On general Stieltjes moment problems

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The problem of moments, as its generalizations, is an important mathematical problem which has attracted much attention for more than a century.

It was first raised and solved by Stieltjes for positive measures.

Problem (Stieltjes, 1894)

Find conditions over $\{a_n\}_{n=0}^{\infty}$ which ensure the existence of solutions μ to the infinity system of equations

$$a_n = \int_0^{\infty} x^n d\mu(x), \quad n = 0, 1, 2, \dots ,$$

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The classical Stieltjes moment problem

Stieltjes found a necessary and sufficient condition for the existence of solutions. Define the sequence of matrices

$$\Delta_n = \begin{pmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \dots & a_{2n} \end{pmatrix} \text{ and } \Delta_n^{(1)} = \begin{pmatrix} a_1 & a_2 & \dots & a_{n+1} \\ a_2 & a_3 & \dots & a_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & a_{n+2} & \dots & a_{2n+1} \end{pmatrix}$$

Theorem (Stieltjes, 1894-1895)

The Stieltjes moment problem

$$a_n = \int_0^\infty x^n d\mu(x), \quad n = 0, 1, 2, \dots,$$

has solution if and only if

$$\det(\Delta_n) > 0 \textit{ and } \det(\Delta_n^{(1)}) > 0, \quad n = 0, 1, 2, \dots .$$

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Stieltjes' influential papers led to many important ideas:

- The theory of Stieltjes integrals

$$a_n = \int_0^\infty x^n dF(x), \quad F \nearrow .$$

- The Stieltjes transform, $\Re e\, z \notin (-\infty, 0]$,

$$S(z) = \int_0^\infty \frac{dF(x)}{x+z} \sim \sum_{n=0}^\infty \frac{(-1)^n a_n}{z^{n+1}} .$$

- Continued fraction approximations.

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- Carleman (1923-1926): connections with the theory of quasi-analytic functions.

Other moment problems:

- Hamburger (1920):

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- Hausdorff (1923):

$$a_n = \int_{b}^{c} x^n dF(x), \quad n = 0, 1, 2, \dots .$$

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Moment problems for arbitrary sequences

Theorem (Boas and Pólya, independently, 1939)

Given an arbitrary sequence $\{a_n\}_{n=0}^{\infty}$, there is always a function of bounded variation F such that

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A. Durán achieved a major improvement to this result:

Theorem (A. Durán, 1989)

Every Stieltjes moment problem

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- A. Durán's proof: Laguerre expansions, Hankel transform.
- A. L. Durán and Estrada found a simple proof (1994):

$$a_n = \int_0^\infty x^n \phi(x)dx, \quad n = 0, 1, 2, \dots, \tag{1}$$

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- Chung-Chung-Kim (2003) exploited the method to show that (1) has solutions $\phi \in \mathcal{S}^\beta(0, \infty)$, $\beta > 1$.
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Abstract moment problem

We want to replace

$$a_n = \int_0^\infty x^n \phi(x)dx, \quad n = 0, 1, 2, \ldots,$$

by the infinite system of linear equations

$$a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \ldots \ , \tag{2}$$

where the sought solution ϕ is an element of a (topological!) vector space E and $f_n \in E'$.

Problem

Conditions over E and $\{f_n\}_{n=0}^\infty$ such that every generalized moment problem (3) has a solution $\phi \in E$.

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Particular cases

- The Borel problem:

$$a_n = \phi^{(n)}(0), \quad n = 0, 1, 2, \dots .$$

Here $E = C^\infty(\mathbb{R})$ and $f_n = (-1)^n \delta^{(n)}$, elements of $\mathcal{E}'(\mathbb{R})$.

- The Borel-Ritt problem. Given a sector $S : \alpha < \arg z < \beta$, $|z| < r$. Find an analytic function ϕ on S such that on any subsector $S_1 : \alpha_1 < \arg z < \beta_1$ one has

$$\phi(z) \sim \sum_{n=0}^{\infty} a_n z^n, \quad z \to 0^+.$$

- Entire functions with prescribed values. Let $\{\omega_n\}_{n=0}^{\infty}$ a sequence of complex numbers. Find ϕ entire such that

$$\phi(\omega_n) = a_n.$$

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Particular case: General Stieltjes moment problems for rapidly decreasing smooth functions

Direct generalization of Pólya-Boas-Durán problem,

$$a_n = \int_0^\infty x^n \phi(x) dx, \quad n = 0, 1, 2, \dots,$$

where $\phi \in \mathcal{S}(0, \infty)$.

Distribution moment problem:

$$a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \dots, \tag{3}$$

where $f_n \in \mathcal{S}'[0, \infty)$ (= $f_n \in \mathcal{S}'(\mathbb{R})$ with supp $f_n \subseteq [0, \infty)$).
Again we seek solutions $\phi \in \mathcal{S}(0, \infty)$.

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Particular cases

1. Continuous generalized moment problem

$$a_n = \int_0^\infty f_n(x)\phi(x)dx, \quad n = 0, 1, 2, \ldots.$$

2. Discrete problem: Let $(B_{k,n})_{(k,n)\in\mathbb{N}^2}$ be an infinite matrix

$$a_n = \sum_{k=1}^\infty B_{k,n}\phi(k), \quad n = 0, 1, 2, \ldots,$$

or, more generally, $0 < \lambda_n \to \infty$,

$$a_n = \sum_{k=1}^\infty B_{k,n}\phi(\lambda_k), \quad n = 0, 1, 2, \ldots.$$

3. Let $\{F_n\}_{n=0}^\infty$ be a sequence of functions of local bounded variation (having at most polynomial growth)

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Back to the abstract moment problem

We now consider the abstract moment problem, $\{f_n\}_{n=0}^{\infty} \subset E'$,

$$a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \dots . \tag{4}$$

where E is B-complete (also called Pták). This means that a linear subspace of E' is weak* closed iff its intersection with every equicontinous set is weak* closed.

