Minicourse:
Tauberian Theorems

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Chuo University, Tokyo
July 7-9, 2025

These are lecture notes of an introductory minicourse on Tauberian theorems that the author delivered at Chuo university in the period July 7-9, 2025. The minicourse consisted of the following three topics:

- Lecture 1 On some classical Tauberian theorems of Hardy and Littlewood.
- Lecture 2 The Wiener-Ikehara theorem
- Lecture 3 Some recent developments on the Wiener-Ikehara theorem.

Preliminaries: Besides standard notation from classical analysis, we will make use of Schwartz distributions in our manipulations. We recall that $\mathcal{D}(I)$ stands for the space of smooth compactly supported test functions on an open subset $I \subseteq \mathbb{R}$. The space of distributions $\mathcal{D}'(I)$ is its topological dual. The dual pairing between a distribution f and a test function φ is denoted by $\langle f, \varphi \rangle$, or by $\langle f(x), \varphi(x) \rangle$, with the use of a dummy variable of evaluation.

We fix constants in the Fourier transform as follows:

$$\hat{f}(t) = \mathcal{F}\{f;t\} = \int_{-\infty}^{\infty} f(x)e^{-itx}\,dx, \quad t \in \mathbb{R}.$$

The Schwartz space of rapidly decreasing smooth functions is denoted by $S(\mathbb{R})$. Its dual is the space of tempered distributions $S'(\mathbb{R})$. We point out that the Fourier transform is automorphism $\mathcal{F}: S(\mathbb{R}) \longrightarrow S(\mathbb{R})$. This lets us define the Fourier transform $\mathcal{F}: S'(\mathbb{R}) \longrightarrow S'(\mathbb{R})$ by duality:

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle, \quad f \in S'(\mathbb{R}) \text{ and } \varphi \in S(\mathbb{R}).$$

We refer to [Vla] for distribution theory.

We need one more spaces, $S'[0,\infty) = \{f \in S'(\mathbb{R}) : \operatorname{supp} f \subseteq [0,\infty)\}$. It can be canonically identified with the dual of $S[0,\infty) = \{\psi \in C^{\infty}[0,\infty) : \exists \varphi \in S(\mathbb{R})$ such that $\psi = \varphi_{|[0,\infty)}\}$. Since $e^{-sx} \in S[0,\infty)$ whenever $\operatorname{Re} s > 0$, the Laplace transform of $f \in S'[0,\infty)$ is well-defined

$$\mathcal{L}\{f;s\} = \langle f(x), e^{-sx} \rangle, \quad s = \sigma + it \in \mathbb{C}, \ \sigma > 0.$$

The next lemma is classical, a proof can be found in [Vla] (see also [Ze]).

Lemma 0.1 Let $f \in S'[0,\infty)$. Then

$$(0.1) \quad \hat{f}(t) = \lim_{\sigma \to 0^+} \mathcal{L}\{f; \sigma + it\} \text{ in } S'(\mathbb{R}),$$

that is, its Fourier transform is the distributionally boundary value of its Laplace transform on the imaginary axis. ///

The limit (0.1) is distributional, so it means that

$$\langle \hat{f}(t), \varphi(t) \rangle = \lim_{\sigma \to 0^+} \int_{-\infty}^{\infty} \varphi(t) \mathcal{L}\{f; \sigma + it\}\, dt,$$

for all test function $\varphi \in S(\mathbb{R})$. The following corollary will be of importance to us in the future.

Corollary 0.2 Let D be a subset of $\{s : \operatorname{Re} s > 0\}$ with an accumulation point in this half plane. Then the linear span of $\{e^{-sx} : s \in D\}$ is dense in $S[0,\infty)$. ///

Proof. We shall use the Hahn-Banach criterion for density of subspaces. So, let $f \in S'[0,\infty)$ be arbitrary and such that $\langle f, \varphi \rangle = 0$, $\forall \varphi \in D$. The conclusion of this corollary would follow if we are able to show that $f = 0$. But our assumption just says that

$$\mathcal{L}\{f; s\} = \langle f(x), e^{-sx} \rangle = 0, \ \forall s \in D.$$

By the uniqueness property of analytic functions,

$$\mathcal{L}\{f; s\} = 0, \ \forall s \text{ with } \operatorname{Re} s > 0.$$ (by Lemma 0.1,

we obtain

$$\hat{f}(t) = \lim_{\sigma \to 0^+} \mathcal{L}\{f; \sigma + it\} = 0.$$

Since $\hat{f}$ vanishes and the Fourier transform is an isomorphism on the space $S'(\mathbb{R})$, we conclude $f = 0$, as required. ///

1. On some classical Tauberian theorems of Hardy and Littlewood

(by Jasson Vindas presented at Chuo University, Tokyo, 7-7-25)

$\boxed{1.1}$ Introduction: More than a century ago, Hardy and Littlewood proved a series of Tauberian theorems for power series. In this first talk we will discuss some of these results and their variants. Some of arguments below are based on the distributional method of R. Estrada and the speaker from [EV07, EV08]. Our proof of the Hardy-Littlewood-Karamata theorem is an adaptation of Vladimirov's treatment, see [Vla] for multi-dimensional generalizations.

For classical accounts on the subject, we refer to the excellent books by Hardy [H] and Korevaar [K].

$\boxed{1.2}$ Tauber's theorem: Everyone who has taken a basic analysis course should be familiar with the next theorem due to Abel:

Theorem 1.1 (Abel): Let $\{a_n\}_{n\in\mathbb{N}}$ be a numerical series. If $\sum_{n=0}^{\infty} a_n = \ell \Rightarrow \lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n r^n = \ell$. ///

The proof of this theorem is easy if we express series as Stieltjes integrals. It is easier to work with $r = e^{-\sigma}$, so that $\sigma \to 0^+$ if $r \to 1^-$. We write

$$S(x) = \sum_{n \le x} a_n ,$$

so that the assumption of Theorem 1 is $S(x) \longrightarrow \mathcal{A}$, $x \longrightarrow \infty$. Therefore,

$$\sum_{n=0}^{\infty} a_n e^{-\sigma n} = \int_{0^-}^{\infty} e^{-\sigma x}\, dS(x) = \sigma \int_0^{\infty} S(x) e^{-\sigma x}\, dx$$

$$= \int_0^{\infty} S\left(\frac{u}{\sigma}\right) e^{-u}\, du \xrightarrow[\sigma \to 0^+]{} \mathcal{A} \int_0^{\infty} e^{-u}\, du = \mathcal{A}.$$

