Generalized Asymptotics and Applications

Jasson Vindas
jvindas@math.lsu.edu

Department of Mathematics
Louisiana State University

Imperial College
Department of Mathematics
London, November 12, 2008

Introduction

- ‘Generalized asymptotics’ refers to asymptotic analysis on spaces of generalized functions
- I will focus on spaces of Schwartz distributions (in one dimension)
- Asymptotic notions lead to pointwise regularity for distributions

Outline

Notation
from distribution theory

- $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ denote the spaces of smooth compactly supported functions and smooth rapidly decreasing functions
- $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ the spaces of distributions and tempered distributions
- The Fourier transform in $\mathcal{S}(\mathbb{R})$ is defined as

$$\widehat{\phi}(x) = \int_{-\infty}^{\infty} \phi(t) e^{ixt} \mathrm{d}t$$

- The evaluation of f at a test function ϕ is denoted by

$$\langle f(x), \phi(x) \rangle$$

Quasiasymptotics

The idea is to study the weak asymptotic behavior of the dilates of f. So we look for asymptotic representations

$$f(\lambda x) \sim \rho(\lambda) g(x) \, .$$

Definition

We say that $f \in \mathcal{D}'(\mathbb{R})$ has quasiasymptotic behavior in $\mathcal{D}'(\mathbb{R})$ with respect to ρ if for some $g \in \mathcal{D}'(\mathbb{R})$ and every $\phi \in \mathcal{D}(\mathbb{R})$,

$$\lim \left\langle \frac{f(\lambda x)}{\rho(\lambda)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle \, .$$

In such a case one writes $f(\lambda x) = \rho(\lambda) g(x) + o(\rho(\lambda))$ in $\mathcal{D}'(\mathbb{R})$.

Łojasiewicz (1957) defined the value of a distribution $f \in \mathcal{D}'(\mathbb{R})$ at the point x_0 as the limit

$$\gamma = \lim_{\varepsilon \to 0} f(x_0 + \varepsilon x)\,,$$

if the limit exists in $\mathcal{D}'(\mathbb{R})$. We use the notation $f(x_0) = \gamma$, distributionally.
It is an average notion:

Theorem

(Łojasiewicz structural theorem, 1957) $f(x_0) = \gamma$, distributionally, if and only if there exist $k \in \mathbb{N}$ a continuous $k-$primitive F of f (i.e. $f = F^{(k)}$) such that F is continuous near x_0 and

$$\lim_{x \to \infty} \frac{k! F(x)}{(x - x_0)^k} = \gamma.$$

$S-$asymptotics

For the $S-$asymptotic, we look at the translates of the distribution.

Definition

We say that $f \in \mathcal{D}'(\mathbb{R})$ has $S-$asymptotic with respect to a function ρ if there exists $g \in \mathcal{D}'(\mathbb{R})$ such that

$$f(x+h) \sim \rho(h)g(x) \quad \text{as } h \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) .$$

Pointwise Fourier inversion formula

The relaship between the value of a function at a point and the convergence or summability of its Fourier transform (or series) is an old problem. The question even makes sense for tempered distributions.

Questions:

- If a tempered distribution has a value at a point, can it be recovered by its Fourier transform?
- Specifically, is it possible to give pointwise sense to

$$f(x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{-itx_0}\mathrm{d}t\,, \quad \text{for } f \in \mathcal{S}'(\mathbb{R})?$$

- Is it possible to characterize the existence of point values by certain type of summability of the Fourier transform?

Pointwise Fourier inversion formula

The relaship between the value of a function at a point and the convergence or summability of its Fourier transform (or series) is an old problem. The question even makes sense for tempered distributions.

Questions:

- If a tempered distribution has a value at a point, can it be recovered by its Fourier transform?
- Specifically, is it possible to give pointwise sense to

$$f(x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(t)e^{-itx_0}\mathrm{d}t\,, \quad \text{for } f \in \mathcal{S}'(\mathbb{R})?$$

- Is it possible to characterize the existence of point values by certain type of summability of the Fourier transform?

Pointwise Fourier inversion formula

The relaship between the value of a function at a point and the convergence or summability of its Fourier transform (or series) is an old problem. The question even makes sense for tempered distributions.

Questions:

- If a tempered distribution has a value at a point, can it be recovered by its Fourier transform?
- Specifically, is it possible to give pointwise sense to

$$f(x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{-itx_0}\,\mathrm{d}t\,, \quad \text{for } f \in \mathcal{S}'(\mathbb{R})?$$

- Is it possible to characterize the existence of point values by certain type of summability of the Fourier transform?

Pointwise Fourier inversion formula

The relaship between the value of a function at a point and the convergence or summability of its Fourier transform (or series) is an old problem. The question even makes sense for tempered distributions.

