Weyl asymptotic formulas for infinite order ΨDOs and Sobolev type spaces. Part I: symbolic calculus, hypoellipticity, semi-boundedness

Bojan Prangoski
joined work with Stevan Pilipović and Jasson Vindas

Preliminaries

By M_p, $p \in \mathbb{N}$, we denote a sequence of positive numbers such that $M_0 = M_1 = 1$ and satisfies the following conditions:
$(M.1)$ (Logarithmic convexity)

$$M_p^2 \leq M_{p-1}M_{p+1}, \quad p \in \mathbb{Z}_+;$$

$(M.2)$ (Stability under ultradifferential operators)

$$M_p \leq c_0 H^p \min_{0\leq q\leq p}\{M_{p-q}M_q\}, \quad p, q \in \mathbb{N}, \quad \text{for some} \quad c_0, H > 0;$$

$(M.3)$ (Strong non-quasi-analyticity)

$$\sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} \leq c_0 q \frac{M_q}{M_{q+1}}, \quad q \in \mathbb{Z}_+.$$

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$$\frac{M_p^2}{p!^2} \leq \frac{M_{p-1}}{(p-1)!} \cdot \frac{M_{p+1}}{(p+1)!}, \; p \in \mathbb{Z}_+.$$

The associated function: $M(\lambda) = \sup_{p\in\mathbb{N}} \ln \frac{\lambda^p}{M_p}$, $\lambda > 0$. ex. $M_p = p!^s$, $s > 1$; $M(\lambda) \asymp \lambda^{1/s}$.

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Test spaces. Ultradistributions

$\mathcal{S}^{M_p,m}(\mathbb{R}^d)$, $m > 0$, is the (B)-space of all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ which satisfy

$$\sigma_m(\varphi) = \sup_{\alpha \in \mathbb{N}^d} \frac{m^{|\alpha|} \| e^{M(m|\cdot|)} D^\alpha \varphi \|_{L^\infty(\mathbb{R}^d)}}{M_\alpha} < \infty,$$

$$\mathcal{S}^{(M_p)} = \lim_{\substack{\longleftarrow \\ m\to\infty}} \mathcal{S}^{M_p,m}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{S}^{\{M_p\}} = \lim_{\substack{\longrightarrow \\ m\to 0}} \mathcal{S}^{M_p,m}(\mathbb{R}^d).$$

The spaces of tempered ultradistributions of Beurling and Roumieu type are their respective strong duals $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$.

The common notation for symbols the (M_p) and $\{M_p\}$ will be $*$.

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Symbol classes

A_p, $p \in \mathbb{N}$, is a sequence of positive numbers; it satisfies

- $(M.1)$, $(M.2)$, $(M.3)$, $(M.4)$;
- $A_p \subset M_p$, i.e. there exists $c, L > 0$ such that $A_p \leq cL^p M_p$;
- $\rho_0 = \inf\{\rho \in \mathbb{R}_+ | \, A_p \subset M_p^\rho\}$; clearly $0 < \rho_0 \leq 1$; ρ is fixed and satisfies $\rho_0 \leq \rho \leq 1$, if the infimum is reached, or, otherwise $\rho_0 < \rho \leq 1$.

$\Gamma_{A_p,\rho}^{M_p,\infty}(\mathbb{R}^{2d}; h, m)$ is the (B)-space of all $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ satisfying (we write $w = (x, \xi) \in \mathbb{R}^{2d}$)

$$\sup_{\alpha \in \mathbb{N}^{2d}} \sup_{w \in \mathbb{R}^{2d}} \frac{|D^\alpha a(w)| \, \langle w \rangle^{\rho|\alpha|} e^{-M(m|w|)}}{h^{|\alpha|} A_\alpha} < \infty.$$

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Symbol classes

As l.c.s., we define

$$\Gamma_{A_p,\rho}^{(M_p),\infty}(\mathbb{R}^{2d}) = \varinjlim_{m\to\infty} \varprojlim_{h\to 0} \Gamma_{A_p,\rho}^{M_p,\infty}(\mathbb{R}^{2d}; h, m)$$