Note that the converse of Abel's theorem is not true in general. <u>Example:</u> $\sum_{n=0}^{\infty} (-1)^n$ diverges but

$$\sum_{n=0}^{\infty} (-1)^n e^{-\sigma n} = \frac{1}{1+e^{-\sigma}} \longrightarrow \frac{1}{2} \text{ as } \sigma \to 0^+.$$

The beginning of Tauberian theory goes back to the following partial converse, whose proof is easy as well:

<u>Theorem 1.2 (Tauber, 1897).</u> Suppose that

$$F(\sigma) := \sum_{n=0}^{\infty} a_n e^{-\sigma n}$$ converges for all $y > 0$ and that $F(\sigma) \longrightarrow \mathcal{A}$ as $\sigma \longrightarrow 0^+$. If the ensuing Tauberian condition is satisfied:

$$a_n = o\left(\frac{1}{n}\right), \quad n \longrightarrow \infty, \qquad (1.1)$$

then $$\sum_{n=0}^{\infty} a_n = \mathcal{A}. \quad ///$$

<u>Proof.</u> We have $|a_n| \le \varepsilon/n$ for, say, $n \geq x_0$. Hence,

$$\left| \sum_{n \le x} a_n - F\left(\frac{1}{x}\right) \right| \le \left| \sum_{n \le x_0} a_n \left(1 - e^{-\frac{n}{x}}\right) \right| + \cdots$$

$$\dots + \varepsilon \sum_{0<\frac{n}{x}\leq 1} (1-e^{-\frac{n}{x}})\cdot\frac{1}{n} + \varepsilon \sum_{1<\frac{n}{x}} e^{-\frac{n}{x}}\cdot\frac{1}{x}$$

$$\leq \Big|\sum_{n\leq x_0} a_n (1-e^{-\frac{n}{x}})\Big| + \varepsilon\cdot\Big[\int_0^1 \frac{(1-e^{-u})}{u}du + \int_1^\infty e^{-u}du\Big]\Big)$$

Therefore, the result follows by taking $\varliminf_{x\to\infty}$ sup and then $\varepsilon \to 0^+$. ///

$\boxed{1.3}$ Littlewood's theorem:

In 1910 Littlewood proved that Tauber's theorem still holds if (1.1) is replaced by the weaker condition

$$a_n = O\left(\frac{1}{n}\right). \qquad (1.2)$$

This change makes the theorem deeper and more difficult to show. There are several variants. For example, Hardy and Littlewood conjectured that the condition (1.2) could still be weakened to the one-sided hypothesis

$$a_n \geqslant -\frac{M}{n} \quad (\text{for some } M>0), \qquad (1.3)$$

and in fact provided a proof of their conjectured in 1914.

We give here a short distributional proof of Littlewood's theorem on condition (1.2).

<u>Theorem 1.3</u> Let $\{a_n\}_{n\in\mathbb{N}}$ satisfy the Tauberian condition (1.2). Then

$$F(\sigma) = \sum_{n=0}^{\infty} a_n e^{-\sigma n} \to \gamma \text{ as } \sigma \to 0^+ \quad (1.4)$$

implies that

$$\sum_{n=0}^{\infty} a_n = \gamma \, . \quad (1.5)$$

Proof. Reasoning as in the proof of Theorem 1.2, one readily shows that (all that is need is that $F(\sigma) = O(1)$)

$$S(x) := \sum_{n \leq x} a_n = O(1) . \quad (1.6)$$

We consider the family of tempered distributions

$$f_\sigma(x) = \sum_{n=0}^{\infty} a_n \delta(x - \sigma \cdot n) \, , \ \sigma > 0 ,$$

namely, $\langle f_\sigma , \varphi \rangle = \sum_{n=0}^{\infty} a_n \varphi(\sigma \cdot n)$.

Our first goal is to show that this family converges to the Dirac delta δ times γ as $\sigma \to 0^+$, i.e., for each Schwartz rapidly decreasing smooth function, that is, for each $\varphi \in \mathcal{S}[0, \infty)$, we claim that

$$\lim_{\to 0^+} \langle f_\sigma , \varphi \rangle = \lim_{\sigma \to 0^+} \sum_{n=0}^{\infty} a_n \varphi(\sigma \cdot n) = \gamma \cdot \varphi(0) \quad (1.7) \, .$$

Our hypothes (1.4) says that (1.7) holds for any linear combination of expenentials in the Schwartz class $\mathcal{S}[0, \infty)$, which forms a dense subspace by Corollary 0.2. We recall from basic functional analysis that a family of continuous linear functionals is weakly* convergent if it is convergent on a dense set and it is equicontinuous. It is enough then to show the latter and we do it via (1.6):

$$\langle f_\sigma, \varphi \rangle = \int_{0^-}^{\infty} \varphi(\sigma \cdot x)\, dS(x) = -\int_{0}^{\infty} S\left(\frac{x}{\sigma}\right) \varphi'(x)\, dx$$

$$= O(1) \cdot \sup_{x \geq 0} (x^2+1) \cdot |\varphi'(x)| ,$$

so, we have establish (1.7) To show (1.5), we make a choice for φ in (1.7). Let $\varepsilon > 0$ and choose M such that

$$|a_n| \leq \frac{M}{n}, \quad n \geq 1. \qquad (1.8)$$

Pick $\varphi \in C^\infty[0,\infty)$ such that $0 \leq \varphi \leq 1$ and

ⓘ $\varphi(x) = 1$ for $x \in [0,1]$ and ⓘⓘ $\varphi(x) = 0$ for $x > 1+\varepsilon$

Now,

$$\varlimsup_{x \to \infty} \left| \sum_{n \leq x} a_n - \gamma \right| =$$

$$\varlimsup_{x \to \infty} \left| \sum_{n=0}^{\infty} a_n \varphi\left(\frac{n}{x}\right) - \gamma \varphi(0) - \sum_{1 < \frac{n}{x} < 1+\varepsilon} a_n \varphi\left(\frac{n}{x}\right) \right|$$

$$\leq \limsup_{x\to\infty}\left|\sum_{n=0}^{\infty} a_n\varphi\left(\frac{n}{x}\right)-\gamma\right| + \limsup_{x\to\infty}\left|\sum_{1<\frac{n}{x}\leq(1+\varepsilon)} a_n\varphi\left(\frac{n}{x}\right)\right|$$

$$\leq \limsup_{x\to\infty}\left|\sum_{x<n\leq x(1+\varepsilon)} a_n\varphi\left(\frac{n}{x}\right)\right|$$

$$\leq M\cdot\lim_{x\to\infty}\sum_{1<\frac{n}{x}\leq 1+\varepsilon}\frac{1}{n}$$

$$= M\log(1+\varepsilon).$$

The proof is now complete upon taking $\varepsilon\to0^+$.

