Pointwise Fourier Inversion Formula for Tempered Distributions

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- What does f at x_0 mean?
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- What is the meaning of $\int_{-\infty}^{\infty} \hat{f}(t)e^{-itx_0}dt$?
  Later...

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- The evaluation of f at a test function ϕ is denoted by

$$\langle f(x), \phi(x) \rangle$$

Distributional Point Values

Lojasiewicz defined the value of a distribution $f \in \mathcal{D}'$ at the point x_0 as the limit

$$f(x_0) = \lim_{\varepsilon \to 0} f(x_0 + \varepsilon x),$$

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In terms of test functions, it means that for all $\phi \in \mathcal{D}$

$$\lim_{\varepsilon \to 0} \frac{1}{\epsilon} \left\langle f(x), \phi\left(\frac{x - x_0}{\epsilon}\right)\right\rangle = f(x_0) \int_{-\infty}^{\infty} \phi(x) dx.$$

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- **Notation:** If $f \in \mathcal{D}'$ has a value γ at x_0, we say that $f(x_0) = \gamma$ in $\mathcal{D}'$. The meaning of $f(x_0) = \gamma$ in $\mathcal{S}'$, ..., must be clear.

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- **Remark:** R.Estrada has shown that if $f \in \mathcal{S}'$, then $f(x_0) = \gamma$ in $\mathcal{D}'$ implies $f(x_0) = \gamma$ in $\mathcal{S}'$.

Characterization of Distributional Point Values

Lojasiewicz showed that $f(x_0) = \gamma$ is equivalent to the existence of $n \in \mathbb{N}$, and a primitive of order n of f which is continuous in a neighborhood of x_0 and satisfies

$$\lim_{x \to x_0} \frac{n! F(x)}{(x - x_0)^n} = \gamma.$$

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Remark: If μ is a measure the above formula reads as

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In particular if f is locally integrable and x_0 is a Lebesgue point of f, then f has a distributional point value at x_0

Abel Summability

We say that $\sum a_n$ is Abel summable to γ, and write

$$\sum_{n=0}^{\infty} a_n = \gamma \; (A)$$

if

$$\lim_{r \to 1^-} \sum_{n=0}^{\infty} a_n r^n = \gamma$$

Remark: We can also define Abel summability with respect to a positive increasing sequence $\{\lambda_n\}$ by replacing the term r^n by r^{λ_n} in such case we write

$$\sum a_n = \gamma \; (A, \lambda_n)$$

Cesaro limits and Cesaro summability

We say that

$$\lim_{n\to\infty} a_n = \gamma \ (C, 1)$$

if

$$\lim_{n\to\infty} \frac{a_0 + a_1 + \ldots + a_n}{n+1} = \gamma.$$

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Remark: $\sum a_n = \gamma \ (C,1)$ means that the limit of the partial sums is equal to γ in the $(C,1)$ sense.
Remark: We can continue taking average and we end up with the (C,k) sense

A very basic result in summability of Fourier Series

Suppose that $f \in L^1[0,2\pi]$ and let $\{c_n\}_{n\in\mathbb{Z}}$ be its Fourier coefficients. Then if f is continuous at $x_0 \in (0,2\pi)$, then

$$\lim_{N\to\infty} \sum_{-N}^{N} c_n e^{ix_0 n} = f(x_0) \ (C,1).$$

Summability of Fourier Series

G.Walter proved the following:

Theorem 1 *Let f be a periodic distribution (and hence tempered) with Fourier Series*

$$\sum_{n=0}^{\infty} c_n e^{inx}.$$

Then, $f(x_0) = \gamma$ *in* $\mathcal{S}'$ *iff*

$$\sum_{n=0}^{\infty} c_n e^{inx_0} = \gamma \ (C, k),$$

for some $k \in \mathbb{N}$.

Summability of Fourier Series

Moreover, he also showed

Theorem 2 *Let f be a periodic distribution (and hence tempered) with Fourier Series*

$$\sum_{-\infty}^{\infty} c_n e^{inx}.$$

If $f(x_0) = \gamma$ *in* $\mathcal{S}'$*, then for some* $k \in \mathbb{N}$

$$\lim_{N\to\infty} \sum_{-N}^{N} c_n e^{inx_0} = \gamma \ (C, k).$$

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- Under certain assumptions on the conjugated series, G.Walter gave a sort of converse of this result.

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- Under certain assumptions on the conjugated series, G.Walter gave a sort of converse of this result.
- If we only assume the (C,k)-summability of the symmetric partial sums, the converse is far from being true as shown by

$$2\sum_{n=1}^{\infty}\frac{sin(nx)}{n} = -i\sum_{n\neq 0}\frac{e^{inx}}{n}.$$

at $x=0$

Characterization of Point Values

R.Estrada has characterized the distributional point values of a periodic distribution in terms of the summability of its Fourier Series.