Questions:

- If a tempered distribution has a value at a point, can it be recovered by its Fourier transform?
- Specifically, is it possible to give pointwise sense to

$$f(x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{-itx_0}\mathrm{d}t\ , \quad \text{for } f \in \mathcal{S}'(\mathbb{R})?$$

- Is it possible to characterize the existence of point values by certain type of summability of the Fourier transform?

Point values and the Fourier transform

(quasi)asymptotic behavior of the Fourier transform

Suppose that $f(x_0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$. Then,

$$f(x_0 + \varepsilon x) = \gamma + o(1) \Leftrightarrow \frac{1}{2\pi} e^{-i\lambda x_0 x} \hat{f}(\lambda x) = \frac{\gamma \delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right)$$

Thus, one is led to study the quasiasymptotic behavior

$$g(\lambda x) = \frac{\gamma \delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty$$

in the space $\mathcal{S}'(\mathbb{R})$.

Point values and the Fourier transform

(quasi)asymptotic behavior of the Fourier transform

Suppose that $f(x_0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$. Then,

$$f(x_0 + \varepsilon x) = \gamma + o(1) \Leftrightarrow \frac{1}{2\pi} e^{-i\lambda x_0 x}\,\hat{f}(\lambda x) = \frac{\gamma\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right)$$

Thus, one is led to study the quasiasymptotic behavior

$$g(\lambda x) = \frac{\gamma\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty$$

in the space $\mathcal{S}'(\mathbb{R})$.

Point values and the Fourier transform
(quasi)asymptotic behavior of the Fourier transform

Suppose that $f(x_0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$. Then,

$$f(x_0 + \varepsilon x) = \gamma + o(1) \Leftrightarrow \frac{1}{2\pi} e^{-i\lambda x_0 x}\, \hat{f}(\lambda x) = \frac{\gamma\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right)$$

Thus, one is led to study the quasiasymptotic behavior

$$g(\lambda x) = \frac{\gamma\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty$$

in the space $\mathcal{S}'(\mathbb{R})$.

Structure of $g(\lambda x) = \lambda^{-1}\gamma\delta(x) + o\left(\lambda^{-1}\right)$

Definition

Let $h \in \mathcal{D}'(\mathbb{R})$, we say that $\lim_{x\to\infty} h(x) = \gamma \quad (\mathrm{C}, k)$, if $\exists F$, continuous, such that $h = F^{(k)}$ and $F(x) \sim \dfrac{\lambda x^k}{k!}$.

Theorem

Let $g \in \mathcal{S}'(\mathbb{R})$. It has the behavior

$$g(\lambda x) = \frac{\gamma\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}),$$

if and only if $\exists k$ such that for a primitive G of g ($G' = g$),

$$\lim_{x\to\infty} (G(ax) - G(-x)) = \gamma \quad (\mathrm{C}, k), \quad \text{for each } a > 0 .$$

Structure of $g(\lambda x) = \lambda^{-1}\gamma\delta(x) + o\left(\lambda^{-1}\right)$

Definition

Let $h \in \mathcal{D}'(\mathbb{R})$, we say that $\lim_{x\to\infty} h(x) = \gamma \quad (\mathrm{C}, k)$, if $\exists F$, continuous, such that $h = F^{(k)}$ and $F(x) \sim \dfrac{\lambda x^k}{k!}$.

Theorem

Let $g \in \mathcal{S}'(\mathbb{R})$. It has the behavior

$$g(\lambda x) = \frac{\gamma\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \, ,$$

if and only if $\exists k$ such that for a primitive G of g ($G' = g$),

$$\lim_{x\to\infty} (G(ax) - G(-x)) = \gamma \quad (\mathrm{C}, k) \, , \quad \text{for each } a > 0 \, .$$

Consequences

Corollary

Let $f \in \mathcal{S}'(\mathbb{R})$, suppose that $\widehat{f} \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then, $f(x_0) = \gamma$, distributionally, if and only if $\exists k \in \mathbb{N}$ such that

$$\lim_{x \to \infty} \frac{1}{2\pi} \int_{-x}^{ax} \widehat{f}(x) e^{-ix_0 x} \mathrm{d}x = \gamma \quad (\mathrm{C}, k)\,, \quad \text{for each } a > 0\,.$$

Corollary

Let $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ be a $2\pi-$periodic distribution. Then, $f(x_0) = \gamma$, distributionally, if and only if $\exists k \in \mathbb{N}$ such that

$$\lim_{x \to \infty} \sum_{-x < n \leq ax} c_n e^{inx_0} = \gamma \quad (\mathrm{C}, k)\,, \quad \text{for each } a > 0\,.$$

