The Prime Number Theorem for Beurling's Generalized Primes. New Cases

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The prime number theorem

The prime number theorem (PNT) states that

$$\pi(x) \sim \frac{x}{\log x} , \quad x \to \infty ,$$

where

$$\pi(x) = \sum_{p \text{ prime},\ p<x} 1 .$$

We will consider in this talk generalizations of the PNT for Beurling's generalized numbers

Outline

Landau's theorem

In 1903, Landau essentially showed the following theorem.

- Let $1 < p_1 \leq p_2, \ldots$ be a non-decreasing sequence tending to infinity.
- Arrange all possible products of the p_j in a non-decreasing sequence $1 < n_1 \leq n_2, \ldots,$ where every n_k is repeated as many times as represented by $p_{\nu_1}^{\alpha_1} p_{\nu_2}^{\alpha_2} \ldots p_{\nu_m}^{\alpha_m}$ with $\nu_j < \nu_{j+1}$.
- Denote $N(x) = \sum_{n_k < x} 1$ and $\pi(x) = \sum_{p_k < x} 1$.

Theorem (Landau, 1903)

If $N(x) = ax + O(x^{\theta})$, $\quad x \to \infty$, where $a > 0$ and $\theta < 1$, then

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Landau's theorem: Examples

- Gaussian integers $\mathbb{Z}[i] := \{a + b\,i \in \mathbb{C} : a, b \in \mathbb{Z}\}$, with Gaussian norm $|a + ib| := a^2 + b^2$. If we define $\{p_k\}_{k=1}^{\infty}$ as the sequence of norms of Gaussian primes, then the sequence $\{n_k\}_{k=1}^{\infty}$ corresponds to the sequence of norms of gaussian numbers such that $|a + ib| > 1$. One can show that

$$N(x) = \sum_{a,b\in\mathbb{Z},\ a^2+b^2<x} 1 = \pi x + O(\sqrt{x})$$

Consequently, the PNT holds for Gaussian primes.

- Laudau actually showed that if the $\{p_k\}_{k=1}^{\infty}$ corresponds to the norms of the distinct prime ideals of the ring of integers in an algebraic number field, then $\pi(x) \sim x/\log x$.

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Beurling's problem

In 1937, Beurling raised the question: Find conditions over N which ensure the validity of the PNT, i.e., $\pi(x) \sim x/\log x$.

Theorem (Beurling, 1937)

if

$$N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right),$$

where $a > 0$ and $\gamma > 3/2$, then the PNT holds.

Theorem (Diamond, 1970)

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Extension of Beurling theorem

We were able to relax the hypothesis of Beurling's theorem.

Theorem (2010, exdending Beurling, 1937)

Suppose there exist constants $a > 0$ and $\gamma > 3/2$ such that

$$N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right) \quad (\mathrm{C})\,, \quad x \to \infty\,,$$

Then the prime number theorem still holds.

The hypothesis means that there exists some $m \in \mathbb{N}$ such that:

$$\int_0^x \frac{N(t) - at}{t}\left(1 - \frac{t}{x}\right)^m \mathrm{d}t = O\left(\frac{x}{\log^{\gamma} x}\right)\,, \quad x \to \infty\,.$$

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Pseudo-functions

- A distribution $f \in \mathcal{S}'(\mathbb{R})$ is called a pseudo-function if $\hat{f} \in C_0(\mathbb{R})$.
- $f \in \mathcal{D}'(\mathbb{R})$ is locally a pseudofunction if for each $\phi \in \mathcal{D}(\mathbb{R})$, the distribution ϕf is a pseudo-function.

f is locally a pseudo-function if and only if the following 'Riemann-Lebesgue lemma' holds: for each ϕ with compact support

$$\lim_{|h|\to\infty} \left\langle f(t), e^{-iht}\phi(t) \right\rangle = 0$$

Corollary

If f belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$, then f is locally a pseudo-function.

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If f belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$, then f is locally a pseudo-function.

Local pseudo-function boundary behavior

Let $G(s)$ be analytic on $\Re e\, s > \alpha$. We say that G has local pseudo-function boundary behavior on the line $\Re e\, s = \alpha$ if it has distributional boundary values in such a line, namely,

$$\lim_{\sigma \to \alpha^+} \int_{-\infty}^{\infty} G(\sigma + it)\phi(t)\mathrm{d}t = \langle f, \phi \rangle \ , \quad \phi \in \mathcal{D}(\mathbb{R}) \ ,$$

and the boundary distribution $f \in \mathcal{S}'(\mathbb{R})$ is locally a pseudo-function.

