Scaling asymptotic properties of distributions and wavelet and non-wavelet transforms

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In this lecture we study local properties of distributions in terms of the boundary properties of transforms:

$$M_{\varphi}^{\mathbf{f}}(x,y) = (\mathbf{f} * \varphi_y)(x), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+, \tag{1}$$

where $\varphi_y(t) = y^{-n}\varphi(t/y)$. Specifically, we aim:

1. To present characterizations of scaling (weak-)asymptotic properties of distributions in terms of (1).
2. To give characterizations of positive measures in terms of extreme angular boundary values of non-wavelet transforms.
3. To discuss how these ideas have recently led to the construction of a new integral for functions of one variable that is more general than that of Denjoy-Perron-Henstock.

First part reports on work with Pilipović, while the second one with Estrada.

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Introduction

Distributions are not pointwisely defined objects. How can one study their behavior at individual points?

Two views of the problem:

1. Local regularity. Fix a global space of functions: a distribution is said to be regular at a point if it coincides near the point with an element of the global space.
2. Pointwise regularity. In several contexts, one is interested in finer pointwise measurements that allow one to distinguish special features in an irregular background.

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Representative example:

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n^2 t)}{n^2}. \tag{2}$$

Its point behavior depends on Diophantine approximations of the point: it radically changes from point to point.

Jaffard and Meyer showed that (2), and other functions, can be fully understood via a refined analysis of scaling and oscillating properties of distributions. Key notion: 2-microlocal spaces.

Zavialov (1973) introduced a natural measure of scaling properties. Closely related to 2-microlocal spaces.

Oscillation is also deeply involved in a new theory of integration!

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Outline

1. Scaling weak-asymptotic properties of distributions
   - Weak-asymptotics and Pointwise weak Hölder spaces
   - Characterizations: Tauberian theorems
   - Application: Pointwise analysis of Riemann type distributions
2. Measures and the $\phi-$transform
   - Characterizations of positive measures
3. A General Integral
   - Motivation: from Denjoy to Łojasiewicz
   - Properties of the distributional integral
   - Examples

General Notation

- $\mathcal{S}_0(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\varphi \in \mathcal{S}_0(\mathbb{R}^n) \;\Leftrightarrow\; \int_{\mathbb{R}^n} t^m \varphi(t)\mathrm{d}t = 0, \;\forall m \in \mathbb{N}^n.$$

- L always denotes a Karamata slowly varying function at the origin

$$\lim_{\varepsilon \to 0^+} \frac{L(a\varepsilon)}{L(\varepsilon)} = 1, \quad \forall a > 0.$$

- For test functions, $\check{\varphi}_y(t) = y^{-n}\varphi(-t/y)$.
- Distributions will be noted by $\mathbf{f}, \mathbf{g}, \dots,$ while functions by $f, g, \dots$.

Weak-asymptotics (by scaling)

Definition

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n)$. We write (as $\varepsilon \to 0^+$):

- $\mathbf{f}(x_0 + \varepsilon t) = O(\varepsilon^\alpha L(\varepsilon))$ in $\mathcal{S}'(\mathbb{R}^n)$ if $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \mathbf{f}\,(x_0 + \varepsilon t)\,, \varphi(t) \rangle = (\mathbf{f} * \check{\varphi}_\varepsilon)(x_0) = O(\varepsilon^\alpha L(\varepsilon)). \tag{3}$$

- $\mathbf{f}(x_0 + \varepsilon t) = O(\varepsilon^\alpha L(\varepsilon))$ in $\mathcal{S}'_0(\mathbb{R}^n)$ if (3) is just assumed to hold $\forall \varphi \in \mathcal{S}_0(\mathbb{R}^n)$
- $\mathbf{f}(x_0 + \varepsilon t) \sim \varepsilon^\alpha L(\varepsilon) \mathbf{g}(t)$ in $\mathcal{S}'(\mathbb{R}^n)$ if

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathbf{f}(x_0 + \varepsilon t) = \mathbf{g}(t) \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

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- $\mathbf{f}(x_0 + \varepsilon t) = O(\varepsilon^\alpha L(\varepsilon))$ in $\mathcal{S}_0'(\mathbb{R}^n)$ if (3) is just assumed to hold $\forall \varphi \in \mathcal{S}_0(\mathbb{R}^n)$
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Examples

- Meyer defined the weak scaling exponent of $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n)$ at $x_0 \in \mathbb{R}^n$ as the supremum over all α such that

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Typical example: $t^{-1/2}\sin(t^{-1})$, its weak scaling exponent is ∞.