Theorem

Let E be B-complete. Then every moment problem (4) admits a solution $\phi \in E$ if and only if

1. $f_0, f_1, f_2, \dots, f_n, \dots$ *are linear independent.*
2. *For any equicontinuous subset $A \subset E'$, the intersection*

$$A \cap \operatorname{span}\{f_n : n \in \mathbb{N}\}$$

is contained in a finite dimensional subspace.

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Abstract moment problem in Fréchet spaces

We rediscovered the following result, originally due to Eidelheit (1936).

Corollary

Let $E = \operatorname{proj\,lim} E_j$ be a Fréchet space, where each E_j is a Banach space, and $E_{j+1} \to E_j$ is dense. Every arbitrary abstract moment problem

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2. $\operatorname{span}\{f_n : n \in \mathbb{N}\} \cap E_j'$ *is finite dimensional,* $\forall j \in \mathbb{N}$.

Applications

- For the Borel problem:

$$a_n = \phi^{(n)}(0) = \langle (-1)^n \delta^{(n)}, \phi \rangle, \quad n = 0, 1, 2, \dots,$$

one takes $E = C^\infty(\mathbb{R}) = \operatorname{proj} \lim C^j[-j, j]$. Since all elements of the dual of $C^j[-j, j]$ are derivatives of order $\leq j + 1$ of measures, the last result implies that every Borel problem has solution.

- A similar argument shows that every Borel-Ritt problem has a solution.
- Employing the Köthe-Silva-Grothendieck representation theorem for analytic functionals and the previous theorem, one can show: Every sampling problem

$$\phi(\omega_n) = a_n$$

has an entire solution ϕ if and only if $|\omega_n| \to \infty$.

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has an entire solution ϕ if and only if $|\omega_n| \to \infty$.

Applications

- For the Borel problem:

$$a_n = \phi^{(n)}(0) = \langle (-1)^n \delta^{(n)}, \phi \rangle, \quad n = 0, 1, 2, \ldots,$$

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Distribution moment problem. Cesàro asymptotics

Let $f \in \mathcal{S}'[0,\infty)$ and $\alpha > -1$. We write

$$f(x) = O(x^{\alpha}) \quad (\mathrm{C}, m), \quad x \to \infty$$

if $f^{(-m)}$, the primitive of order m of f, is continuous on $[0,\infty)$ and

$$f^{(-m)}(x) = O(x^{\alpha+m}), \quad x \to \infty,$$

in the ordinary sense.

Here f is the convolution of f with $x_+^{m-1}/(m-1)!$, so that if f is locally integrable

$$\frac{1}{x}\int_0^x f(t)\left(1-\frac{t}{x}\right)^{m-1} \mathrm{d}t = O(x^{\alpha})$$

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General Stieltjes moment problem for sequences of distributions

Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of distributions with supp $f_n \subseteq [0, \infty)$.

Theorem

Every generalized Stieltjes moment problem

$$a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \ldots,$$

has a solution $\phi \in \mathcal{S}(0, \infty)$ if:

1. $f_1, f_2, f_3 \ldots, f_n, \ldots,$ *are linearly independent.*
2. $\operatorname{span}\{f_n : n \in \mathbb{N}\} \cap \operatorname{span}\{\delta^{(j):j\mathbb{N}}\} = \{0\}$.
3. *There is an increasing sequence of integers $\{m_j\}_{j=0}^{\infty}$ such that for every j and $\alpha > 0$ there exists $\nu_{j,\alpha}$ such that if $N \geq \nu$*

$$\sum_{n=0}^{\infty} b_n f_n(x) = O(x^{\alpha})(\mathrm{C}, m_j) \implies b_{\nu} = b_{\nu+1} = \cdots = b_N = 0.$$

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The weighted Stieltjes moment problem

Let $0 \leq F \nearrow$ on $[0, \infty)$ with $F(x) = O(x^k)$ and let $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ with

$$\lim_{n\to\infty} \Re e\, \alpha_n = \infty.$$

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$$a_n = \int_0^\infty \phi(x) x^{\alpha_n} dF(x), \quad n = 0, 1, 2 \ldots,$$

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Examples

The following generalized moment problem is not always solvable:

$$a_n = \sum_{k=1}^{\infty} 2^{-k} k^n \phi(k), \quad n = 0, 1, 2, \ldots .$$

Let $\{\alpha_n\}_{n=0}^{\infty}$ be such that $\Re e\, \alpha_n \nearrow \infty$. The following generalized moment problems always have a solution $\phi \in \mathcal{S}(0, \infty)$.

$$a_n = \sum_{p \text{ prime}} p^{\alpha_n} \phi(p), \quad n = 0, 1, 2, \ldots .$$

$$a_n = \int_0^{\infty} x^{\alpha_n} \sin\left(\frac{1}{x^{\beta}}\right) \phi(x) \mathrm{d}x, \quad n = 0, 1, 2, \ldots, \quad (\beta \geq 0).$$

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