties.
- Integration of the quasiasymptotic and relationship with asymptotically and associate asymptotically homogeneous functions.
- Structural Theorems for quasiasymptotics at infinity and at the origin.
- Particular case: the quasiasymptotic of order -1 at infinity.
- **Consequence**: Characterization of jump behavior of Fourier series in terms of Cesaro summability.
- Evaluation of distributions in the e.v. Cesaro sense.
- **Consequence**: Pointwise Fourier inversion formula.

Notation

- $\mathcal{D}$ and $\mathcal{D}'$ denote the Schwartz spaces of test functions and distributions.
- $\mathcal{S}$ and $\mathcal{S}'$ are the spaces of rapidly decreasing functions and the space of tempered distributions.
- All of our functions and distributions are over the real line.
- The Fourier transform in $\mathcal{S}$ is defined as

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \phi(t) e^{ixt} \mathrm{d}t.$$

Slowly Varying Functions

Recall that real-valued measurable function defined in some interval of the form $[A, \infty)$, $A > 0$, is called *slowly varying function at infinity* if L is positive for large arguments and

$$\lim_{x \to \infty} \frac{L(ax)}{L(x)} = 1,$$

for each $a > 0$.
Similarly one defines slowly varying functions at the origin.

Quasiasymptotic at infinity

Let L be slowly varying. We say that $f \in \mathcal{D}'$ has *quasiasymptotic behavior at infinity in* $\mathcal{D}'$ *with respect to* $\lambda^{\alpha} L(\lambda)$, $\alpha \in \mathbb{R}$, if for some $g \in \mathcal{D}'$ and every $\phi \in \mathcal{D}$,

$$\lim_{\lambda \to \infty} \left\langle \frac{f(\lambda x)}{\lambda^{\alpha} L(\lambda)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle .$$

We also say that f has *quasiasymptotic of order* α *at infinity with respect to* L.
We also express this by

$$f(\lambda x) = \lambda^{\alpha} L(\lambda) g(x) + o(\lambda^{\alpha} L(\lambda)),\ \lambda \to \infty \ \text{ in } \mathcal{D}'.$$

We may also have

$$f(\lambda x) = \lambda^{\alpha} L(\lambda) g(x) + o(\lambda^{\alpha} L(\lambda)),\ \lambda \to \infty \ \text{ in } \mathcal{S}'.$$

Quasiasymptotic at the Origin

Similarly, one defines the quasiasymptotic in $\mathcal{D}'$ and $\mathcal{S}'$ at the origin.

- By shifting, one can define the quasiasymptotic of distributions at any point.
- For example, Łojasiewicz defined the value of a distribution $f \in \mathcal{D}'$ at the point x_0 as the limit

  $$f(x_0) = \lim_{\varepsilon \to 0} f(x_0 + \varepsilon x)\,,$$

  if the limit exists in $\mathcal{D}'$.
- **Notation:** If $f \in \mathcal{D}'$ has a value γ at x_0, we say that $f(x_0) = \gamma$ in $\mathcal{D}'$. The meaning of $f(x_0) = \gamma$ in $\mathcal{S}'$, ..., must be clear.

Previous known properties at infinity

- If $f \in \mathcal{D}'$ has quasiasymptotic at infinity in $\mathcal{D}'$. Then, $f \in \mathcal{S}'$.
- Structural theorems when the order of the quasiasymptotic $\alpha \notin -\mathbb{N}$.
- If f has quasiasymptotic in $\mathcal{D}'$ whose order is not a negative integer, then f has the same quasiasymptotic in $\mathcal{S}'$. For $\alpha \in -\mathbb{N}$ the result was known only under the assumption L bounded.

Previous known properties at the origin

- Structural Theorem for $\alpha > 0$.
- Structural Theorem for $\alpha \in (-1, 0]$ under the assumption L bounded.
- If $f \in \mathcal{S}'$ has quasiasymptotic at the origin in $\mathcal{D}'$, then it has the same quasiasymptotic in $\mathcal{S}'$ in the following two cases,
    - $\alpha \leq 0$ and $\alpha \notin -\mathbb{N}$.
    - $\alpha > 0$ and L bounded.