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A class of pseudo-differential operators

The τ-quantisation ($\tau \in \mathbb{R}$) of the symbol $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ is the ΨDO $\mathrm{Op}_\tau(a) : \mathcal{S}^*(\mathbb{R}^d) \to \mathcal{S}'^*(\mathbb{R}^d)$ defined by

$$\langle \mathrm{Op}_\tau(a)u, v\rangle = \langle \mathcal{F}^{-1}_{\xi\to x-y}(a)((1-\tau)x + \tau y, \xi), v \otimes u\rangle,\ u, v \in \mathcal{S}^*(\mathbb{R}^d).$$

Proposition

For each $\tau \in \mathbb{R}$, the bilinear mapping $(a, \varphi) \mapsto \mathrm{Op}_\tau(a)\varphi$, $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) \times \mathcal{S}^*(\mathbb{R}^d) \to \mathcal{S}^*(\mathbb{R}^d)$, is hypocontinuous and it extends to the hypocontinuous bilinear mapping $(a, T) \mapsto \mathrm{Op}_\tau(a)T$, $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) \times \mathcal{S}'^*(\mathbb{R}^d) \to \mathcal{S}'^*(\mathbb{R}^d)$. The mappings $a \mapsto \mathrm{Op}_\tau(a)$, $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) \to \mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^d), \mathcal{S}^*(\mathbb{R}^d))$, $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) \to \mathcal{L}_b(\mathcal{S}'^*(\mathbb{R}^d), \mathcal{S}'^*(\mathbb{R}^d))$ are continuous.

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- symbolic calculus;
- notion of hypoellipticity and construction of parametrices; global regularity;
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Hypoellipticity and global regularity

Definition

Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$. We say that a is $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic (or, in short, simply hypoelliptic) if

i) there exists $B > 0$ such that there are $c, m > 0$ (resp. for every $m > 0$ there is $c > 0$) such that

$$|a(x,\xi)| \geq ce^{-M(m|x|)-M(m|\xi|)}, \quad (x,\xi) \in Q^c_B = \mathbb{R}^{2d} \backslash \{(x,\xi) | \langle x \rangle < B, \langle \xi \rangle < B\},$$

ii) there exists $B > 0$ such that for every $h > 0$ there is $C > 0$ (resp. there are $h, C > 0$) such that

$$|D^\alpha a(w)| \leq Ch^{|\alpha|} A_\alpha |a(w)| \langle w \rangle^{-\rho|\alpha|}, \ \alpha \in \mathbb{N}^{2d}, \ w \in Q^c_B.$$

- hypoelliptic operators have parametrices;

- hypoelliptic operators are globally regular; i.e. $\mathrm{Op}_\tau(a)u \in \mathcal{S}^*(\mathbb{R}^d)$ implies $u \in \mathcal{S}^*(\mathbb{R}^d)$

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Realisation in $L^2(\mathbb{R}^d)$

- for $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, a^w stands for the Weyl quantisation ($\tau = 1/2$);
- A stands for the unbounded operator on $L^2(\mathbb{R}^d)$ with domain $\mathcal{S}^*(\mathbb{R}^d)$ defined as $A\varphi = a^w\varphi$, $\varphi \in \mathcal{S}^*(\mathbb{R}^d)$;
- the restriction of a^w to $\{g \in L^2(\mathbb{R}^d) | \, a^w g \in L^2(\mathbb{R}^d)\}$ is closed: the maximal realisation of A;
- $\overline{A}$ is the closure of A: the minimal realisation of A;
- A^* coincides with the maximal realisation of $(a^w)^* = \bar{a}^w$, i.e.

$$D(A^*) = \{g \in L^2(\mathbb{R}^d) | \, (a^w)^* g \in L^2(\mathbb{R}^d)\} \text{ and } A^* g = (a^w)^* g, \ \forall g \in D(A^*)$$