- Let $x_0 \in \mathbb{R}^n$. We say that $\mathbf{f}$ has Łojasiewicz point value $\gamma \in \mathbb{C}$ at x_0, and write $\mathbf{f}(x_0) = \gamma$, distributionally, if

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Łojasiewicz point values

Łojasiewicz concept is an average notion. For instance, if $\mathbf{f} \in \mathcal{S}(\mathbb{R})$, one can show that $\mathbf{f}(x_0) = \gamma$, distributionally, if and only if there exist $k \in \mathbb{N}$ and a continuous function F such that $\mathbf{F}^{(k)} = \mathbf{f}$, near x_0, and

$$F(x) = \gamma \frac{(x - x_0)^k}{k!} + o(|x - x_0|^k), \quad x \to x_0.$$

The average nature can be explained with Fourier series: If $\mathbf{f}(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$, then $\mathbf{f}(x_0) = \gamma$, distributionally, if and only if $\exists m$ such that

$$\lim_{x\to\infty} \sum_{-x \le n \le ax} c_n e^{inx_0} = \gamma \quad (\mathrm{C}, m), \quad \forall a > 0.$$

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Classical Pointwise Hölder spaces

Let $x_0 \in \mathbb{R}^n$ and $\alpha > 0$.
We say $f \in C^\alpha(x_0)$ if there is a polynomial P such that

$$|f(x_0 + h) - P(h)| \leq C|h|^\alpha,$$

for small h.

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2. $\mathbf{f} \in C_w^{\alpha,L}(x_0)$ if there is a polynomial P such that $\mathbf{f} - \mathbf{P} \in \mathcal{O}^{\alpha,L}(x_0)$.
3. $\mathbf{f} \in C_{*,w}^{\alpha,L}(x_0) \iff \mathbf{f}(x_0 + \varepsilon t) = O(\varepsilon^\alpha L(\varepsilon))$ in $\mathcal{S}_0'(\mathbb{R}^n)$.

If $L \equiv 1$, we omit it from the notation. Meyer denotes $C_{*,w}^{\alpha}(x_0) = \Gamma^\alpha(x_0)$.

Connection with 2-microlocal spaces: $C_{*,w}^{\alpha}(x_0) = \bigcup_{s\in\mathbb{R}} C_{x_0}^{\alpha,s}$.

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Properties of these pointwise spaces

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In fact $\mathbf{f} \in C_{*,w}^{\alpha,L}(x_0)$ if and only if it has a weak asymptotic expansion

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The $\phi-$ and wavelet transforms

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n)$. We denote $\mathbb{H}^{n+1} = \mathbb{R}^n \times (0, \infty)$.
The moments of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ are denoted by

$$\mu_m(\varphi) = \int_{\mathbb{R}^n} t^m \varphi(t) \mathrm{d}t, \quad m \in \mathbb{N}^n.$$

$\phi-$transform: We always assume $\mu_0(\phi) = \int_{\mathbb{R}^n} \phi(t)\mathrm{d}t = 1$.

$$F_\phi \mathbf{f}(x,y) := \langle \mathbf{f}(x+yt), \phi(t) \rangle = (\mathbf{f} * \check{\phi}_y)(x), \quad (x,y) \in \mathbb{H}^{n+1}.$$

Wavelet transform: Assume ψ is a wavelet, meaning $\mu_0(\psi) = \int_{\mathbb{R}^n} \psi(t)\mathrm{d}t = 0$.

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Non-degenerate wavelets

Definition

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. It is said to be degenerate if there is a ray through the origin along which φ identically vanishes. In contrary case, the test function it is said to be non-degenerate.

