Point Behavior of Fourier Series and Conjugate Series

Jasson Vindas
jvindas@cage.ugent.be

Department of Mathematics
Ghent University

Aalto University
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The study of the relationship between the local behavior of periodic functions and convergence or summability properties of Fourier series is a very classical problem in Analysis.

There are essentially three main aspects:

1. Conclude convergence or summability of the series from local behavior (Abelian problem)
2. Extract local information about functions from convergence or summability (usually a Tauberian problem)
3. Beyond the Abel-Tauber problem: Obtain precise characterizations of point behavior in terms of certain summability properties of the series

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Outline

Fatou's Theorem (1906)

Fatou's theorem states that if $f \in L^1[-\pi,\pi]$ with Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta \ ,$$

and its primitive is differentiable at the point $\theta = \theta_0$, i.e.,

$$\lim_{\theta \to \theta_0} \frac{1}{\theta - \theta_0} \int_{\theta_0}^{\theta} f(t)\mathrm{d}t = \gamma \ ,$$

then

$$\lim_{r \to 1^-} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta_0 + b_n \sin n\theta_0) r^n \right) = \gamma \ .$$

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Abel Summability

Definition

A numerical series $\sum_{n=0}^{\infty} c_n$ is called **Abel summable** to γ if

$$\lim_{r\to 1^-} \sum_{n=1}^{\infty} c_n r^n = \gamma \, .$$

One then writes $\sum_{n=0}^{\infty} c_n = \gamma \quad (\mathrm{A})$.

With this notation, the conclusion of Fatou's theorem becomes

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta_0 + b_n \sin n\theta_0 = \gamma \quad (\mathrm{A}) \, .$$

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Harmonic Representations and Fatou's Theorem

For $z = re^{i\theta}$,

$$U(re^{i\theta}) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta_0 + b_n \sin n\theta_0)r^n \ ,$$

then, $U(z)$ is harmonic on $|z| < 1$. Since the primitive of f is differentiable almost everywhere with derivative $f(\theta_0)$, Fatou's theorem tells us:

Corollary

If $f \in L^1[-\pi, \pi]$, then we have almost everywhere

$$f(\theta_0) = \lim_{r \to 1^-} U(re^{i\theta_0}) \ .$$

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Loomis Converse to Fatou's Theorem (1943)

Loomis gave a converse to Fatou theorem in 1943.

Theorem

If f is a positive function and its Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta_0 + b_n \sin n\theta_0 = \gamma \quad (\mathrm{A}) , \tag{1}$$

then the symmetric derivative of the primitive of f exits and equals γ, i.e.,

$$\lim_{\theta \to 0} \frac{1}{2\theta} \int_{\theta_0-\theta}^{\theta_0+\theta} f(t)\mathrm{d}t = \gamma \, . \tag{2}$$

Conversely, (2) implies (1).

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A Theorem of Hardy and Littlewood

Cesàro summability

One says that a series is (C,κ) summable to γ and writes

$$\sum_{n=0}^{\infty} c_n = \gamma \quad (\mathrm{C},\kappa)\ ,$$

if

$$\lim_{n\to\infty} \frac{\kappa!}{n^{\kappa}} \sum_{m=0}^{n} \binom{m+\kappa}{\kappa} c_{n-m} = \gamma\ .$$

The latter is equivalent, by a theorem of M. Riesz (1911), to

$$\lim_{x\to\infty} \sum_{0\leq n<x} c_n \left(1-\frac{n}{x}\right)^{\kappa} = \gamma\ .$$

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Conjugate Series

Let $f \in \mathcal{D}'(\mathbb{R})$, a periodic distribution with Fourier series

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta \ ,$$

the conjugate series is defined as

$$\tilde{f}(\theta) = \sum_{n=1}^{\infty} a_n \sin n\theta - b_n \cos n\theta$$

it gives a well defined distribution.

Remark Even if $f \in L^1[-\pi, \pi]$, $\tilde{f}$ is not a function. One can show the existence of f such that the conjugate distribution $\tilde{f}$ is not integrable on any finite interval.

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Conjugate series and Conjugate Harmonics

Set

$$V(re^{i\theta}) = \sum_{n=1}^{\infty}(a_n \sin n\theta - b_n \cos n\theta)r^n ,$$

the harmonic representation of $\tilde{f}(\theta)$.