Integration of the Quasiasymptotic

Suppose

$$f\left(\lambda x\right)=L(\lambda)g(\lambda x)+o\left(\lambda^{\alpha}L(\lambda)\right),\ \text{in } \mathcal{D}',$$

(here $\lambda\to\infty$ or 0). Suppose that g admits a primitive G_k of order k which is homogeneous of degree $k+\alpha$. Then, for any given k-primitive F_k of f, there exist functions $b_0,\ldots,b_{k-1}$, such that

$$F_k\left(\lambda x\right)=L(\lambda)G_k(\lambda x)+\sum_{j=0}^{k-1}\lambda^{\alpha+k}b_j(\lambda)\frac{x^{k-1-j}}{(k-1-j)!}+o\left(\lambda^{\alpha+k}L(\lambda)\right),\ \text{in } \mathcal{D}',$$

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where

$$b_j(a\lambda)=a^{-\alpha-j-1}b_j(\lambda)+o(L(\lambda)).$$

Asymptotically Homogeneous Functions

Definition A function b is called **asymptotically homogeneous of degree α at infinity** (resp. at 0) if

$$b(ax) = a^{\alpha} b(x) + o(L(x)).$$

Properties

- In the case at infinity when $\alpha < 0$, or at 0 when $\alpha > 0$,

$$b(x) = o(L(x)).$$

- In the case at infinity when $\alpha > 0$, or at 0 when $\alpha < 0$,

$$b(x) = \beta x^{\alpha} + o(L(x)).$$

Structural Theorem for Some Cases

Theorem 1 *Let $f \in \mathcal{D}'$ have quasiasymptotic behavior at infinity (resp. at the origin) in $\mathcal{D}'$,*

(1) $$f(\lambda x) = C_{-} L(\lambda) \frac{(\lambda x)_{-}^{\alpha}}{\Gamma(\alpha+1)} + C_{+} L(\lambda) \frac{(\lambda x)_{+}^{\alpha}}{\Gamma(\alpha+1)} + o\left(\lambda^{\alpha} L(\lambda)\right).$$

If $\alpha \notin \{-1,-2,\ldots\}$, then there exist a positive integer m, a m-primitive F of f such that F is continuous (resp. continuous in [-1,1]) and

(2) $$\lim \frac{\Gamma(\alpha+m+1) F(x)}{|x|^{\alpha+m} L(|x|)} = C \pm .$$

Conversely, if these conditions hold, then (by differentiation) (1) follows.

Associate Asymptotically Homogeneous Functions

In the case of negative integer order, the main coefficient of integration of the quasiasymptotic satisfies the following definition.

Definition A function b is called **associate asymptotically homogeneous of degree 0 at infinity** (resp. at 0) with respect to L if

$$b(ax) = b(x) + \beta L(x) \log a + o(L(x)).$$

Structural Theorem for the Other Cases

Theorem 2 *f has the quasiasymptotic behavior in $\mathcal{D}'$ at infinity (resp. at the origin),*

$$f(\lambda x) = \gamma\lambda^{-k}L(\lambda)\delta^{(k)}(x) + \frac{(-1)^{k-1}\beta}{(k-1)!}\lambda^{-k}L(\lambda)Pf\left(\frac{1}{x^k}\right) + o\left(\lambda^{-k}L(\lambda)\right),$$

if and only if there exist $m \in \mathbb{N}, m \geq k$, a function b satisfying $b(a\lambda) = b(\lambda) + \beta \log aL(\lambda) + o(L(\lambda))$ and a m-primitive F, which is continuous (resp. continuous in $[-1,1]$) such that

$$F(x) = b\left(|x|\right)\frac{x^{m-k}}{(m-k)!} + \gamma L\left(|x|\right)\frac{x^{m-k}}{2(m-k)!}sgn x$$

$$-\beta L\left(|x|\right)\frac{x^{m-k}}{(m-k)!}\sum_{j=1}^{m-k}\frac{1}{j} + o\left(|x|^{m-k}L\left(|x|\right)\right)$$