Our Tauberian kernels are the non-degenerate test functions.

- In Wiener Tauberian theory the Tauberian kernels are those φ such that $\hat{\varphi}$ do not vanish at any point.
- In our theory the Tauberian kernels will be those φ such that $\hat{\varphi}$ does not identically vanish on any ray through the origin.

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Comments on the Tauberian theorems

The Tauberians to be presented improve several results of Drozhzhinov and Zavilov, and Y. Meyer (see references at the end).

Main improvements:

- Enlargement of the Tauberian kernels. Actually, our class of non-degenerate wavelets is the optimal one.
- Analysis of critical degrees, i.e., $\alpha \in \mathbb{N}$.

Extensions (not presented here):

- There are corresponding versions for asymptotics at infinity
- The results are valids for distributions with values in Banach spaces, and more generally in DFS spaces.
- The vector-valued case is very important in applications to local and global regularity of distributions.

Characterization of $C^{\alpha,L}_{*,w}(x_0)$

Let ψ be non-degenerate with moments $\mu_m(\psi) = 0$, $\forall\, |m| \leq [\alpha]$.

Theorem

The following are equivalent:

- $\mathbf{f} \in C^{\alpha,L}_{*,w}(x_0)$
- *There exists $k \in \mathbb{N}$ such that*

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2 + y^2 = 1,\, y > 0} \frac{y^k}{\varepsilon^{\alpha} L(\varepsilon)} \left| \mathcal{W}_{\psi} \mathbf{f}\,(x_0 + \varepsilon x, \varepsilon y) \right| < \infty.$$

The number k may be arbitrarily large!

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Characterization of $\mathcal{O}^{\alpha,L}(x_0)$

Let ϕ have $\int_{\mathbb{R}^n} \phi(t)\mathrm{d}t = \mu_0(\phi) = 1$.

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Weak-asymptotic behavior

Tauberian theorem for the $\phi-$transform

Theorem

$\mathbf{f}(x_0+\varepsilon t)\sim\varepsilon^{\alpha}L(\varepsilon)\mathbf{g}(t)$ in $\mathcal{S}'(\mathbb{R}^n)$ if and only if

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{\alpha}L(\varepsilon)}F_{\phi}\mathbf{f}(x_0+\varepsilon x,\varepsilon y)=F_{x,y},\quad \forall (x,y)\in\mathbb{S}^n\cap\mathbb{H}^{n+1},$$

and the Tauberian condition: $\exists k\in\mathbb{N}$ such that

$$\limsup_{\varepsilon\to 0^+}\sup_{|x|^2+y^2=1,\ y>0}\frac{y^k}{\varepsilon^{\alpha}L(\varepsilon)}\left|F_{\phi}\mathbf{f}\left(x_0+\varepsilon x,\varepsilon y\right)\right|<\infty.$$

In such a case, $\mathbf{g}$ is completely determined by $F_{\phi}\mathbf{g}(x,y)=F_{x,y}$.

$\mathbb{S}^n$ is the unit sphere in $\mathbb{H}^{n+1}$. As usual $\mu_0(\phi)=1$.

Weak-asymptotic behavior

Tauberian theorem for the wavelet transform

What do the following conditions tell us about pointwise behavior?

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathcal{W}_\psi \mathbf{f}(x_0+\varepsilon x, \varepsilon y) = W_{x,y}, \quad \forall (x,y) \in \mathbb{S}^n \cap \mathbb{H}^{n+1} \tag{4}$$

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Assume ψ is non-degenerate with $\mu_m(\psi)=0$, $|m| \leq [\alpha]$.

Theorem

If $\alpha \notin \mathbb{N}$. Condition (4) and (5) are necessary and sufficient for the existence of $\mathbf{g}$ and a polynomial P such that

$$\mathbf{f}(x_0+\varepsilon t) - \mathbf{P}(\varepsilon t) \sim \varepsilon^\alpha L(\varepsilon) \mathbf{g}(t) \quad \mathcal{S}'(\mathbb{R}^n).$$

$\mathbf{g}$ homogeneous and completely determined by $\mathcal{W}_\psi \mathbf{g}(x,y) = W_{x,y}$.