Theorem 3 *Let $f \in \mathcal{S}'$ be a periodic distribution of period 2π and let $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ be its Fourier series. Let $x_0 \in \mathbb{R}$. Then*

$$f(x_0) = \gamma \ in \ \mathcal{D}'$$

if and only if there exists k such that

$$\lim_{x\to\infty} \sum_{-x\le n\le ax} a_n e^{inx_0} = \gamma \ (\mathrm{C}, k)$$

for each $a > 0$.

Needed for a generalization

The last Theorem admits a generalization to tempered distribution which "looks" like

$$f(x_0) = \lim_{x\to\infty} \int_{-x}^{ax} \hat{f}(t) e^{-itx_0} dt \ (C).$$

Cesaro behavior of Distributions

Let $f \in \mathcal{D}'$ and and $\alpha \in \mathbb{R} - \{-1, -2, -3, ...\}$, then we say that

$$f(x) = O\left(x^{\alpha}\right) \ (C, N) \ as \ x \to \infty,$$

if every primitive F of order N is an ordinary function(locally integrable) for large arguments and satisfies the ordinary order relation,

$$F(x) = p(x) + O\left(x^{\alpha+N}\right) \ as \ x \to \infty$$

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Note that if $\alpha > -1$, then the polynomial p is irrelevant.

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- The definitions when $x \to -\infty$ are clear.
- One can define the limit at ∞ in the Cesàro sense for distribution. We say that $f \in \mathcal{D}'$ has a limit L at infinity in the Cesàro sense and write

$$\lim_{x\to\infty} f(x) = L \ (\mathrm{C}) \, ,$$

if $f(x) = L + o(1) \ (\mathrm{C})$, as $x \to \infty$.

Parametric Behavior

The Cesàro behavior of a distribution f at infinity is related to the parametric behavior of $f(\lambda x)$ as $\lambda \to \infty$(To be interpreted in the weak sense, i.e evaluating at test functions)

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In fact, one can show that if $\alpha > -1$, then $f(x) = O\left(x^{\alpha}\right)$ (C) as $x \to \infty$ and $f(x) = O\left(|x|^{\alpha}\right)$ (C) as $x \to -\infty$ if and only if

$$f(\lambda x) = O\left(\lambda^{\alpha}\right) \ as \ \lambda \to \infty \, ,$$

Special values of distributional evaluations

Definition 1 *Let $g \in \mathcal{D}'$, and $k \in \mathbb{N}$. We say that the evaluation $\langle g(x), \phi(x) \rangle$ exists in the $\mathrm{e.v.}$ Cesàro sense, and write*

(1)
$$\mathrm{e.v.} \langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}, k),$$

if for some primitive G of $g\phi$ and $\forall a > 0$ we have

$$\lim_{x \to \infty} (G(ax) - G(-x)) = \gamma \; (\mathrm{C}, k).$$

If g is locally integrable then we write (1) as

$$\mathrm{e.v.} \int_{-\infty}^{\infty} g(x) \phi(x) \, \mathrm{d}x = \gamma \quad (\mathrm{C}, k).$$

Remark:*In this definition the evaluation of g at ϕ does not have to be defined, we only require that $g\phi$ is well defined.*

Example

Suppose that $\{\lambda_n\}$ is positive increasing sequence. If $g \in \mathcal{S}'$ is given by $g(x) = \sum_{n=0}^{\infty} a_n \delta(x - \lambda_n)$, then

$$\text{e.v}\left\langle \sum_{n=0}^{\infty} a_n \delta(x - \lambda_n), 1 \right\rangle = \gamma\left(C, k\right)$$

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if and only if

$$\sum a_n = \gamma \ \ (R, \lambda_n, k)$$

Pointwise Inversion Formula

Now, we characterize the point values of a distribution in $\mathcal{S}'$ by using Fourier transforms.

Theorem 4 *Let $f \in \mathcal{S}'$. We have $f(x_0) = \gamma$ in $\mathcal{S}'$ if and only if there exists a $k \in \mathbb{N}$ such that*

$$\frac{1}{2\pi}\mathrm{e.v.}\left\langle \hat{f}\left(t\right), e^{-ix_0 t}\right\rangle = \gamma \quad \left(C, k\right),$$

which in case $\hat{f}$ is locally integrable means that

$$\frac{1}{2\pi}\mathrm{e.v.}\int_{-\infty}^{\infty} \hat{f}(t)e^{-ix_0 t}\mathrm{d}t = \gamma \quad \left(C, k\right).$$

Consequences

Estrada Theorem on Fourier Series follows at one by looking at the form of the Fourier transforms of periodic distributions. Moreover,

Theorem 5 *Let* $\{\lambda_n\}_{n=0}^{\infty}$ *be an increasing sequence of positive real numbers. Let*

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{isgn(n)\lambda_n x} \; in \; \mathcal{S}'. \tag{2}$$

Then, $f(x_0) = \gamma \; in \; \mathcal{D}'$,
if and only if there exists $k \in \mathbb{N}$ *such that*

$$\lim_{x\to\infty} \sum_{-x\leq n\leq ax} a_n e^{isgn(n)\lambda_n x_0} = \gamma \left(\mathrm{R}, \lambda_n, k\right),$$

for each $a > 0$.