Pointwise Fourier inversion for tempered distributions

Definition

Let $g \in \mathcal{D}'(\mathbb{R})$, $\phi \in C^{\infty}(\mathbb{R})$, and $k \in \mathbb{N}$. We say that $\mathrm{e.v}\,\langle f(x), \phi(x)\rangle = \gamma \quad (\mathrm{C}, k)$ if for a primitive G_{ϕ} of ϕg,

$$\lim_{x \to \infty} \left(G_{\phi}(ax) - G_{\phi}(-x)\right) = \gamma \quad (\mathrm{C}, k)\,.$$

Theorem

Let $f \in \mathcal{S}'(\mathbb{R})$. Then $f(x_0) = \gamma$, distributionally, if and only if there exists $k \in \mathbb{N}$ such that

$$\frac{1}{2\pi}\mathrm{e.v.}\left\langle \widehat{f}\,(x)\,, e^{-ix_0 x}\right\rangle = \gamma \quad (\mathrm{C}, k)\,.$$

Pointwise Fourier inversion for tempered distributions

Definition

Let $g \in \mathcal{D}'(\mathbb{R})$, $\phi \in C^{\infty}(\mathbb{R})$, and $k \in \mathbb{N}$. We say that $\mathrm{e.v}\left\langle f(x), \phi(x)\right\rangle = \gamma \quad (\mathrm{C}, k)$ if for a primitive G_{ϕ} of ϕg,

$$\lim_{x\to\infty} \left(G_{\phi}(ax) - G_{\phi}(-x)\right) = \gamma \quad (\mathrm{C}, k)\,.$$

Theorem

Let $f \in \mathcal{S}'(\mathbb{R})$. Then $f(x_0) = \gamma$, distributionally, if and only if there exists $k \in \mathbb{N}$ such that

$$\frac{1}{2\pi}\mathrm{e.v.}\left\langle \widehat{f}\,(x)\,, e^{-ix_0 x}\right\rangle = \gamma \quad (\mathrm{C}, k)\,.$$

The prime number theorem

I will present a distributional proof of the Prime Number Theorem

$$\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty,$$

where

$$\pi(x) = \sum_{p \text{ prime},\ p<x} 1 .$$

The proof is based on:

- Chebyshev's elementary estimate
- The non-vanishing of the Riemann zeta function on $\Re e\, z = 1$
- Arguments from generalized asymptotics

The prime number theorem

I will present a distributional proof of the Prime Number Theorem

$$\pi(x) \sim \frac{x}{\log x} , \quad x \to \infty ,$$

where

$$\pi(x) = \sum_{p \text{ prime}, \ p<x} 1 .$$

The proof is based on:

- Chebyshev's elementary estimate
- The non-vanishing of the Riemann zeta function on $\Re e\, z = 1$
- Arguments from generalized asymptotics

Preliminaries

Some well known facts

- $\zeta(z)$ denotes the Riemann zeta function
  - $\zeta(z) - (1/(z-1))$ continues beyond $\Re e\, z = 1$
  - $\zeta(1+ix)$, $x \neq 0$, is free of zeros
- von Mangoldt function: $\Lambda(n) = \begin{cases} 0\,, & \text{if } n = 1 \\ \log p\,, & \text{if } n = p^m \\ 0\,, & \text{otherwise} \end{cases}$
- Chebyshev function: $\psi(x) = \sum_{n<x} \Lambda(n)$
  - The PNT is equivalent to $\psi(x) \sim x$
  - Chebyshev's elementary estimate: $\exists M > 0$ such that $\psi(x) < Mx$

Preliminaries

Some well known facts

- $\zeta(z)$ denotes the Riemann zeta function
  - $\zeta(z) - (1/(z-1))$ continues beyond $\Re e\, z = 1$
  - $\zeta(1+ix)$, $x \neq 0$, is free of zeros
- von Mangoldt function: $\Lambda(n) = \begin{cases} 0\,, & \text{if } n = 1 \\ \log p\,, & \text{if } n = p^m \\ 0\,, & \text{otherwise} \end{cases}$
- Chebyshev function: $\psi(x) = \sum_{n<x} \Lambda(n)$
  - The PNT is equivalent to $\psi(x) \sim x$
  - Chebyshev's elementary estimate: $\exists M > 0$ such that $\psi(x) < Mx$