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What do the following conditions tell us about pointwise behavior?

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathcal{W}_\psi \mathbf{f}(x_0+\varepsilon x, \varepsilon y) = W_{x,y}, \quad \forall (x,y)\in \mathbb{S}^n \cap \mathbb{H}^{n+1} \qquad (4)$$

$$\limsup_{\varepsilon\to 0^+} \sup_{|x|^2+y^2=1,\ y>0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} \left|\mathcal{W}_\psi \mathbf{f}\left(x_0+\varepsilon x, \varepsilon y\right)\right| < \infty \qquad (5)$$

Assume ψ is non-degenerate with $\mu_m(\psi)=0$, $|m|\leq[\alpha]$.

Theorem

If $\alpha\notin\mathbb{N}$. Condition (4) and (5) are necessary and sufficient for the existence of $\mathbf{g}$ and a polynomial P such that

$$\mathbf{f}(x_0+\varepsilon t) - \mathbf{P}(\varepsilon t) \sim \varepsilon^\alpha L(\varepsilon)\mathbf{g}(t) \quad \mathcal{S}'(\mathbb{R}^n).$$

$\mathbf{g}$ *homogeneous and completely determined by* $\mathcal{W}_\psi \mathbf{g}(x,y) = W_{x,y}$.

Weak-asymptotic behavior

Tauberian theorem for the wavelet transform (continuation)

Theorem

If $\alpha \in \mathbb{N}$. Condition (4) and (5) are necessary and sufficient for the existence of $\mathbf{g}$, a polynomial P, and continuous functions c_m such that (in $\mathcal{S}'(\mathbb{R}^n)$)

$$\mathbf{f}(x_0 + \varepsilon t) = \mathbf{P}(\varepsilon t) + \varepsilon^{\alpha} L(\varepsilon)\mathbf{g}(t) + \varepsilon^{\alpha} \sum_{|m|=\alpha} t^m c_m(\varepsilon) + o\left(\varepsilon^{\alpha} L(\varepsilon)\right).$$

- $\mathbf{g}$ determined by $\mathcal{W}_{\psi}\mathbf{g}(x,y) = W_{x,y}$ up to homogeneous polynomials of degree α.
- The c_m satisfy for some constants $\beta_m \in \mathbb{C}$

$$c_m(a\varepsilon) = c_m(\varepsilon) + \beta_m L(\varepsilon)\log a + o(L(\varepsilon)), \quad \forall a > 0.$$

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Riemann type distributions

Using our Tauberian theorems, we fully described the pointwise weak properties of the family of Riemann distributions

$$R_{\beta}(t)=\sum_{n=1}^{\infty}\frac{e^{i\pi n^{2}t}}{n^{2\beta}}\in\mathcal{S}'(\mathbb{R}),\quad \beta\in\mathbb{C},$$

at points of $\mathbb{Q}$.

We split $\mathbb{Q}$ into two disjoint subsets S_0 and S_1 where

$$S_0=\left\{\frac{2\nu+1}{2j}:\ \nu,j\in\mathbb{Z}\right\}\cup\left\{\frac{2j}{2\nu+1}:\ \nu,j\in\mathbb{Z}\right\}$$

and

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Generalized Riemann zeta function

Interestingly, the pointwise behavior of R_β is intimately related to the analytic continuation properties of the zeta-type function

$$\zeta_r(z) := \sum_{n=1}^{\infty} \frac{e^{i\pi r n^2}}{n^z}, \quad \Re e\, z > 1, \tag{6}$$

where $r \in \mathbb{Q}$. If $r = 0$, (6) reduces to $\zeta_0 = \zeta$, the familiar Riemann zeta function.

Case $r \in S_1 = \left\{ \frac{2\nu+1}{2j+1} : \nu, j \in \mathbb{Z} \right\}$

Point behavior of Riemann distributions

Theorem

Let $r \in S_1$. The following Dirichlet series is entire in z,

$$\zeta_r(z) = \sum_{n=1}^{\infty} \frac{e^{i\pi r n^2}}{n^z} \quad (\mathrm{C}), \quad z \in \mathbb{C}, \tag{7}$$

where the sums for $\Re e\, z < 1$ are taken in the Cesàro sense.