One can easily show that V is harmonic conjugate to

$$U(re^{i\theta}) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\, r^n .$$

Therefore, $f(\theta) + i\tilde{f}(\theta)$ is the boundary value of an analytic function from the unit disk.

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Another Classical Result: Abel Summability of Conjugate Series

Version of Fatou theorem for the conjugate series: Let now $f \in L^1[-\pi, \pi]$ if

$$\lim_{\theta \to \theta_0} \frac{1}{\theta - \theta_0} \int_{\theta_0}^{\theta} f(t)\mathrm{d}t = \gamma ,$$

and the principal value integral exists, i.e.,

$$\beta = -\frac{1}{2\pi}\mathrm{p.v.} \int_{-\pi}^{\pi} f(t + \theta_0) \cot\left(\frac{t}{2}\right) \mathrm{d}t ,$$

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Problem of Simultaneous Abel Summability for Fourier and Conjugate Series

Assuming

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Average Point Values of Functions

We shall say that $f \in L^1_{\mathrm{loc}}$ has an average point value of order k at $\theta = \theta_0$ if

$$\lim_{\theta \to \theta_0} \frac{k}{(\theta - \theta_0)^k} \int_{\theta_0}^{\theta} f(t)(\theta - t)^{k-1} \mathrm{d}t = \gamma \, .$$

We write for this $f(\theta_0) = \gamma \quad (\mathrm{C}, k)$.

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Łojasiewicz Point Values

Let now $f \in \mathcal{D}'(\mathbb{R})$. We say that $f(\theta_0) = \gamma$, distributionally, if there exist a non-negative integer k and a function F such that $F^{(k)} = f$ near θ_0 and

$$\lim_{\theta \to \theta_0} \frac{k!F(\theta)}{(\theta - \theta_0)^k} = \gamma \, . \tag{3}$$

- Then, γ is the value of f at $\theta = \theta_0$.
- If (3) holds we say that the point value is of order k and we may write again $f(\theta_0) = \gamma \ \ (\mathrm{C}, k)$.

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Distributional Boundedness at a Point

Let $f \in \mathcal{D}'(\mathbb{R})$. We say that f is distributionally bounded at $\theta = \theta_0$ if there exist a non-negative integer k and a function F such that $F^{(k)} = f$ near θ_0 and

$$F(\theta) = O(|\theta - \theta_0|^k).$$

- Distributional boundedness is often a Tauberian hypothesis.
- If f is locally integrable this is equivalent to have

$$\int_{\theta_0}^{\theta} f(t)(\theta - t)^{k-1} \mathrm{d}t = O(|\theta - \theta_0|^k) .$$

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A Tauberian Theorem

The main tool studying simultaneous Abel summability is the following Tauberian result:

Theorem

Let f be a 2π-periodic distribution. Suppose that

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta_0+b_n\sin n\theta_0)=\gamma \quad (\mathrm{A}) \ ,$$

and

$$\sum_{n=1}^{\infty}(a_n\sin n\theta_0-b_n\cos n\theta_0)=\beta \quad (\mathrm{A}) \ .$$

If either f or $\tilde{f}$ is distributionally bounded at $\theta=\theta_0$, then $f(\theta_0)=\gamma$ and $\tilde{f}(\theta_0)=\beta$, distributionally.

Simultaneous (A) Summability and Functions Bounded from Below

Theorem Let $f \in L^1[-\pi,\pi]$ be bounded from below (or above) in some neighborhood of $\theta=\theta_0$. The following are equivalent:

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- Both series are (C,κ) summable for any $\kappa>0$.
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Furthermore,

$$\gamma=\lim_{\theta\to\theta_0}\frac{1}{\theta-\theta_0}\int_{\theta_0}^{\theta}f(t)\mathrm{d}t\,; \quad \beta=-\frac{1}{2\pi}\mathrm{p.v.}\int_{-\pi}^{\pi}f(t+\theta_0)\cot\left(\frac{t}{2}\right)\mathrm{d}t$$

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More general results are also valid for distributions and positive measures. This talk is based on a joint work with R. Estrada:

On the Point Behavior of Fourier Series and Conjugate Series, Zeitschrift fur Analysis und Ihre Anwendungen (2010), to appear soon