Preliminaries

Some well known facts

- $\zeta(z)$ denotes the Riemann zeta function
  - $\zeta(z) - (1/(z-1))$ continues beyond $\Re e\, z = 1$
  - $\zeta(1+ix)$, $x \neq 0$, is free of zeros
- von Mangoldt function: $\Lambda(n) = \begin{cases} 0\,, & \text{if } n = 1 \\ \log p\,, & \text{if } n = p^m \\ 0\,, & \text{otherwise} \end{cases}$
- Chebyshev function: $\psi(x) = \sum_{n<x} \Lambda(n)$
  - The PNT is equivalent to $\psi(x) \sim x$
  - Chebyshev's elementary estimate: $\exists M > 0$ such that $\psi(x) < Mx$

Preliminaries

Some well known facts

- $\zeta(z)$ denotes the Riemann zeta function
  - $\zeta(z) - (1/(z-1))$ continues beyond $\Re e\, z = 1$
  - $\zeta(1 + ix)$, $x \neq 0$, is free of zeros
- von Mangoldt function: $\Lambda(n) = \begin{cases} 0\,, & \text{if } n = 1 \\ \log p\,, & \text{if } n = p^m \\ 0\,, & \text{otherwise} \end{cases}$
- Chebyshev function: $\psi(x) = \sum_{n<x} \Lambda(n)$
  - The PNT is equivalent to $\psi(x) \sim x$
  - Chebyshev's elementary estimate: $\exists M > 0$ such that $\psi(x) < Mx$

Preliminaries

Some well known facts

- $\zeta(z)$ denotes the Riemann zeta function
  - $\zeta(z) - (1/(z-1))$ continues beyond $\Re e\, z = 1$
  - $\zeta(1+ix)$, $x \neq 0$, is free of zeros
- von Mangoldt function: $\Lambda(n) = \begin{cases} 0\,, & \text{if } n = 1 \\ \log p\,, & \text{if } n = p^m \\ 0\,, & \text{otherwise} \end{cases}$
- Chebyshev function: $\psi(x) = \sum_{n<x} \Lambda(n)$
  - The PNT is equivalent to $\psi(x) \sim x$
  - Chebyshev's elementary estimate: $\exists M > 0$ such that $\psi(x) < Mx$

The distribution $v(x)$

We shall study the (S-)asymptotic properties of the distribution

$$v(x) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \delta(x - \log n) .$$

clearly $v \in \mathcal{S}'(\mathbb{R})$. Let us take the Fourier-Laplace transform of v, that is, for $\Im m\, z > 0$

$$\left\langle v(t), e^{izt} \right\rangle = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1-iz}} = -\frac{\zeta'(1-iz)}{\zeta(1-iz)} ,$$

a formula that Riemann obtained by logarithmic differentiation of the Euler product for the zeta function. Then,

$$\hat{v}(x) = -\frac{\zeta'(1-ix)}{\zeta(1-ix)} .$$

The distribution $v(x)$

We shall study the (S-)asymptotic properties of the distribution

$$v(x) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \delta(x - \log n) \ .$$

clearly $v \in \mathcal{S}'(\mathbb{R})$. Let us take the Fourier-Laplace transform of v, that is, for $\Im m\, z > 0$

$$\left\langle v(t), e^{izt} \right\rangle = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1-iz}} = -\frac{\zeta'(1-iz)}{\zeta(1-iz)} \ ,$$

a formula that Riemann obtained by logarithmic differentiation of the Euler product for the zeta function. Then,

$$\hat{v}(x) = -\frac{\zeta'(1-ix)}{\zeta(1-ix)} \ .$$

The distribution $v(x)$

We shall study the (S-)asymptotic properties of the distribution

$$v(x) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \delta(x - \log n) \ .$$

clearly $v \in \mathcal{S}'(\mathbb{R})$. Let us take the Fourier-Laplace transform of v, that is, for $\Im m\, z > 0$

$$\left\langle v(t), e^{izt} \right\rangle = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1-iz}} = -\frac{\zeta'(1-iz)}{\zeta(1-iz)} \ ,$$

a formula that Riemann obtained by logarithmic differentiation of the Euler product for the zeta function. Then,

$$\hat{v}(x) = -\frac{\zeta'(1-ix)}{\zeta(1-ix)} \ .$$

Properties of $v(x)$ to be used

It follows from the properties of ζ that the distributional boundary value of $\hat{v}(z) - \frac{i}{z}$ is a function, i.e.,

- $\hat{v}(x) - \frac{i}{(x + i0)} \in L^1_{\text{loc}}(\mathbb{R})$

In addition, we will make use of Chebyshev's estimate:

- $\psi(x) < Mx$

Properties of $v(x)$ to be used

It follows from the properties of ζ that the distributional boundary value of $\hat{v}(z) - \dfrac{i}{z}$ is a function, i.e.,

- $\hat{v}(x) - \dfrac{i}{(x + i0)} \in L^1_{\text{loc}}(\mathbb{R})$

In addition, we will make use of Chebyshev's estimate:

- $\psi(x) < Mx$

Properties of $v(x)$ to be used

It follows from the properties of ζ that the distributional boundary value of $\hat{v}(z) - \frac{i}{z}$ is a function, i.e.,

- $\hat{v}(x) - \dfrac{i}{(x + i0)} \in L^1_{\text{loc}}(\mathbb{R})$

In addition, we will make use of Chebyshev's estimate:

- $\psi(x) < Mx$

The plan
Steps

1. To show that

$$\lim_{h\to\infty} v(x+h) = 1 , \quad \text{in } \mathcal{S}'(\mathbb{R})$$

2. To show that

$$\lim_{\lambda\to\infty} \psi'(\lambda x) = \lim_{\lambda\to\infty} \sum_{n=1}^{\infty} \Lambda(n)\delta(\lambda x - n) = H(x) , \quad \text{in } \mathcal{D}'(0,\infty)$$

3. Final step, Step 2 is used to conclude

$$\psi(x) \sim x$$

The plan
Steps

1. To show that

$$\lim_{h\to\infty} v(x+h) = 1\ , \quad \text{in } \mathcal{S}'(\mathbb{R})$$

2. To show that

$$\lim_{\lambda\to\infty} \psi'(\lambda x) = \lim_{\lambda\to\infty} \sum_{n=1}^{\infty} \Lambda(n)\delta(\lambda x - n) = H(x)\ , \quad \text{in } \mathcal{D}'(0,\infty)$$

3. Final step, Step 2 is used to conclude

$$\psi(x) \sim x$$

The plan

Steps

1. To show that

$$\lim_{h\to\infty} v(x+h) = 1\ , \quad \text{in } \mathcal{S}'(\mathbb{R})$$

2. To show that

$$\lim_{\lambda\to\infty} \psi'(\lambda x) = \lim_{\lambda\to\infty} \sum_{n=1}^{\infty} \Lambda(n)\delta(\lambda x - n) = H(x)\ , \quad \text{in } \mathcal{D}'(0,\infty)$$

3. Final step, Step 2 is used to conclude

$$\psi(x) \sim x$$

$\lim_{h\to\infty} v(x+h) = 1$ in $\mathcal{S}'(\mathbb{R})$

Step 1

- First, $v(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$, as $h \to \infty$

Proof.

Set $g(x) = e^{-x}\psi(e^{x})$, by Chebyshev estimate $g(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$. Next, $g'(x+h) = O(1)$, but
$g'(x) = -g(x) + e^{-x}\sum \Lambda(n)\delta(x - \log n) = -g(x) + v(x)$. □

- Second, $\lim_{h\to\infty} \langle v(x+h), \phi(x)\rangle = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x$, for ϕ in a dense subspace of $\mathcal{S}(\mathbb{R})$

$\lim_{h\to\infty} v(x+h) = 1$ in $\mathcal{S}'(\mathbb{R})$

Step 1

- First, $v(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$, as $h \to \infty$

Proof.

Set $g(x) = e^{-x}\psi(e^x)$, by Chebyshev estimate $g(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$. Next, $g'(x+h) = O(1)$, but
$g'(x) = -g(x) + e^{-x}\sum \Lambda(n)\delta(x - \log n) = -g(x) + v(x)$. □

- Second, $\lim_{h\to\infty} \langle v(x+h), \phi(x)\rangle = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x$, for ϕ in a dense subspace of $\mathcal{S}(\mathbb{R})$

$\lim_{h\to\infty} v(x+h) = 1$ in $\mathcal{S}'(\mathbb{R})$

Step 1

- First, $v(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$, as $h \to \infty$

Proof.

Set $g(x) = e^{-x}\psi(e^x)$, by Chebyshev estimate $g(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$. Next, $g'(x+h) = O(1)$, but

$$g'(x) = -g(x) + e^{-x}\sum \Lambda(n)\delta(x - \log n) = -g(x) + v(x).$$

□

- Second, $\lim_{h\to\infty} \langle v(x+h), \phi(x)\rangle = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x$, for ϕ in a dense subspace of $\mathcal{S}(\mathbb{R})$

$\lim_{h\to\infty} v(x+h) = 1$ in $\mathcal{S}'(\mathbb{R})$

Step 1

- First, $v(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$, as $h \to \infty$

Proof.