$\boxed{1.4}$ The Hardy-Littlewood condition

The same technique used in the previous section can be used to show Theorem 1.3 with (1.2) replaced by (1.3) if one is able to establish the following lemma.

Lemma 1.4 Suppose that (1.3) holds. If

$$F(\sigma)=\sum_{n=0}^{\infty} a_n e^{-\sigma n}=O(1),\quad \sigma\to0^+, \qquad (1.9)$$

then

$$\sum_{n\leq x} a_n = O(1),\quad x\to\infty \qquad (1.10).$$

Proof. Let $S(x)=\sum_{n\leq x} a_n$. Consider an exponential polynomial $P(u)=\sum_{k=1}^{M} b_k e^{-ku}$.

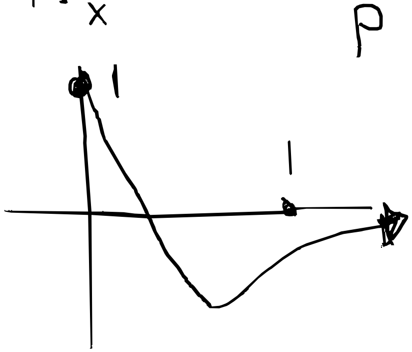
Note that

$$F_p(r) = \sum_{n=0}^{+\infty} a_n P(rn) = \sum_{k=1}^{M} b_k F(k\cdot r) = O(1),\ r\to 0.$$

We compare $S(x)$ with $F_p(\frac{1}{x})$:

$$S(x) - F_p(x) = \sum_{\frac{n}{x}\leq 1}\left[1-P\left(\frac{n}{x}\right)\right]a_n + \sum_{1\leq\frac{n}{x}} a_n\left(-P\left(\frac{n}{x}\right)\right). \quad (1.11)$$

If we choose P s.t.



① $P(0)=1$

② $P(u)\leq 1$ for $u\in[0,1]$

③ $P(u)\leq 0$ for $u\geq 1$,

we obtain

$$S(x)-F_p(x) \geq -M\left[\sum_{n\leq x}\frac{1-P(\frac{n}{x})}{n/x}\cdot\frac{1}{x} + \sum_{n>x}\frac{-P(\frac{n}{x})}{n/x}\cdot\frac{1}{x}\right]$$

$-4x+1$

$$= -M\left[\int_0^1\frac{1-P(u)}{u}\,du + \int_1^\infty\frac{-P(u)}{u}\,du\right] + O(1)$$

$$= O(1).$$

An example of the polynomial P is $P(u) = 2e^{-2u} - e^{-u}$.

Our second choice is a polynomial Q such that:

Ⓐ $Q(0)=1$ Ⓑ $Q(u)\geq 1$ for $u\in[0,1]$ and Ⓒ $P(u)\geq 0$ for $u\in[1,\infty)$

(An example of such polynomial is

$$Q(u)=\frac{e^2+1}{1+e}e^{-u}-\left[\frac{e^2+1}{1+e}-1\right]e^{-2u}.)$$

Then inserting it into (1.11),

$$S(x)-F_p(x)\leq M\left(\sum_{n\leq x}\frac{Q\left(\frac{n}{x}\right)-1}{n}+\sum_{n>x}P\left(\frac{n}{x}\right)\frac{x}{n}\cdot\frac{1}{x}\right)$$

$$=O(1). \quad ///$$

We can now adapt the proof of Theorem 1.3.

Theorem 1.5 Suppose (1.3) and (1.4) are satisfied. Then (1.5) must hold.

Proof. In view of Lemma 1.4, the same argument as in the proof of Theorem 1.3 lead to

$$\lim_{r\to 0^+}\sum_{n=0}^{\infty}a_n\varphi(r\cdot n)=\gamma\varphi(0) \qquad (1.12)$$

for every $\varphi\in S[0,\infty)$. We choose first φ in (1.12) such that $\operatorname{supp}\varphi\subseteq[0,1+\varepsilon]$, $0\leq\varphi\leq 1$, and $\varphi(x)=1$ for $x\in[0,1]$. We might assume that $a_0=0$. Using that

$$a_n+\frac{M}{n}\geq 0,$$

$$\limsup_{x\to+\infty} \sum_{n\le x} a_n \le \limsup_{x\to+\infty} \sum_{n=0}^{\infty} \left(a_n+\frac{M}{n}\right)\varphi\left(\frac{n}{x}\right) - M\sum_{n\le x}\frac{1}{n}$$

$$\le \gamma + \lim_{x\to\infty} M\cdot\left[\sum_{x\le n\le x(1+\varepsilon)} \frac{1}{n}\right]$$

$$= \gamma + M\cdot\log(1+\varepsilon).$$

Taking $\varepsilon\to 0^+$, we deduce that

$$\limsup_{x\to\infty} \sum_{n\le x} a_n \le \gamma.$$

Similarly, we now select φ such that $\operatorname{supp}\varphi \subseteq [0,1]$, $0\le\varphi\le 1$, and $\varphi(x)=1$ for $x\in[0,1-\varepsilon]$. Then

$$\liminf_{x\to\infty} \sum_{n\le x} a_n \ge \liminf_{x\to\infty} \sum_{n=0}^{\infty} \left(a_n+\frac{M}{n}\right)\varphi\left(\frac{n}{x}\right) - M\sum_{n\le x}\frac{1}{n}$$

$$= \gamma + \liminf_{n\to\infty} M \sum_{(1-\varepsilon)x\le n\le x} -\frac{1}{n}$$

$$= \gamma + M\log(1-\varepsilon),$$

so that we get $\liminf_{x\to\infty} \sum_{n\le x} a_n \ge \gamma$ after letting $\varepsilon\to 0^+$. ////

1.5 The Hardy-Littlewood-Karamata theorem.

Hardy and Littlewood also considered Tauberian theorems, where the conclusion is asymptotic behavior of the partial sums. In fact, they showed that if

$$F(\sigma)=\sum_{n=0}^{\infty} a_n e^{-\sigma n} \sim \frac{A}{\sigma^\alpha}, \quad \sigma \to 0^+ \quad (\alpha>0)$$

and if $a_n \geq 0$, then

$$\sum_{n\leq x} a_n \sim \frac{A}{\Gamma(\alpha+1)}\cdot x^\alpha, \quad x\to 0^+$$

In the years 30s, Karamata found a simple proof of this Hardy-Littlewood theorem and very much generalized, leading to the so-called theory of regularly varying functions [BGT, K]. We state and prove here a particular case of Karamata's theorem.