A Tauberian theorem
for local pseudo-function boundary behavior

Theorem

Let $\{\lambda_k\}_{k=1}^{\infty}$ be such that $0 < \lambda_k \nearrow \infty$.
Assume $\{c_k\}_{k=1}^{\infty}$ satisfies: $c_k \geq 0$ and $\sum_{\lambda_k < x} c_k = O(x)$.
If there exists β such that

$$G(s) = \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k^s} - \frac{\beta}{s-1}\,, \quad \Re e\, s > 1\,, \tag{1}$$

has local pseudo-function boundary behavior on $\Re e\, s = 1$, then

$$\sum_{\lambda_k < x} c_k \sim \beta x\,, \quad x \to \infty\,. \tag{2}$$

Functions related to generalized primes

The zeta function is the analytic function (under our hypothesis)

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{n_k^s}, \quad \Re e\, s > 1 .$$

For ordinary integers it reduces to the Riemann zeta function.

One has an Euler product representation

$$\zeta(s) = \prod_{k=1}^{\infty} \frac{1}{1 - \left(\frac{1}{p_k}\right)^s}, \quad \Re e\, s > 1 .$$

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Define the von Mangoldt function

$$\Lambda(n_k) = \begin{cases} \log p_j \, , & \text{if } n_k = p_j^m \, , \\ 0 \, , & \text{otherwise} \, . \end{cases}$$

The Chebyshev function is

$$\psi(x) = \sum_{p_k^m < x} \log p_k = \sum_{n_k < x} \Lambda(n_k) \, .$$

On can show the PNT is equivalent to $\psi(x) \sim x$. We also have the identity

$$\sum_{k=1}^{\infty} \frac{\Lambda(n_k)}{n_k^s} = -\frac{\zeta'(s)}{\zeta(s)} \, , \quad \Re e \, s > 1 \, .$$

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S-asymptotics

Let $\mathcal{A}(\mathbb{R})$ be a topological vector space of functions.

Definition (Pilipović-Stanković)

$f \in \mathcal{A}'(\mathbb{R})$ has S-asymptotic behavior with respect to ρ if

$$\langle f(x+h), \phi(x)\rangle = (f * \check{\phi})(h) \sim \rho(h) \langle g(x), \phi(x)\rangle, \quad \phi \in \mathcal{A}(\mathbb{R}).$$

We write in short: $f(x+h) \sim \rho(h) g(x)$, $h \to \infty$ in $\mathcal{A}'(\mathbb{R})$.

$f \in \mathcal{A}'(\mathbb{R})$ is S-asymptotic bounded with respect to ρ if

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S-asymptotics and the zeta function

- Special distribution: $\sum_{k=1}^{\infty} \frac{1}{n_k} \delta(x - \log n_k)$
- Observe: $\mathcal{L}\{v; s\} = \langle v(x), e^{-sx} \rangle = \zeta(s+1)$

The condition

$$N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right) \quad \text{(C)}$$

is equivalent to

$$v(x+h) = aH(x+h) + O\left(\frac{1}{h^{\gamma}}\right), \quad |h| \to \infty, \quad \text{in } \mathcal{S}'(\mathbb{R})$$

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Under $N(x) = ax + O(x/\log^{\gamma} x)$ (C)

Using 'generalized distributional asymptotics', we translated the Cesàro estimate into:

- For $\gamma > 1$, $\zeta(s) - \dfrac{a}{s-1}$ has continuous extension to $\Re e\, s = 1$.
- For $\gamma > 3/2$
  - $(s-1)\zeta(s)$ is free of zeros on $\Re e\, s = 1$.
  - $-\dfrac{\zeta'(s)}{\zeta(s)} - \dfrac{1}{s-1}$ has local pseudo-function boundary behavior on the line $\Re e\, s = 1$
  - A Chebyshev upper estimate: $\sum_{n_k < x} \Lambda(n) = \psi(x) = O(x)$
  - So, the Tauberian theorem implies the PNT ($\gamma > 3/2$)

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Other related results ($\gamma > 3/2$)

Theorem

Our theorem is a proper extension of Beurling's PNT, namely, there is a set of generalized numbers satisfying the Cesàro estimate but not Beurling's one.

Theorem

Let μ be the Möbius function. Then,

$$\sum_{k=1}^{\infty} \frac{\mu(n_k)}{n_k} = 0 \text{ and } \lim_{x\to\infty} \frac{1}{x} \sum_{n_k<x} \mu(n_k) = 0 .$$

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