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- the restriction of a^w to $\{g \in L^2(\mathbb{R}^d) |\, a^w g \in L^2(\mathbb{R}^d)\}$ is closed: the maximal realisation of A;
- $\overline{A}$ is the closure of A: the minimal realisation of A;
- A^* coincides with the maximal realisation of $(a^w)^* = \bar{a}^w$, i.e.

$$D(A^*) = \{g \in L^2(\mathbb{R}^d) |\, (a^w)^* g \in L^2(\mathbb{R}^d)\} \text{ and } A^* g = (a^w)^* g,\ \forall g \in D(A^*)$$

Realisation in $L^2(\mathbb{R}^d)$

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Realisation in $L^2(\mathbb{R}^d)$ of hypoelliptic operators. Semi-boundedness

Proposition

Let a be hypoelliptic and A be the corresponding unbounded operator on $L^2(\mathbb{R}^d)$ defined above. Then the minimal realisation $\overline{A}$ coincides with the maximal realisation. Moreover, $\overline{A}$ coincides with the restriction of a^w on the domain of $\overline{A}$.

Proposition

Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be positive hypoelliptic symbol. Then, there exists $C > 0$ such that $(a^w\varphi, \varphi) \geq -C\|\varphi\|^2_{L^2(\mathbb{R}^d)}$, $\forall \varphi \in \mathcal{S}^*(\mathbb{R}^d)$.

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The spectrum of operators with positive hypoelliptic Weyl symbols

Proposition

Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be a hypoelliptic real-valued symbol such that $|a(w)| \to \infty$ as $|w| \to \infty$ and let A be the unbounded operator on $L^2(\mathbb{R}^d)$ defined by a^w. Then the closure $\overline{A}$ of A is a self-adjoint operator having spectrum given by a sequence of real eigenvalues either diverging to $+\infty$ or to $-\infty$ according to the sign of a at infinity. The eigenvalues have finite multiplicities and the eigenfunctions belong to $\mathcal{S}^*(\mathbb{R}^d)$. Moreover, $L^2(\mathbb{R}^d)$ has an orthonormal basis consisting of eigenfunctions of $\overline{A}$.

Construction of the heat parametrix

Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be hypoelliptic and real-valued such that $a(w)/\ln|w| \to \infty$ as $|w| \to \infty$.

The spectrum of the self-adjoint operator $\overline{A}$ is given by a sequence of real eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$, $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$, which tends to $+\infty$, where the multiplicities are taken into account, and $L^2(\mathbb{R}^d)$ has an orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}}$ consisting of eigenfunctions of $\overline{A}$ which all belong to $\mathcal{S}^*(\mathbb{R}^d)$.

Let $N(\lambda) = \#\{j \in \mathbb{N} | \lambda_j \leq \lambda\}$. What is the asymptotic behaviour of $N(\lambda)$, as $\lambda \to \infty$?

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$$T(t)g = \sum_{j=0}^{\infty} e^{-t\lambda_j}(g,\varphi_j)\varphi_j,\ \ g \in L^2(\mathbb{R}^d),\ \ t \geq 0.$$

Theorem

We have $T(t) \in \mathcal{L}(\mathcal{S}^*(\mathbb{R}^d), \mathcal{S}^*(\mathbb{R}^d))$ for each $t \geq 0$. Moreover, the mapping $t \mapsto T(t)$ belongs to $\mathcal{C}^\infty([0,\infty); \mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^d), \mathcal{S}^*(\mathbb{R}^d)))$ and $(d^k/dt^k)T(t) = (-1)^k(a^w)^k T(t)$, $t \geq 0$, $k \in \mathbb{Z}_+$.

$$t \mapsto T(t) \text{ solves } \begin{cases} (\partial_t + a^w)T(t) = 0,\ t \in [0,\infty), \\ T(0) = \mathrm{Id}, \end{cases}$$