<u>Theorem 1.6</u> (Hardy-Littlewood-Karamata). Let $\alpha>0$ and let S be a non-decreasing function on $[0,\infty)$. Then

$$S(x)\sim A\cdot\frac{x^\alpha}{\Gamma(\alpha+1)}, \quad x\to\infty \qquad (1.13)$$

if and only if

$$\mathcal{L}\{dS;\sigma\}=\int_{0^-}^{\infty} e^{-\sigma x}\, dS(x)\sim\frac{A}{\sigma^\alpha}, \quad \sigma\to 0^+ \qquad (1.14)$$

Proof. Assume first that (1.13) holds. Integrating by parts

$$\mathcal{L}\{dS;\sigma\} = \sigma \int_0^\infty S(x)\, e^{-\sigma x}\, dx$$

$$= \frac{1}{\sigma^\alpha} \int_0^\infty \frac{S(\frac{x}{\sigma})}{(\frac{x}{\sigma})^\alpha} \cdot x^\alpha e^{-x}\, dx$$

$$\sim \frac{1}{\sigma^\alpha} \frac{A}{\Gamma(\alpha+1)} \cdot \int_0^\infty x^\alpha e^{-x}\, dx = \frac{A}{\sigma^\alpha}.$$

Conversely, suppose that (1.14) is satisfied. We first prove that

$$S(x) = O(x^\alpha) \qquad (1.15).$$

This follows from an elementary trick. In fact,

$$S(x) = \int_{0^-}^{x} dS(u) \le \int_{0^-}^{x} e^{1-\frac{u}{x}}\, dS(u) \le e \cdot \mathcal{L}\{dS, \tfrac{1}{x}\},$$

whence (1.15) follows.

In the next step we introduce an auxiliary family of distributions $\{f_x\}_{x>0}$, given by

$$\langle f_x, \varphi \rangle = \frac{1}{x^\alpha} \int_{0^-}^{\infty} \varphi\left(\frac{u}{x}\right) dS(u).$$

Using (1.15) and integration by parts,

$$\langle f_x, \varphi \rangle = \frac{1}{x^\alpha} \int_{0^-}^{\infty} S(xu)\, \varphi(u)\, dx = O(1), \quad x \to \infty,$$

which, together with the Banach-Steinhaus theorem, tells us that $\{f_x\}_{x\geq 1}$ is equicontinuous (as a family of elements of $S'[0,\infty)$). If we take $\varphi(u) = e^{-\lambda u}$

$$\langle f_x, \varphi \rangle = \frac{1}{x^\alpha}\mathcal{L}\{dS, \frac{\lambda}{x}\} \sim \frac{A}{\lambda^\alpha} = \frac{A}{\Gamma(\alpha)}\int_0^\infty \varphi(u)\, u^{\alpha-1}\, du$$

Corollary 0.2 and the equicontinuity of $\{f_x\}$ imply that

$$\lim_{x\to\infty} f_x(u) = \frac{A}{\Gamma(\alpha)} \cdot u^{\alpha-1} \quad \text{in } S'[0,\infty).$$

Namely, $\forall \varphi \in S[0,\infty)$,

$$\lim_{x\to\infty} \frac{1}{x^\alpha}\int_{0^-}^\infty \varphi\left(\frac{u}{x}\right) dS(u) = \frac{A}{\Gamma(\alpha)}\int_0^\infty \varphi(u)\, u^{\alpha-1}\, du. \quad (1.16).$$

Pick $\varphi \in S[0,\infty)$ such that $0 \leq \varphi \leq 1$, $\operatorname{supp}\varphi \subseteq [0, 1+\varepsilon]$, and $\varphi(u) = 1$ on $[0,1]$,

$$\limsup_{x\to\infty} \frac{S(x)}{x^\alpha} = \limsup_{x\to\infty} \frac{1}{x^\alpha}\int_{0^-}^x dS(u)$$

$$\leq \lim_{x\to\infty} \frac{1}{x^\alpha}\int_{0^-}^\infty \varphi\left(\frac{u}{x}\right) dS(u)$$

$$= \frac{A}{\Gamma(\alpha)} \cdot \int_{0^-}^{1+\varepsilon} \varphi(u) \cdot u^{\alpha-1}\, du \leq \frac{A}{\Gamma(\alpha)}\int_0^{1+\varepsilon} u^{\alpha-1}\, du$$

$$= \frac{A}{\Gamma(\alpha+1)}(1+\varepsilon)^\alpha.$$

Taking $\varepsilon \to 0$, $\limsup\limits_{x\to\infty} \dfrac{S(x)}{x^{\alpha}} \leq \dfrac{A}{\Gamma(\alpha+1)}$.

The inequality $\liminf\limits_{x\to\infty} \dfrac{S(x)}{x^{\alpha}} \geq \dfrac{A}{\Gamma(\alpha+1)}$ is deduced

Similarly by choosing $0 \leq \varphi \leq 1$ with supp $\varphi \subseteq [0,1]$ and $\varphi(u) = 1$ on $[1, 1-\varepsilon]$. ///

2. The Wiener-Ikehara theorem

(by Jasson Vindas, presented at Chuo university, Tokyo, 8-7-25)

2.1 Introduction:

In contrast to the Hardy-Littlewood Tauberian theorems from the previous lectures, complex Tauberian theorems make use of information on the Laplace transform in the complex domain.

The Wiener-Ikehara and Ingham-Karamata theorems re two cornerstones of complex Tauberian theory. We discuss here some versions of this theorems.

We refer to [K] for historical and bibliographical facts about these theorems.

2.2 The Wiener-Ikehara theorem:

In 1908 Landau deviced[1] the following theorem to deduce the prime number theorem:

Theorem 2.1 (Landau, 1908). Let S be a non-decreasing function whose Laplace-Stieltjes transform

$$\mathcal{L}\{dS;s\} = \int_{0^-}^{\infty} e^{-sx}\, dS(x), \quad s=\sigma+it,$$

---

[1] Landau's theorem was formulated for Mellin-Stieltjes transforms.

converges for $\operatorname{Re} s > 1$. Suppose that, for some $A \geq 0$, the function

$$G(s) = \mathcal{L}\{dS; s\} - \frac{A}{s-1} \qquad (2.1)$$

has analytic continuation to a region $\Omega \supseteq \{s : \operatorname{Re} s \geq 0\}$ and that $G(s) = O(|s|^k)$, for some $k \in \mathbb{N}$, for $|s| \to \infty$ through this region. Then,

$$S(x) \sim A e^x, \quad x \to \infty. \qquad (2.2)$$ ///

In 1931, Ikehara, a student of Wiener, significantly improved Landau's theorem by removing the superfluous hypothesis $G(s) = O(|s|^k)$. We state here a version from Wiener's book [W].