Theorem

Let $r \in S_1$. Then $R_\beta \in C_w^\infty(r)$ for any $\beta \in \mathbb{C}$. Moreover,

$$R_\beta(r + \varepsilon t) \sim \sum_{m=0}^{\infty} \frac{\zeta_r(2\beta - 2m)}{m!} (i\varepsilon\pi t)^m \quad as\ \varepsilon \to 0^+ \ in\ \mathcal{S}'(\mathbb{R}).$$

Case $r \in S_0$

Analytic continuation of generalized Riemann zeta function

Theorem

Let $r \in S_0$. Then, ζ_r admits an analytic continuation to $\mathbb{C} \setminus \{1\}$, it has a simple pole at $z = 1$ with residue $\mathfrak{p}_r$, and the entire function

$$A_r(z) = \zeta_r(z) - \frac{\mathfrak{p}_r}{z-1}$$

can be expressed as the Cesàro limit

$$A_r(z) = \lim_{x \to \infty} \sum_{1 \leq n < x} \frac{e^{i\pi r n^2}}{n^z} - \mathfrak{p}_r \int_1^x \frac{\mathrm{d}\xi}{\xi^z} \quad (\mathrm{C}).$$

The $\mathfrak{p}_r$ are completely determined by the transformation equations:

$$\mathfrak{p}_0 = 1, \quad \mathfrak{p}_{r+2} = \mathfrak{p}_r, \quad \text{and} \quad \mathfrak{p}_{-\frac{1}{r}} = \sqrt{-\frac{i}{r}}\, \mathfrak{p}_r.$$

Case $r \in S_0$

Point behavior of Riemann distributions

We define the generalized gamma constant as

$$\gamma_r := A_r(1).$$

Observe that in fact $\gamma_0 = \gamma$, the familiar Euler gamma constant.

Theorem. Let $r \in S_0$. We have the expansions as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R})$.
(i) If $\beta \in \mathbb{C} \setminus \{1/2\}$, then

$$R_\beta(r+\varepsilon t) \sim \frac{(-i\pi)^{\beta-\frac{1}{2}}\Gamma\left(\frac{1}{2}-\beta\right)\mathfrak{p}_r}{2}(\varepsilon t+i0)^{\beta-\frac{1}{2}}+\sum_{m=0}^{\infty}\frac{\zeta_r(2\beta-2m)}{m!}(i\varepsilon\pi t)^m.$$

(ii) When $\beta = 1/2$, we have

$$R_{\frac{1}{2}}(r+\varepsilon t) \sim \gamma_r+\frac{\mathfrak{p}_r}{2}\left(-\log\left(\frac{\varepsilon\left|t\right|}{\pi}\right)+\frac{i\pi}{2}\operatorname{sgn} t-\gamma\right)+\sum_{m=1}^{\infty}\frac{\zeta_r(1-2m)}{m!}(i\varepsilon\pi t)^m.$$

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Measures and the $\phi-$transform

We shall present characterizations of positive measures in terms of the $\phi-$transform,

$$F_\phi \mathbf{f}(x,y) = (\mathbf{f} * \check{\phi}_y)(x), \quad (x,y) \in \mathbb{H} = \mathbb{R} \times \mathbb{R}_+,$$

where we always assume that $\phi \in \mathcal{D}(\mathbb{R})$ is positive and normalized, i.e.,

$$\int_{-\infty}^{\infty} \phi(t)\mathrm{d}t = 1.$$

Observation: It is easy to show that $\mathbf{f} \in \mathcal{D}'(\mathbb{R})$ is a positive measure $\Leftrightarrow$ $F_\phi \mathbf{f}(x,y) \geq 0$, $\forall (x,y) \in \mathbb{H}$.

Question: It is possible to characterize positive measure by mere knowledge of boundary extreme data of the $\phi-$transform?