Consequences

The inversion formula can be specialized as follows.

Theorem 6 *Let $f \in \mathcal{S}'$. Suppose that $supp\ \hat{f} \subseteq [0, \infty)$. We have $f(x_0) = \gamma$ in $\mathcal{S}'$ if and only if there exists a $k \in \mathbb{N}$ such that every k-primitive of $e^{-ixx_0}\hat{f}$ is locally integrable and*

$$(e^{-ixx_0}\hat{f}) * x_+^k = \gamma x^k + o(x^k)\ as\ x \longrightarrow \infty.$$

Example

Suppose that

$$f(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} e^{-ig(n,m)x},$$

then, $f(x_0) = \gamma$ in $\mathcal{S}'$ iff there is a k such that

$$\lim_{x\to\infty} \sum_{g(n,m)\le x} a_{n,m} \left(1 - \frac{g(n,m)}{x}\right)^k = \gamma.$$

Order of Point Values

Definition 2 *We say that* $f(x_0) = \gamma$ *in* $\mathcal{D}'$ *has order* k*, if* k *is the minimum integer such that there exists a primitive of order* k *of* f*,* F*, such that* F *is locally integrable in a neighborhood of* x_0 *and F satisfies*

$$\lim_{x \to x_0} \frac{n!F(x)}{(x - x_0)^n} = \gamma.$$

Remark: Lojasiewicz had defined the order of the point value in a different way, but I propose this new definition to be consistent with the following Theorems.

Order of inversion Formula

Theorem 7 *Let $f \in \mathcal{S}'$. Suppose that there exists a $m \in \mathbb{N}$, such that every m-primitive h of f, i.e., $h^{(m)} = f$, is locally integrable and $h(x) = O\left(|x|^{m-1}\right)$. Let m_0 be the smallest natural number with this property. If f has a distributional point value γ at x_0, whose order is n, then*

$$\frac{1}{2\pi} e.v. \left\langle \hat{f}(x), e^{-ix_0 x} \right\rangle = \gamma \; (C, k+1),$$

where $k = max\{m_0, n\}$.

Two Remarkable Cases

Define

$$\phi_a^\beta(t) = (1+t)^\beta \chi_{[-1,0]}(t) + \left(1 - \frac{t}{a}\right)^\beta \chi_{[0,a]}(t).$$

Two Remarkable Cases

Theorem 8 *Let f be a distribution with compact support and order n. Suppose that $f(x_0) = \gamma$ in $\mathcal{D}'$ with order k. Let $\beta > max\{k, n+1\}$. Then for each $a > 0$*

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-x}^{ax} \hat{f}(t) e^{-ix_0 t} dt = \gamma \ (C, \beta)$$

or which is the same

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_a^\beta \left(\frac{t}{x}\right) \hat{f}(t) e^{-ix_0 t} dt = \gamma,$$

Moreover, these relations hold uniformly for a in compact subsets of $(0, \infty)$.

Two Remarkable Cases

Theorem 9 *Let f be a 2π-periodic distribution of order n, with Fourier series $\sum_{-\infty}^{\infty} c_n e^{ixn}$. If $f(x_0) = \gamma$ in $\mathcal{D}'$ with order k. Let $\beta > max\{k, n+1\}$. Then for each $a > 0$,*

$$\lim_{x\to\infty} \sum_{-x\leq n\leq ax} c_n e^{ix_0 n} = \gamma \ (C, \beta),$$

or equivalently

$$\lim_{x\to\infty} \sum_{-x\leq n\leq ax} \phi_a^{\beta}\left(\frac{n}{x}\right) c_n e^{ix_0 n} = \gamma.$$

Moreover, these relations hold uniformly for a in compact subsets of $(0, \infty)$.

A special case of interest

Theorem 10 *Suppose that $f \in \mathcal{S}'$ is such that $supp\,\hat{f}$ is bounded at the left. If $f(x_0) = \gamma$ in $\mathcal{D}'$ with order k, and f is the derivative of order k of a locally integrable function which is $O\left(x^{k-1}\right)$, then*

$$\left(t_+^k * \left(\hat{f}(t)e^{-ix_0 t}\right)\right)$$

is locally integrable and for every $\beta > k$,

$$\left(t_+^\beta * \left(\hat{f}(t)e^{-ix_0 t}\right)\right)(x) = 2\pi\gamma x^\beta + o\left(x^\beta\right) \; as \; x \to \infty.$$

Order of Point Value

Theorem 11 *Let $f \in \mathcal{S}'$. Suppose that*

$$\frac{1}{2\pi} e.v. \left\langle \hat{f}(x), e^{-ixx_0} \right\rangle = \gamma \ (C, k);$$

then, $f(x_0) = \gamma$ in $\mathcal{S}'$ f is the derivative of order $k+1$ of a locally integrable function and the order of $f(x_0)$ is less or equal to $k+2$.