Theorem 2.2 (Wiener-Ikehara). Let S be a non-decreasing function on $[0, \infty)$ such that its Laplace Stieltjes transform converges on $\operatorname{Re} s > 1$. If, for some $A \geq 0$, the function (2.1) has continuous extension to the boundary line $\operatorname{Re} s = 1$, then (2.2) holds true. ///

We discuss a proof of this theorem in Section 2.4.

2.3 From Wiener-Ikehara to the Prime Number Theorem

The motivation of the Wiener-Ikehara theorem was to give a deduction of the PNT solely based on the non-vanishing of the Riemann zeta function on Re $s=1$. We discuss such deduction in this section. Let us start by considering a specialized version of Theorem 2.2.

Corollary 2.3. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers and assume that the Dirichlet series $\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ converges for Re $s>1$. If

$$\sum_{n=0}^{\infty}\frac{a_n}{n^s}-\frac{A}{s-1}$$ has continuous extension to

Re $s=1$, then $\sum_{n\leq x} a_n \sim Ax$, $x\to\infty$. ///

Proof. Let $\tilde{S}(y)=\sum_{n\leq e^y} a_n=\sum_{\log n\leq y} a_n$. Then

$\tilde{S}'(y)=d\tilde{S}(y)=\sum_{n=1}^{\infty} a_n\delta(y-\log n)$, so that its Laplace-Stieltjes transform is $\mathcal{L}\{d\tilde{S};s\}=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$.

Theorem 2.2 then yields $\tilde{S}(y)\sim Ae^y$, which amounts to the same as $\sum_{n\leq x} a_n\sim Ax$. ///

Let us now study some properties of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \operatorname{Re} s > 1.$$

Lemma 2.4 ⓘ ζ has analytic continuation to $\operatorname{Re} s > 0$ with the exception of $s=1$, where it has a simple pole with residue 1

(ii) $$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}} = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{1}{p^{s \cdot k}}\right)$$

Proof. For ⓘ,

$$\zeta(s) - \frac{1}{s-1} = \int_{1^-}^{\infty} \frac{d\lfloor x \rfloor}{x^s} - \frac{1}{s-1} = \int_{1^-}^{\infty} \frac{d(\lfloor x \rfloor - x)}{x^s}$$

$$= 1 - \int_{1^-}^{\infty} \frac{\{x\}}{x^{s+1}}\, dx,$$ which converges for $\operatorname{Re} s > 0$.

For (ii), the product is absolutely convergent because $(s = \sigma + it)$

$$\sum_p \left| \sum_{k=1}^{\infty} \frac{1}{p^{s \cdot k}} \right| \leq \sum_p \frac{1}{p^\sigma} \cdot \frac{1}{1 - \frac{1}{p^\sigma}} \leq \frac{1}{1 - \frac{1}{2^\sigma}} \cdot \sum \frac{1}{n^\sigma}.$$

For the equality, let $N \in \mathbb{N}^+$ and $A_N = \{n \in \mathbb{N}^+ : p \mid n \Rightarrow p \leq N\}$.

Then, $$P_N(s) = \prod_{p \leq N} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \sum_{n \in A_N} \frac{1}{n^s}.$$

Now, $$|P_N(s) - \zeta(s)| = \left| \sum_{n \notin A_N} \frac{1}{n^s} \right| \leq \sum_{n > N} \frac{1}{n^\sigma} < \infty, \quad \sigma > 1.$$

Lemma 2.5 (Hadamard) $|\zeta(\sigma)|^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)|\geq 1$,
$\sigma>1$ & $t\in\mathbb{R}$.

Proof. The following trigonometric polynomial is non-negative

$$P(\theta)=3+4\cos\theta+\cos2\theta=2+4\cos\theta+2\cos^2\theta$$
$$=2(1+\cos\theta)^2\geq 0.$$

Also,

$$|\zeta(s)|=e^{\mathrm{Re}\log\prod_p\frac{1}{1-\frac{1}{p^s}}}=e^{\mathrm{Re}\sum_p\sum_{m\geq1}\frac{1}{mp^{ms}}}=e^{\sum_p\sum_{m\geq1}\frac{\cos(t\log p^m)}{mp^{m\sigma}}}$$

So,

$$|\zeta(\sigma)|^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)|$$
$$=e^{\sum_p\sum_{m\geq1}\frac{1}{mp^{m\sigma}}[3+4\cos(t\log p^m)+\cos(t\log p^m)]}$$
$$\geq 1. //$$

Lemma 2.6 (Hadamard) $\zeta(1+it)\neq 0$, $\forall t\neq 0$.

Proof. Suppose on the contrary that $\zeta(1+it_0)=0$. Since ζ is analytic, $\zeta(\sigma+it_0)=\zeta(1+it_0)+O(\sigma-1)$
$=O(\sigma-1)$.
By continuity $\zeta(\sigma+2it_0)=O(1)$. Also, by Lemma 2.4 (i), $\zeta(\sigma)=O(\frac{1}{\sigma-1})$. So, we would have, by Lemma 2.5,

$$1\leq O\left(\frac{1}{(\sigma-1)^3}\right)\cdot O((\sigma-1)^4)\cdot O(1)=O(\sigma-1),$$

$\sigma\to1^+$, which is absurd. //

The last ingredient in the proof of the PNT is the Chebyshev function

$$\psi(x)=\sum_{n\le x}\Lambda(n),$$

where

$$\Lambda(n)=\begin{cases}1, & n=p^m,\ p \text{ prime}\\ 0 & \text{otherwise}\end{cases}$$ (Von Mangoldt function)

<u>Lemma 2.7</u> (Chebyshev) If

$$\psi(x)\sim x \tag{2.3},$$

then, the PNT holds, namely,

$$\pi(x)=\sum_{\substack{p\le x\\ p \text{ prime}}} 1\sim\frac{x}{\log x} \tag{2.4}.$$

<u>Proof.</u> Define the intermidiate function (p is always prime)

$$\Theta(x)=\sum_{p\le x}\log p.$$

We have

$$\psi(x)=\Theta(x)+R(x),$$ where

$$R(x)=\sum_{\substack{p^m\le x\\ m\ge 2}}\log p=\sum_{p\le\sqrt{x}}\left\lfloor\frac{\log x}{\log p}\right\rfloor\cdot\log p=O(\sqrt{x}\cdot\log x)$$
$$=o(x).$$

So, $\Theta(x)\sim x$ under (2.3). Now,

$$d\Theta(u)=\log u\cdot d\pi(u),$$ so

$$\pi(x) = \int_{2^-}^{x} \frac{d\theta(u)}{\log u} = \frac{\theta(x)}{\log x} + \int_{2^-}^{x} \frac{\theta(u)}{u \log^2 u}\, du$$

$$= \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) + O\left(\int_{2^-}^{x} \frac{du}{\log^2 u}\right)$$

$$= \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) + O\left(\frac{x}{\log^2 x}\right) \quad ///$$

Theorem 2.8 (PNT) The relation (2.4) holds true.