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Lower angular values of the $\phi-$transform

If $x_0 \in \mathbb{R}$, denote by $C_{x_0,\theta}$ the cone in $\mathbb{H}$ starting at x_0 of angle θ,

$$C_{x_0,\theta} = \{(x,t) \in \mathbb{H} : |x - x_0| \leq (\tan\theta)t\} .$$

If $\mathbf{f} \in \mathcal{D}'(\mathbb{R})$, then lower angular values of its $\phi-$transform are

$$\mathbf{f}^{-}_{\phi,\theta}(x_0) = \liminf_{\substack{(x,y)\to(x_0,0) \\ (x,t)\in C_{x_0,\theta}}} F_\phi \mathbf{f}(x,y) .$$

For $\theta = 0$, we obtain the lower radial values.

Theorem

Let U be an open set. Then $\mathbf{f}$ is a positive measure in U if and only if its $\phi-$transform satisfies

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for each angle θ.

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Characterization of positive measures in terms of the $\phi-$transform

Questions:

- Can we replaced angular values by radial ones?
- Can the everywhere condition from the last theorem be relaxed to an a.e one?

Theorem

If the lower radial values satisfy

$$\mathbf{f}^{-}_{\phi,0}(x) \geq 0\,, \quad \text{almost everywhere in} U\,,$$

and for each angle and each $x \in U$ there is $M_x > 0$ such that

$$\mathbf{f}^{-}_{\phi,\theta}(x) \geq -M_x\,, \tag{8}$$

then $\mathbf{f}$ *is a positive measure in* U.

Conditions on the primitive

Question: Can the global assumption (8) be relaxed to an nearly everywhere condition?

A distribution is said to be a Łojasiewicz distribution if their Łojasiewicz point values exist everywhere.

Theorem

Assume that $\mathbf{f}^{-}_{\phi,0}(x) \geq 0$ *almost everywhere in U, and that there exists a countable set E such that there are constants* $M_x > 0$ *such that*

$$\mathbf{f}^{-}_{\phi,\theta}(x) \geq -M_x\,, \quad x \in U \setminus E, \quad \forall \theta$$

If the primitives of $\mathbf{f}$ *are Łojasiewicz distributions, then* $\mathbf{f}$ *is a positive measure in U.*

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If the primitives of $\mathbf{f}$ are Łojasiewicz distributions, then $\mathbf{f}$ is a positive measure in U.

We now discuss properties of a new integral, the distributional integral that integrates functions of one variable.

The construction of such an integral is based upon the characterizations of measures in terms of the $\phi-$transform. Scaling pointwise limits and oscillations are also important.

Recall the main drawbacks of the Riemann integral:

1. The class of Riemann integrable functions is too small.
2. Lack of convergence theorems.
3. The fundamental theorem of calculus

$$\int_a^x f(t)\mathrm{d}t = F(x)$$

where $F'(t) = f(t)$, *for all* t, is not always valid.

Lebesgue integral solves the first and second problem. Unfortunately, it does not solve the third one.

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Denjoy integral

In 1912 Denjoy constructed an integral with the properties:

- It is more general than the Lebesgue integral .
- The fundamental theorem of calculus is always valid.

For example, Denjoy integral integrates

$$\int_0^1 \frac{1}{x} \sin\left(\frac{1}{x^2}\right) \mathrm{d}x$$

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Peano differentials

In 1935 Denjoy went beyond integration of first order derivates and studied the problem of integration of higher order differential coefficients.

Let F be continuous on $[a, b]$, we say that F has a Peano n^{th} derivative at $x \in (a, b)$ if there are n numbers $F_1(x), \ldots, F_n(x)$ such that

$$F(x+h) = F(x) + F_1(x)h + \cdots + F_n(x)\frac{h^n}{n!} + o(h^n) , \quad \text{as } h \to 0 .$$

We call each $F_j(x)$ its Peano j^{th} derivative at x.

If $n > 1$ and this holds at every point, then $F'(x)$ exists everywhere, but this does not even imply that $F \in C^1[a, b]$.