Set $g(x) = e^{-x}\psi(e^x)$, by Chebyshev estimate $g(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$. Next, $g'(x+h) = O(1)$, but
$g'(x) = -g(x) + e^{-x}\sum \Lambda(n)\delta(x - \log n) = -g(x) + v(x)$. □

- Second, $\lim_{h\to\infty} \langle v(x+h), \phi(x)\rangle = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x$, for ϕ in a dense subspace of $\mathcal{S}(\mathbb{R})$

$\lim_{h\to\infty} v(x+h) = 1$ in $\mathcal{S}'(\mathbb{R})$
Step 1

- First, $v(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$, as $h \to \infty$

Proof.

Set $g(x) = e^{-x}\psi(e^x)$, by Chebyshev estimate $g(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$. Next, $g'(x+h) = O(1)$, but
$g'(x) = -g(x) + e^{-x}\sum \Lambda(n)\delta(x - \log n) = -g(x) + v(x)$. □

- Second, $\lim\limits_{h\to\infty} \langle v(x+h), \phi(x)\rangle = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x$, for ϕ in a dense subspace of $\mathcal{S}(\mathbb{R})$

$\lim_{h\to\infty} v(x+h) = 1$ in $\mathcal{S}'(\mathbb{R})$

Step 1 (continuation)

Proof.

Let $\phi = \widehat{\phi_1}$ with $\operatorname{supp}\phi_1$ compact.

$$\langle v(x+h), \phi(x)\rangle = \int_{-h}^{\infty} \phi(x)\mathrm{d}x + \left\langle v(x+h) - H(x+h)\, , \widehat{\phi_1}(x) \right\rangle$$

$$= \int_{-h}^{\infty} \phi(x)\mathrm{d}x + \left\langle \hat{v}(x) - \frac{i}{(x+i0)}\, , e^{-ihx}\phi_1(x) \right\rangle$$

$$= \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x + o(1), \quad h \to \infty$$

□

- Banach-Steinhaus theorem immediately gives the result

$\lim_{h\to\infty} v(x+h) = 1$ in $\mathcal{S}'(\mathbb{R})$

Step 1 (continuation)

Proof.

Let $\phi = \widehat{\phi_1}$ with $\operatorname{supp}\phi_1$ compact.

$$\langle v(x+h), \phi(x)\rangle = \int_{-h}^{\infty}\phi(x)\mathrm{d}x + \left\langle v(x+h) - H(x+h)\,, \widehat{\phi_1}(x)\right\rangle$$

$$= \int_{-h}^{\infty}\phi(x)\mathrm{d}x + \left\langle \hat{v}(x) - \frac{i}{(x+i0)}\,, e^{-ihx}\phi_1(x)\right\rangle$$

$$= \int_{-\infty}^{\infty}\phi(x)\mathrm{d}x + o(1), \quad h\to\infty$$

□

- Banach-Steinhaus theorem immediately gives the result

$\lim_{\lambda\to\infty} \psi'(\lambda x) = H(x)$, in $\mathcal{D}'(0,\infty)$
Step 2

Proof.

Step 2 implies that $e^{x+h}v(x+h) \sim e^{x+h}$, in $\mathcal{D}'(\mathbb{R})$, explicitely,

$$\sum_{n=1}^{\infty} \Lambda(n)\phi(\log n - h) \sim e^h \int_{-\infty}^{\infty} e^x \phi(x)\mathrm{d}x , \quad \forall \phi \in \mathcal{D}(\mathbb{R})$$

Changing variable in the last integral and writing $\lambda = e^h$,

$$\langle \psi'(\lambda x), \phi_1(x) \rangle = \frac{1}{\lambda} \sum_{n=1}^{\infty} \Lambda(n)\phi_1\left(\frac{n}{\lambda}\right) \sim \int_0^{\infty} \phi_1(x)\mathrm{d}x , \qquad (1)$$

where $\phi_1(x) = \phi(\log x)$. Thus, (1) holds $\forall \phi_1 \in \mathcal{D}(0,\infty)$. □

$\lim_{\lambda\to\infty}\psi'(\lambda x)=H(x)$, in $\mathcal{D}'(0,\infty)$

Step 2

Proof.

Step 2 implies that $e^{x+h}v(x+h)\sim e^{x+h}$, in $\mathcal{D}'(\mathbb{R})$, explicitely,

$$\sum_{n=1}^{\infty}\Lambda(n)\phi(\log n-h)\sim e^{h}\int_{-\infty}^{\infty}e^{x}\phi(x)\mathrm{d}x\ ,\quad \forall\phi\in\mathcal{D}(\mathbb{R})$$

Changing variable in the last integral and writing $\lambda=e^{h}$,

$$\langle\psi'(\lambda x),\phi_1(x)\rangle=\frac{1}{\lambda}\sum_{n=1}^{\infty}\Lambda(n)\phi_1\left(\frac{n}{\lambda}\right)\sim\int_{0}^{\infty}\phi_1(x)\mathrm{d}x\ , \qquad (1)$$

where $\phi_1(x)=\phi(\log x)$. Thus, (1) holds $\forall\phi_1\in\mathcal{D}(0,\infty)$. □

$\lim_{\lambda\to\infty}\psi'(\lambda x) = H(x)$, in $\mathcal{D}'(0,\infty)$

Step 2

Proof.