Proof. We want to apply Corollary 2.3 to deduce (2.3) and then conclude the proof via Lemma 2.7. For this, It suffices to show that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} - \frac{1}{s-1}$$

has analytic continuation beyond $\mathrm{Re}\, s = 1$. We calculate $(\log \zeta(s))'$ via Lemma 2.4 (ii):

$$\frac{\zeta'(s)}{\zeta(s)} = (\log \zeta(s))' = \left(-\sum_p \log\left(1 - \frac{1}{p^s}\right)\right)'$$

$$= \left(\sum_p \sum \frac{1}{m p^{sm}}\right)'$$

$$= -\sum_p \frac{\log p}{p^{sm}} = -\sum_{m=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

By Lemma 2.6, $(s-1)\zeta(s)$ has no zeros on $\operatorname{Re} s=1$, so

$$-\frac{d}{ds}\left(\log (s-1)\zeta(s)\right) = -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$

$$= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} - \frac{1}{s-1}$$

is analytic in a neighborhood of $\operatorname{Re} s=1$, which completes the proof of the PNT.

$\boxed{2.4}$ **Proof of the Wiener-Ikehara theorem:** We follow here the approach from [DV16] (see also [DV19]). We first show a "boundedness Tauberian" proposition

Proposition 2.7. Let S be a non-decreasing function on $[0,\infty)$ with convergent Laplace-Stieltjes transform on $\operatorname{Re} s>1$. If the function G given by (2.1) has continuous extension to a boundary neighborhood of $s=1$, then

$$S(x)=O(e^{x}), \quad x\to\infty \qquad (2.5)$$

Proof. Assume that $g(t)=\lim_{\sigma\to 1^+} G(\sigma+it)$ uniformly for $t\in(-\lambda,\lambda)$. Integrating by parts,

$$F(s) = \int_0^{\infty} e^{-sx} S(x)\,dx - \frac{A}{s-1}$$

$$= \frac{1}{s}\mathcal{L}\{dS;s\} - \frac{A}{s-1}$$

$$= \frac{1}{s}\left(G(s) + \frac{A}{s-1} \right) - \frac{A}{s-1}$$

$$= \frac{G(s)}{s} - \frac{A}{s},$$

which also extends continuously to $1+i(-\lambda,\lambda)$. Let then

$$f(t) = \lim_{\sigma \to 1} F(\sigma+it) \quad (\text{uniformly on } (-\lambda,\lambda)).$$

We may assume without loss of generality that $S \geq 0$.

Let $\Delta(x) = e^{-x} S(x)$, so that we want to show that $\Delta(x) = O(1)$. Let $\varphi \in \mathcal{D}(-\lambda,\lambda)$ be arbitrary, such that φ is even and $\hat{\varphi} \geq 0$. We claim that $\Delta * \varphi$ exists and

$$(\Delta * \hat{\varphi})(h) = O(1). \qquad (2.6)$$

By the Beppo Levi theorem (monotone convergence), we have

$$(\Delta * \hat{\varphi})(h) = \lim_{\sigma \to 0^+} \int_0^\infty \Delta(x) \hat{\varphi}(x-h) \cdot e^{-\sigma x} \, dx$$

$$= \lim_{\sigma \to 0^+} \int_0^\infty S(x) e^{-(\sigma+1)x} \hat{\varphi}(x-h) \, dx \qquad (2.7)$$

$$= \lim_{\sigma \to 0^+} \int_{-\infty}^{\infty} \mathcal{L}\{S; \sigma+1+it\} e^{iht} \varphi(t) \, dt.$$

$$= \int_{-\infty}^{\infty} g(t) \varphi(t) e^{iht} dt + \lim_{\sigma \to 0^+} \int_{-\infty}^{\infty} \frac{A}{\sigma+it} \varphi(t) e^{iht} dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{iht}\varphi(t) + A \lim_{\sigma \to 0^+} \int_0^{\infty} e^{-\sigma x}\hat{\varphi}(x-h)\,dx$$

$$= \int_{-\infty}^{\infty} f(t)e^{iht}\varphi(t) + A\int_0^{\infty} \hat{\varphi}(x-h)\,dx$$

(Riemann-Lebesgue lemma)

$$= o(1) + A\int_{-h}^{\infty}\hat{\varphi}(x)\,dx = o(1) + A\int_{-\infty}^{\infty}\hat{\varphi}(x)\,dx.$$

This proves (2.6). Let now $B = \int_0^{\infty} e^{-x}\hat{\varphi}(x)\,dx > 0$.

Using (2.6) and the fact that S is non-decreasing, we obtain $(h>0)$

$$e^{-h}S(h) = \frac{1}{B}\int_0^{\infty} e^{-x-h} S(h)\hat{\varphi}(x)\,dx$$

$$\le \frac{1}{B}\int_0^{\infty} e^{-x-h} S(x+h)\hat{\varphi}(x)\,dx$$

$$\le \frac{1}{B}\int_0^{\infty} \Delta(x)\hat{\varphi}(x-h) = O(1). \quad ///$$

We are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Since we now know that $\Delta \in S'(\mathbb{R})$, similarly to the calculation (2.7), we have, for arbitrary $\varphi \in D(\mathbb{R})$

$$= \int_{-\infty}^{\infty} \Delta(x+h)\hat{\varphi}(x)\,dx = \langle \hat{\Delta}(t), \varphi(t)e^{iht} \rangle$$

$$= \lim_{\sigma\to 1^+} \int_{-\infty}^{\infty} \mathcal{Z}\{S;\sigma+it\}\varphi(t)e^{iht}dt$$

$$= o(1) + A\int_{-\infty}^{\infty}\hat{\varphi}(t)dt.$$

This means: that the family $\{g_h\}_{h>0}$, with $g_h(x) = \Delta(x+h)$, satisfies.