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Construction of the heat parametrix

Theorem

There exists a vector-valued mapping $\mathbf{u} : t \mapsto u(t, \cdot)$, $[0,\infty) \to \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, *belonging to* $\mathcal{C}^\infty([0,\infty); \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}))$ *such that the operator-valued mapping* $t \mapsto (\mathbf{u}(t))^w$ *belongs to both* $\mathcal{C}^\infty([0,\infty); \mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^d), \mathcal{S}^*(\mathbb{R}^d)))$ *and* $\mathcal{C}^\infty([0,\infty); \mathcal{L}_b(\mathcal{S}'^*(\mathbb{R}^d), \mathcal{S}'^*(\mathbb{R}^d)))$, *and* $(\mathbf{u}(t))^w$ *satisfies*

$$\begin{cases} (\partial_t + a^w)(\mathbf{u}(t))^w = \mathbf{K}(t),\ t \in [0,\infty), \\ (\mathbf{u}(0))^w = \mathrm{Id}, \end{cases} \tag{1}$$

where $\mathbf{K} \in \mathcal{C}^\infty([0,\infty); \mathcal{L}_b(\mathcal{S}'^*(\mathbb{R}^d), \mathcal{S}^*(\mathbb{R}^d)))$.
Moreover, $u(t,w)$ *satisfies: for every* $h > 0$ *there exists* $C > 0$ *(resp. there exist* $h, C > 0$*) such that*

$$|D^n_t D^\alpha_w u(t,w)| \leq C n! h^{|\alpha|} A_\alpha (a(w))^n \langle w\rangle^{-\rho|\alpha|} e^{-\frac{t}{4}a(w)},$$

for all $\alpha \in \mathbb{N}^{2d}$, $n \in \mathbb{N}$, $(t,w) \in [0,\infty) \times \mathbb{R}^{2d}$.

Construction of the heat parametrix

Moreover, there exist $u_j \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^{2d})$, $j \in \mathbb{N}$, with $u_0(t, w) = e^{-ta(w)}$ satisfying: for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$|D_t^n D_w^\alpha u_j(t, w)| \leq Cn! h^{|\alpha|+2j} A_{|\alpha|+2j}(a(w))^n \langle w\rangle^{-\rho(|\alpha|+2j)} e^{-\frac{t}{4} a(w)}$$

and

$$\sup_{k\in\mathbb{Z}_+} \sup_{\substack{\alpha\in\mathbb{N}^{2d}\\ n\in\mathbb{N}}} \sup_{\substack{w\in Q^c_{3Rm_k}\\ t\in[0,\infty)}} \frac{\left|D_t^n D_w^\alpha \left(u(t, w) - \sum_{j<k} u_j(t, w)\right)\right| \langle w\rangle^{\rho(|\alpha|+2k)} e^{\frac{t}{4} a(w)}}{n! h^{|\alpha|+2k} A_{|\alpha|+2k}(a(w))^n} \leq C$$

The trace of the heat parametrix

$T(t)$ and $(\mathbf{u}(t))^w$ are the same, modulo a smooth $*$-regularising family, more precisely

$$(\mathbf{u}(t))^w\varphi - T(t)\varphi = \int_0^t T(t-s)\mathbf{K}(s)\varphi ds,\ \ \varphi \in \mathcal{S}^*(\mathbb{R}^d),$$

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Theorem

Let a be a hypoelliptic real-valued symbol in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ *such that* $a(w)/\ln|w| \to \infty$ *as* $|w| \to \infty$. *Then*

$$\sum_{j=0}^{\infty} e^{-t\lambda_j} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{-ta(w)}dw + O\left(\int_{\mathbb{R}^{2d}} \frac{e^{-\frac{t}{4}a(w)}}{\langle w\rangle^{2\rho}}dw\right),\ \ t \to 0^+.$$

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THANK YOU FOR YOUR ATTENTION