Step 2 implies that $e^{x+h}v(x+h) \sim e^{x+h}$, in $\mathcal{D}'(\mathbb{R})$, explicitely,

$$\sum_{n=1}^{\infty}\Lambda(n)\phi(\log n - h) \sim e^{h}\int_{-\infty}^{\infty} e^{x}\phi(x)\mathrm{d}x \ , \ \ \forall\phi\in\mathcal{D}(\mathbb{R})$$

Changing variable in the last integral and writing $\lambda = e^{h}$,

$$\langle\psi'(\lambda x),\phi_1(x)\rangle = \frac{1}{\lambda}\sum_{n=1}^{\infty}\Lambda(n)\phi_1\left(\frac{n}{\lambda}\right) \sim \int_{0}^{\infty}\phi_1(x)\mathrm{d}x \ , \qquad (1)$$

where $\phi_1(x) = \phi(\log x)$. Thus, (1) holds $\forall\phi_1\in\mathcal{D}(0,\infty)$. □

$\lim_{\lambda\to\infty}\psi'(\lambda x) = H(x)$, in $\mathcal{D}'(0,\infty)$

Step 2

Proof.

Step 2 implies that $e^{x+h}v(x+h) \sim e^{x+h}$, in $\mathcal{D}'(\mathbb{R})$, explicitly,

$$\sum_{n=1}^{\infty}\Lambda(n)\phi(\log n - h) \sim e^{h}\int_{-\infty}^{\infty} e^{x}\phi(x)\mathrm{d}x \ , \ \ \forall\phi\in\mathcal{D}(\mathbb{R})$$

Changing variable in the last integral and writing $\lambda = e^{h}$,

$$\langle\psi'(\lambda x),\phi_1(x)\rangle = \frac{1}{\lambda}\sum_{n=1}^{\infty}\Lambda(n)\phi_1\left(\frac{n}{\lambda}\right) \sim \int_{0}^{\infty}\phi_1(x)\mathrm{d}x \ , \qquad (1)$$

where $\phi_1(x) = \phi(\log x)$. Thus, (1) holds $\forall\phi_1\in\mathcal{D}(0,\infty)$. □

Final Step: $\psi(x) \sim x$

Proof

Formally,

$$\frac{1}{\lambda}\sum_{n\leq\lambda}\Lambda(n) = \left\langle \psi'(\lambda x), \chi_{[0,1]}(x)\right\rangle .$$

We approximate $\chi_{[0,1]}$ by elements of $\mathcal{D}(0,\infty)$.

- Let ε be an arbitrary small positive number
- Choose ϕ_1 and ϕ_2 with the properties:
  - $0 \leq \phi_1, \phi_2 \leq 1$
  - $\operatorname{supp}\phi_1 \subseteq (0,1]$, $\phi_1(x) = 1$ on $[\varepsilon, 1-\varepsilon]$
  - $\operatorname{supp}\phi_2 \subseteq (0,1+\varepsilon]$, and $\phi_2(x) = 1$ on $[\varepsilon, 1]$

Final Step: $\psi(x) \sim x$

Proof

Formally,

$$\frac{1}{\lambda}\sum_{n\leq\lambda}\Lambda(n) = \left\langle \psi'(\lambda x), \chi_{[0,1]}(x)\right\rangle \ .$$

We approximate $\chi_{[0,1]}$ by elements of $\mathcal{D}(0,\infty)$.

- Let ε be an arbitrary small positive number
- Choose ϕ_1 and ϕ_2 with the properties:
  - $0 \leq \phi_1, \phi_2 \leq 1$
  - $\operatorname{supp}\phi_1 \subseteq (0,1]$, $\phi_1(x) = 1$ on $[\varepsilon, 1-\varepsilon]$
  - $\operatorname{supp}\phi_2 \subseteq (0,1+\varepsilon]$, and $\phi_2(x) = 1$ on $[\varepsilon, 1]$

Final Step: $\psi(x) \sim x$

Proof

Formally,

$$\frac{1}{\lambda}\sum_{n\leq\lambda}\Lambda(n) = \left\langle \psi'(\lambda x), \chi_{[0,1]}(x)\right\rangle \ .$$

We approximate $\chi_{[0,1]}$ by elements of $\mathcal{D}(0,\infty)$.