① $\|g_h\|_{L^\infty} = O(1)$, $h\to\infty$.

② $\langle g_h,\psi\rangle = \int_{-\infty}^{\infty} g_h(x)\psi(x)dx \xrightarrow{h\to\infty} \int_{-\infty}^{\infty}\psi(x)dx$, $\forall\psi\in\mathcal{F}(\mathcal{D}(\mathbb{R}))$.

The image under the Fourier transform of $\mathcal{D}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$. A standard density argument[3] shows that ② remains valid for all $\psi\in L^1(\mathbb{R})$, that is,

$$\int_{-\infty}^{\infty}\Delta(x+h)\psi(x)dx = A\int_{-\infty}^{\infty}\psi(x)dx + o(1),\ \forall\psi\in L^1(\mathbb{R}) \quad (2.8).$$

To prove $\Delta(h)\to A$, we select first ψ in (2.8) with: $\psi\geq 0$, $\int_{-\infty}^{\infty}\psi(x)dx = 1$, and $\operatorname{supp}\psi\subseteq[0,\varepsilon]$.

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[3] Let $\varepsilon>0$. If $\psi_n\in\mathcal{D}(\mathbb{R})$ approximates ψ in $L^1(\mathbb{R})$,
$\left|\int_{-\infty}^{\infty}(\Delta(x+h)-A)\psi(x)dx\right| \leq (\|\Delta\|_{L^\infty}+A)\cdot\|\psi_n-\psi\|_{L^1} + \left|\int_{-\infty}^{\infty}(\Delta(x+h)-A)\psi_n(x)dx\right|$

Using that S is non-decreasing[4],

$$\Delta(h) = \int_0^{\varepsilon} e^{-h} S(h)\psi(x)\,dx \leq \int_0^{\varepsilon} e^{-h} S(x+h)\psi(x)\,dx$$

$$\leq \int_0^{\varepsilon} e^{x} \Delta(x+h)\psi(x)\,dx \leq e^{\varepsilon} \int_0^{\infty} \Delta(x+h)\psi(x)\,dx$$

$$= e^{\varepsilon}(A + o(1)).$$

Taking $\limsup_{h\to\infty}$ and then $\varepsilon \to 0^+$, we obtain

$$\limsup_{h\to\infty} \Delta(h) \leq A.$$

The inequality $\liminf_{h\to\infty} \Delta(h) \geq A$ is shown similarly by picking $\psi \geq 0$ such that $\operatorname{supp}\psi \subseteq [-\varepsilon, 0]$ and still $\int_{-\infty}^{\infty} \psi(x)\,dx = 1$. ///

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[4] Without loss of generality, we assume again $S \geq 0$.

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Some developments on the Wiener-Ikehara theorem

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Minicourse: Tauberian theorems
Lecture # 3
Chuo University, Tokyo, July 9, 2025

In this third lecture we will discuss recent developments on the Wiener-Ikehara theorem.

Main questions we will address:

1. Relax boundary requirements to a minimum.
2. Exact form (if and only if form).
3. Absence of remainders.
4. Some remainder terms and optimality.

The classical Wiener-Ikehara theorem

$\mathcal{L}\{S;s\} = \int_0^\infty S(x)e^{-sx}\mathrm{d}x$ and $\mathcal{L}\{\mathrm{d}S;s\} = \int_{0^-}^\infty e^{-sx}\mathrm{d}S(x)$; $s = \sigma + it$.

Theorem (Wiener-Ikehara, Laplace transforms)

Let S be a non-decreasing function (Tauberian hypothesis) such that $\mathcal{L}\{S;s\}$ converges for $\Re e\, s > 1$. If

$$\mathcal{L}\{S;s\} - \frac{A}{s-1} \qquad \left(\text{or equivalently } \mathcal{L}\{\mathrm{d}S;s\} - \frac{A}{s-1}\right)$$

has analytic continuation through $\Re e\, s = 1$, then $S(x) \sim Ae^x$.

Boundary behavior hypothesis on Laplace transform can be relaxed:

1. G(s) has continuous extension to $\Re e\, s = 1$.
2. L^1_{loc}-boundary behavior: $\lim_{\sigma\to 1^+} G(\sigma + it) \in L^1(I)$ for every finite interval I.
3. Local pseudofunction boundary behavior (Korevaar, 2005).
4. “if and only if version” (Debruyne and V., 2016).

Pseudofunctions

The concept of pseudofunctions naturally arises in harmonic analysis.

- $C_0(\mathbb{R})$: the space of continuous functions vanishing at $\pm\infty$.
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in \mathcal{S}'(\mathbb{R}) : \widehat{g} \in C_0(\mathbb{R})\}$.

Given an open set $I \subseteq \mathbb{R}$, we define the local space:

- $PF_{loc}(I)$: g such that for all bounded open subinterval $I' \subset I$ there is $f \in PF(\mathbb{R})$ such that $g = f$ on I'.
- $L^1_{loc}(I) \subset PF_{loc}(I)$.

Let G be analytic on $\Re e\, s > 1$ and $I \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $1 + iI$ if it has distributional boundary values there, i.e.

$$\lim_{\sigma \to 1^+} G(\sigma + it) = g(t) \text{ in } \mathcal{D}'(I)$$

and $g \in PF_{loc}(I)$.

A Tauberian condition: log-linear slow decrease

Classically, a function is called slowly decreasing in the sense of Schmidt (i.e. in multiplicative form) if

$$\liminf_{\lambda\to 1^+}\liminf_{x\to\infty}\inf_{1\le a\le\lambda}(f(ax)-f(x))\ge 0$$

It is called linearly slowly decreasing if

$$\liminf_{\lambda\to 1^+}\liminf_{x\to\infty}\inf_{1\le a\le\lambda}\frac{f(ax)-f(x)}{x}\ge 0$$

Definition

We call a function S log-linearly slowly decreasing if $S(\log x)$ is linearly slowly decreasing, that is, for each $\varepsilon>0$ there are $\delta, x_0>0$ such that

$$\frac{S(x+h)-S(x)}{e^x}\ge-\varepsilon$$

holds for all $x\ge x_0$ and $0<h<\delta$.

Extension of the Korevaar-Wiener-Ikehara theorem

Theorem (Debruyne and V., 2016)

Let $S \in L^1_{loc}[0,\infty)$. Then,

$$S(x) \sim Ae^x$$

if and only if

1. $\mathcal{L}\{S;s\}$ *converges for* $\Re e\, s > 1$,
2. $\mathcal{L}\{S;s\} - \dfrac{A}{s-1}$ *has local pseudofunction boundary behavior on* $\Re e\, s = 1$*, and*
3. S *is log-linearly slowly decreasing.*

- This and related Tauberians have found many recent applications in the theory of Beurling generalized primes.
- Such applications are out of reach for Tauberian theorems with weaker boundary behavior hypotheses.