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Denjoy higher order integration problem

Suppose that F has a Peano n^{th} derivative $\forall x \in (a, b)$. Denjoy asked:

1. If $F_n(x) = 0$ for all $x \in [a, b]$, is F a polynomial of degree at most $n - 1$?
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Denjoy solved these two problems with an extremely difficult "totalization procedure" (involving transfinite induction).

- In 1957, Łojasiewicz found, using distribution theory, a more transparent solution to the first problem. His gave a solution by identifying a new class of functions with distributions: the so-called Łojasiewicz functions.

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Properties of the distributional integral

We have contructed an integral, the distributional integral, that enjoys the following properties:

- Distributionally integrable functions are true functions: measurable and finite almost everywhere.
- The integrals of functions that are equal (a.e) coincide.
- Any Denjoy-Perron-Henstock integrable function, in particular Lebesgue integrable, is distributionally integrable, and the two integrals coincide within this class of functions.
- The distributional integral integrates higher order differential coefficients, and thus solves Denjoy's second problem in a constructive manner.
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- It enjoys all useful properties of the standard integrals, including:
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  - Mean value theorems.
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- If $\beta > 0$, it integrates unbounded functions such as

$$\frac{1}{|x|^{\gamma}} \sin\left(\frac{1}{|x|^{\beta}}\right) \quad \text{for all } \gamma \in \mathbb{R}$$

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- It identifies in a precise fashion a new class of functions with distributions.

If f is distributionally integrable over compacts, it can be identified with a distribution $\mathbf{f}$ in a natural way:

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Given $\{c_n\}_{n=1}^{\infty}$, define the function

$$f(x) = \begin{cases} 0\,, & \text{if } x \leq 0 \text{ or } x \geq 1\,, \\ c_n\,, & \text{if } \frac{1}{n+1} \leq x < \frac{1}{n}\,. \end{cases} \tag{9}$$

Let $a_n = c_n \left(\frac{1}{n} - \frac{1}{n+1} \right)$, so that

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Then f is, on the interval $[0,1]$,

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$$\int_x^1 f(t)\mathrm{d}t = \sum_{n \le x^{-1}} a_n + c_{[1/x]} \left(\frac{1}{[1/x]} - x \right), \quad x \in (0,1).$$

Then f is, on the interval $[0,1]$,

- Lebesgue integrable if and only if $\sum_{n=1}^{\infty} |a_n| < \infty$.
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In case $\sum_{n=1}^{\infty} a_n$ is Cesàro summable, we have

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} a_n \quad (\mathrm{C})\,.$$

For example, if $c_n = (-1)^n n(n+1)$, so that $a_n = (-1)^n$, we obtain

$$\int_0^1 f(x)\,\mathrm{d}x = -1/2$$

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Example

Consider the functions $s_\alpha(x) := |x|^\alpha \sin(1/x)$ for $\alpha \in \mathbb{C}$. Near $x = 0$:

- If $-1 < \Re e\, \alpha$, then it is Lebesgue integrable.
- If $-2 < \Re e\, \alpha \leq -2$, then it is not Lebesgue integrable but Denjoy-Perron-Henstock integrable.
- If $\Re e\, \alpha \leq -2$, it is not Denjoy-Perron-Henstock integrable, but distributional integrable.

The family of distributions $\mathbf{s}_\alpha$, where $\mathbf{s}_\alpha \leftrightarrow s_\alpha$, is analytic in α.

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References

For details about the first part see my preprint with Pilipović:

- Multidimensional Tauberian theorems for wavelets and non-wavelet transforms, preprint (arXiv:1012.5090v2).

For the distributional integral see my joint article with Estrada:

- A General integral, to appear in Dissertationes Mathematicae (preprint at arXiv:1109.2958v1).

See also:

- Drozhzhinov, Zavialov, Multidimensional Tauberian theorems for Banach-space valued generalized functions, Sb. Math. 194 (2003), 1599–1646.
- Meyer, Wavelets, vibrations and scalings, CRM Monograph series 9, A.M.S, Providence, 1998.
- Vindas, Pilipović, Rakić, Tauberian theorems for the wavelet transform, J. Fourier Anal. Appl. 17 (2011), 65–95.