Second version of the Structural Theorem

This is a version free of b

Theorem 3 *Let $f \in \mathcal{D}'$. Then f has quasiasymptotic at infinity (resp. at the origin) of order $-k$, $k \in \{1, 2, ...\}$ if and only if there exists a continuous m-primitive F of f (resp. continuous in [-1,1]), $m > k$, such that for each $a > 0$,*

(3) $$\lim_{x \to \infty} \frac{(m-k)!\left(a^{k-m}F(ax) - (-1)^{m-k}F(-x)\right)}{x^{m-k}L(x)} = I(a).$$

In such case I has the form $I(a) = \gamma + \beta \log a$.

A Particular Case

Recall the definition of Cesaro limits of distributions. Let $g \in \mathcal{D}'$,

$$\lim_{x \to \infty} g(x) = \eta \ (\mathrm{C}, k),$$

if there exists a k-primitive G of g, being a regular distribution, such that $G(x) = \eta x^k / k! + o(x^k)$, as $x \to \infty$. Then the structural theorem for the quasiasymptotic of order -1 is the following:

$$f(\lambda x) = \gamma \delta(\lambda x) + \beta \mathcal{P} f \left(\frac{1}{\lambda x} \right) + o \left(\frac{1}{\lambda} \right) \text{ as } \lambda \to \infty$$

if and only there is $k \in \mathbb{N}$ such that for all 1-primitive F of f and $a > 0$

$$\lim_{x \to \infty} F(ax) - F(-x) = \gamma + \beta \log a \ (\mathrm{C}, k).$$

First Consequence: Local behavior of Fourier Series

f is said to have a **jump behavior** at x_0 if

$$f(x_0 + \epsilon x) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \text{ in } \mathcal{D}' \text{ as } \epsilon \to 0^+.$$

Suppose that $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, then it has this jump behavior at x_0 if and only if there exists $k \in \mathbb{N}$ such that for each $a > 0$

$$\lim_{N\to\infty} \sum_{-N \le n \le aN} a_n e^{inx_0} = \frac{\gamma_+ + \gamma_-}{2} + \frac{i}{2\pi}(\gamma_+ - \gamma_-)\log a \quad (\mathrm{C}, k).$$

e.v Cesaro evaluations

Definition 1 *Let* $g \in \mathcal{D}'$*, and* $k \in \mathbb{N}$*. We say that the evaluation* $\langle g(x), \phi(x) \rangle$ *exists in the* $\mathrm{e.v.}$ *Cesàro sense, and write*

$$\mathrm{e.v.} \langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}, k), \tag{4}$$

if for some primitive G *of* $g\phi$ *and* $\forall a > 0$ *we have*

$$\lim_{x \to \infty} (G(ax) - G(-x)) = \gamma \ (\mathrm{C}, k).$$

If g *is locally integrable then we write (4) as*

$$\mathrm{e.v.} \int_{-\infty}^{\infty} g(x) \phi(x) \, \mathrm{d}x = \gamma \quad (\mathrm{C}, k).$$

Remark: *In this definition the evaluation of* g *at* ϕ *does not have to be defined, we only require that* $g\phi$ *is well defined.*

Pointwise Fourier Inversion Formula

Now, we characterize the point values of a distribution in $\mathcal{S}'$ by using Fourier transforms.

Theorem 4 *Let $f \in \mathcal{S}'$. We have $f(x_0) = \gamma$ in $\mathcal{S}'$ if and only if there exists a $k \in \mathbb{N}$ such that*

$$\frac{1}{2\pi}\mathrm{e.v.}\left\langle \hat{f}\left(t\right), e^{-ix_0 t}\right\rangle = \gamma \quad \left(\mathrm{C}, k\right),$$

which in case $\hat{f}$ is locally integrable means that

$$\frac{1}{2\pi}\mathrm{e.v.}\int_{-\infty}^{\infty}\hat{f}(t)e^{-ix_0 t}\mathrm{d}t = \gamma \quad \left(\mathrm{C}, k\right).$$