Remainders in the Wiener-Ikehara theorem

Let S be non-decreasing.

If one wishes to attain a stronger relative remainder than $o(1)$, i.e.,

$$\frac{S(x)}{e^x} = A + O(\rho(x)) \qquad \text{with} \quad \rho(x) = o(1),$$

it is natural to strengthen the assumptions on

$$\mathcal{L}\{\mathrm{d}S; s\} - \frac{A}{s-1}. \tag{1}$$

It is folklore that remainders can be obtained from:

1. quantified information on the shape of the region of analytic continuation;
2. bounds for (1) on such a region.

In this part of the talk we explore:

- whether one could drop the second point here and get some error term from merely analytic continuation (short answer: no);
- some quantitative results when both hypotheses are satisfied.

A conjecture under merely analytic continuation

M. Müger raised the question of whether it is still possible to obtain error terms without assuming bounds on the analytic continuation of $\mathcal{L}\{\mathrm{d}S; s\} - A/(s-1)$ to a half-plane. He actually conjectured one could get the following error term:

Conjecture (Müger, 2018)

Let $0 < \alpha < 1$ and $A > 0$. If $\mathcal{L}\{\mathrm{d}S; s\} - A/(s-1)$ has analytic continuation to $\Re e\, s > \alpha$, then

$$S(x) = Ae^x + O_\varepsilon(e^{x(\frac{\alpha+2}{3}+\varepsilon)}), \quad \forall \varepsilon > 0.$$

We refuted this conjecture; in fact:

Negative general answer

No remainder can be expected in the Wiener-Ikehara theorem, even from entire continuation.

Absence of remainders in Wiener-Ikehara theorem

Theorem (Debruyne and V., 2018; Broucke, Debruyne, and V., 2021; Callewaert, Neyt, and V., 2025)

Let ρ *be an arbitrary positive function tending to 0. There is a non-decreasing function S on* $[0,\infty)$ *such that*

$$\mathcal{L}\{\mathrm{d}S;s\} = \int_{0^-}^{\infty} e^{-sx}dS(x) \qquad \text{converges for } \Re e\, s > 1$$

and

$$\mathcal{L}\{dS;s\} - \frac{A}{s-1}$$

admits extension to $\mathbb{C}$ *as an entire function for some* $A>0$*, but such that*

$$\limsup_{x\to\infty} \frac{|S(x)-Ae^{x}|}{\rho(x)e^{x}} = \infty.$$

Remainders under analytic continuation hypotheses I

We consider the following situation, with S, M, and K positive non-decreasing

- $\mathcal{L}\{\mathrm{d}S; s\} - \dfrac{A}{s-1}$ has analytic continuation to

$$\Omega_M = \left\{ s = \sigma + it \in \mathbb{C} : |1-\sigma| \le \frac{1}{M(|t|)} \right\}.$$

- Bound $\mathcal{L}\{\mathrm{d}S; s\} - \dfrac{A}{s-1} = O(K(|t|))$ for $s = \sigma + it \in \Omega_M$.
- $M_{K,log}(t) = M(t)[\log(t \cdot K(t) \cdot \log t)]$.

Theorem (Debruyne, 2024; improving upon Stahn, 2018)

For any $0 < c < 1$*, we always have*

$$\frac{S(x)}{e^x} = A + O\left(\frac{1}{M_{K,\log}^{-1}(cx)} \right).$$

Remainders under analytic continuation hypotheses II

- $\mathcal{L}\{\mathrm{d}S; s\} - \dfrac{A}{s-1}$ has analytic continuation to

$$\Omega_M = \left\{ s = \sigma + it \in \mathbb{C} : |1-\sigma| \leq \frac{1}{M(|t|)} \right\}.$$

- Bound $\mathcal{L}\{\mathrm{d}S; s\} - \dfrac{A}{s-1} = O(K(|t|))$ for $s = \sigma + it \in \Omega_M$.
- $M_K(t) = M(t)[\log(t \cdot K(t))]$.

Theorem (Debruyne, 2024; improving upon Stahn, 2018)

If K is of positive increase, i.e., $\liminf_{x\to\infty} \dfrac{K(\lambda x)}{K(x)} > 1$ for $\lambda > 1$,

$$\frac{S(x)}{e^x} = A + O\left(\frac{1}{M_K^{-1}(x)}\right)$$

Optimality of the remainders

With the previous notation:

Theorem (Debruyne, 2024; improving upon Debruyne-Seifert 2019)

Assume that, for $C > 0$,

$$M(x) = O\left(\exp\left(\exp\left(CxK(x)\right)\right)\right).$$

Suppose that ρ is a non-increasing function such that

$$\frac{S(x)}{e^x} = A + O\left(\rho(x)\right)$$

for all non-decreasing S such that $\mathcal{L}\{\mathrm{d}S; s\} - \dfrac{A}{s-1}$ analytically extends to Ω_M with bound $O(K(|\Im m\, s|))$ there. Then,

$$\frac{1}{M_K^{-1}(x)} \ll \rho(x)$$

with as before M_K^{-1} the inverse function of $M_K(t) = M(t)[\log(t \cdot K(t))]$.

Further developments

- (Debruyne, 2024): More general remainder theory; including non-analytic extension hypotheses.
- (Koga 2021; Chen & V., 2025): Conditions merely on real part of Laplace transform with applications in renewal theory.
- Finite form (Graham & Vaaler, 1981): For non-decreasing S, if

$$G(s) = \mathcal{L}\{S; s\} - \frac{A}{s-1}$$

has pseudofunction boundary behavior on $1 + i(-\lambda, \lambda)$, then

$$A\frac{2\pi/\lambda}{e^{2\pi/\lambda}-1} \leq \liminf_{x\to\infty} \frac{S(x)}{e^x} \leq \limsup_{x\to\infty} \frac{S(x)}{e^x} \leq A\frac{2\pi/\lambda}{1-e^{-2\pi/\lambda}}$$

- (Tenenbaum): Effective Wiener-Ikehara theorem in terms of

$$\eta(\sigma,\lambda) = \int_{-\lambda}^{\lambda} |G(2\sigma+it) - G(\sigma+it)|\mathrm{d}t$$

- (Révész & de Roton, 2013): Further refinements on the effective form of the Wiener-Ikehara theorem.

Some references

The talk is mostly based on a series of collaborative works:

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