- Let ε be an arbitrary small positive number
- Choose ϕ_1 and ϕ_2 with the properties:
  - $0 \leq \phi_1, \phi_2 \leq 1$
  - $\operatorname{supp}\phi_1 \subseteq (0,1]$, $\phi_1(x) = 1$ on $[\varepsilon, 1-\varepsilon]$
  - $\operatorname{supp}\phi_2 \subseteq (0, 1+\varepsilon]$, and $\phi_2(x) = 1$ on $[\varepsilon, 1]$

Final Step: $\psi(x) \sim x$

Proof

Formally,

$$\frac{1}{\lambda}\sum_{n\leq\lambda}\Lambda(n) = \left\langle \psi'(\lambda x), \chi_{[0,1]}(x) \right\rangle \ .$$

We approximate $\chi_{[0,1]}$ by elements of $\mathcal{D}(0,\infty)$.

- Let ε be an arbitrary small positive number
- Choose ϕ_1 and ϕ_2 with the properties:
  - $0 \leq \phi_1, \phi_2 \leq 1$
  - $\operatorname{supp}\phi_1 \subseteq (0,1]$, $\phi_1(x) = 1$ on $[\varepsilon, 1-\varepsilon]$
  - $\operatorname{supp}\phi_2 \subseteq (0,1+\varepsilon]$, and $\phi_2(x) = 1$ on $[\varepsilon, 1]$

Final Step: $\psi(x) \sim x$

Proof (continuation)

- Evaluating at ϕ_2 and using Chebyshev's estimate:

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda}\sum_{x<\lambda}\Lambda(n) \leq \limsup_{\lambda\to\infty}\left(\frac{1}{\lambda}\sum_{x<\varepsilon\lambda}\Lambda(n) + \frac{1}{\lambda}\sum_{n=1}^{\infty}\Lambda(n)\phi_2\left(\frac{n}{\lambda}\right)\right)$$
$$\leq M\varepsilon + \lim_{\lambda\to\infty}\left\langle \psi'(\lambda x), \phi_2(x)\right\rangle$$
$$= M\varepsilon + \int_0^{1+\varepsilon}\phi_2(x)\mathrm{d}x \leq 1 + \varepsilon(M+1)$$

- Likewise, $1 - 2\varepsilon \leq \liminf_{\lambda\to\infty} \frac{1}{\lambda}\sum_{n<\lambda}\Lambda(n)$
- Therefore, $\psi(\lambda) = \sum_{n<\lambda}\Lambda(n) \sim \lambda$

Final Step: $\psi(x) \sim x$

Proof (continuation)

- Evaluating at ϕ_2 and using Chebyshev's estimate:

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda}\sum_{x<\lambda}\Lambda(n) \leq \limsup_{\lambda\to\infty}\left(\frac{1}{\lambda}\sum_{x<\varepsilon\lambda}\Lambda(n) + \frac{1}{\lambda}\sum_{n=1}^{\infty}\Lambda(n)\phi_2\left(\frac{n}{\lambda}\right)\right)$$
$$\leq M\varepsilon + \lim_{\lambda\to\infty}\left\langle \psi'(\lambda x), \phi_2(x)\right\rangle$$
$$= M\varepsilon + \int_0^{1+\varepsilon}\phi_2(x)\mathrm{d}x \leq 1+\varepsilon(M+1)$$

- Likewise, $1-2\varepsilon \leq \liminf_{\lambda\to\infty}\frac{1}{\lambda}\sum_{n<\lambda}\Lambda(n)$
- Therefore, $\psi(\lambda) = \sum_{n<\lambda}\Lambda(n) \sim \lambda$

Final Step: $\psi(x) \sim x$

Proof (continuation)

- Evaluating at ϕ_2 and using Chebyshev's estimate:

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda}\sum_{x<\lambda} \Lambda(n) \leq \limsup_{\lambda\to\infty} \left(\frac{1}{\lambda}\sum_{x<\varepsilon\lambda} \Lambda(n) + \frac{1}{\lambda}\sum_{n=1}^{\infty} \Lambda(n)\phi_2\left(\frac{n}{\lambda}\right)\right)$$
$$\leq M\varepsilon + \lim_{\lambda\to\infty} \left\langle \psi'(\lambda x), \phi_2(x)\right\rangle$$
$$= M\varepsilon + \int_0^{1+\varepsilon} \phi_2(x)\mathrm{d}x \leq 1 + \varepsilon(M+1)$$

- Likewise, $1 - 2\varepsilon \leq \liminf_{\lambda\to\infty} \frac{1}{\lambda}\sum_{n<\lambda} \Lambda(n)$
- Therefore, $\psi(\lambda) = \sum_{n<\lambda} \Lambda(n) \sim \lambda$