# Pointwise scaling properties of distributions and the wavelet transform

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We will define new pointwise spaces of distributions with values in Banach spaces. The aims of this lecture are:

1. To present their characterization through the wavelet transform.
2. To illustrate how they are effective tools in the study of pointwise, asymptotic, and local regularity properties of functions and distributions.

# Introduction

Distributions are not pointwisely defined objects. How can one study their behavior at individual points?

Two views of the problem:

1. Local regularity. Fix a global space of functions: a distribution is said to be regular at a point if it coincides near the point with an element of the global space.
2. Pointwise regularity. In several contexts, one is interested in finer pointwise measurements that allow one to distinguish special features in an irregular background.

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Representative example: Riemann's "non-differentiable" function,

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n^2 t)}{n^2}. \tag{1}$$

Its point behavior depends on Diophantine approximations of the point, and radically changes from point to point.

Jaffard and Meyer showed that (1), and other functions, can be fully understood via a refined analysis of scaling and oscillating properties of distributions. Their key notion: 2-microlocal spaces, introduced by Bony in the context of PDE.

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# Outline

1. Pointwise weak spaces
   - Weak-asymptotics
   - Pointwise weak Hölder spaces
2. Tauberian theorems: Wavelet characterization
   - Characterizations of pointwise weak spaces
   - Tauberians for weak-asymptotic behavior
3. Several applications
   - Pointwise analysis of Riemann distributions
   - Application to regularity in generalized function algebras
   - Characterization of distributionally small distributions

# General Notation

- $E$ always denotes a fixed Banach space with norm $\|\cdot\|$.
- $\mathcal{S}'(\mathbb{R}^n, E) = L_b(\mathcal{S}(\mathbb{R}^n), E)$, $E$-valued tempered distributions.
- $\mathcal{S}_0(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\varphi \in \mathcal{S}_0(\mathbb{R}^n) \ \Leftrightarrow \ \int_{\mathbb{R}^n} t^m \varphi(t) \mathrm{d}t = 0, \ \forall m \in \mathbb{N}^n.$$

- $\mathcal{S}_0'(\mathbb{R}^n, E) = L_b(\mathcal{S}_0(\mathbb{R}^n), E)$, the space of $E$-valued tempered distributions modulo $E$-valued polynomials.
- $L$ always denotes a slowly varying function at the origin

$$\lim_{\varepsilon \to 0^+} \frac{L(a\varepsilon)}{L(\varepsilon)} = 1, \quad \forall a > 0.$$

- For test functions, $\varphi_y(t) = y^{-n}\varphi(t/y)$.

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# Weak-asymptotics (by scaling)

## Definition

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$. We write (as $\varepsilon \to 0^+$):

- $\mathbf{f}(x_0 + \varepsilon t) = O(\varepsilon^\alpha L(\varepsilon))$ in $\mathcal{S}'(\mathbb{R}^n, E)$ if $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\|\langle \mathbf{f}\,(x_0 + \varepsilon t)\,, \varphi(t)\rangle\| = \|(\mathbf{f} * \check{\varphi}_\varepsilon)(x_0)\| = O(\varepsilon^\alpha L(\varepsilon)). \qquad (2)$$

- $\mathbf{f}(x_0 + \varepsilon t) = O(\varepsilon^\alpha L(\varepsilon))$ in $\mathcal{S}_0'(\mathbb{R}^n, E)$ if (2) is just assumed to hold $\forall \varphi \in \mathcal{S}_0(\mathbb{R}^n)$
- $\mathbf{f}(x_0 + \varepsilon t) \sim \varepsilon^\alpha L(\varepsilon)\mathbf{g}(t)$ in $\mathcal{S}'(\mathbb{R}^n, E)$ if

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathbf{f}(x_0 + \varepsilon t) = \mathbf{g}(t) \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E).$$

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# Examples

- Let $x_0 \in \mathbb{R}^n$ and $\mathbf{v} \in E$. We say that $\mathbf{f}$ has Łojasiewicz point value $\mathbf{v} \in E$ at $x_0$, and write $\mathbf{f}(x_0) = \mathbf{v}$, distributionally, if

$$\lim_{\varepsilon \to 0^+} \mathbf{f}(x_0 + \varepsilon t) = \mathbf{v} \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E),$$

i.e.,

$$\lim_{\varepsilon \to 0^+} \langle \mathbf{f}(x_0 + \varepsilon t), \varphi(t) \rangle = \mathbf{v} \int_{\mathbb{R}^n} \varphi(t) \mathrm{d}t, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

- Meyer defined the weak scaling exponent of $f \in \mathcal{S}'(\mathbb{R}^n)$ at $x_0 \in \mathbb{R}^n$ as the supremum over all $\alpha$ such that

$$f(x_0 + \varepsilon t) = O(\varepsilon^{\alpha}) \quad \text{in } \mathcal{S}'_0(\mathbb{R}).$$

# Classical Pointwise Hölder spaces

Let $x_0 \in \mathbb{R}^n$ and $\alpha > 0$.
We say $f \in C^\alpha(x_0)$ if there is a polynomial $P$ such that

$$|f(x_0 + h) - P(h)| \leq C|h|^\alpha,$$

for small $h$.

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Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$. For $x_0 \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we write:

1. $\mathbf{f} \in \mathcal{O}^{\alpha,L}(x_0, E) \Leftrightarrow \mathbf{f}(x_0 + \varepsilon t) = O(\varepsilon^\alpha L(\varepsilon))$ in $\mathcal{S}'(\mathbb{R}^n, E)$.
2. $\mathbf{f} \in C_w^{\alpha,L}(x_0, E)$ if there is an $E$-valued polynomial $\mathbf{P}$ such that $\mathbf{f} - \mathbf{P} \in \mathcal{O}^{\alpha,L}(x_0, E)$.
3. $\mathbf{f} \in C_{*,w}^{\alpha,L}(x_0, E) \Leftrightarrow \mathbf{f}(x_0 + \varepsilon t) = O(\varepsilon^\alpha L(\varepsilon))$ in $\mathcal{S}_0'(\mathbb{R}^n, E)$.

In the scalar-valued case we write $\mathcal{O}^{\alpha,L}(x_0)$, $C_w^{\alpha,L}(x_0)$ and $C_{*,w}^{\alpha,L}(x_0)$. If $L \equiv 1$, we omit it from the notation.
Meyer denotes $C_{*,w}^{\alpha}(x_0) = \Gamma^\alpha(x_0)$.

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# Properties of these pointwise spaces

- $\mathcal{O}^{\alpha,L}(x_0,E) \subseteq C^{\alpha,L}_{w}(x_0,E) \subseteq C^{\alpha,L}_{*,w}(x_0,E)$.
- If $\alpha < 0$, $\mathcal{O}^{\alpha,L}(x_0,E) = C^{\alpha,L}_{w}(x_0,E) = C^{\alpha,L}_{*,w}(x_0,E)$.
- If $\alpha > 0$, $\mathcal{O}^{\alpha,L}(x_0,E) \subsetneq C^{\alpha,L}_{w}(x_0,E)$. But actually $\mathbf{f} \in C^{\alpha,L}_{w}(x_0,E)$ if and only if the following "Taylor formula" holds,

$$\mathbf{f}(t) - \sum_{|m|<\alpha} \frac{\mathbf{f}^{(m)}(x_0)}{m!}(t-x_0)^m \in \mathcal{O}^{\alpha,L}(x_0,E).$$

  where $\mathbf{f}^{(m)}(x_0)$ are its Łojasiewicz point values.
- If $\alpha \notin \mathbb{N}$, then $C^{\alpha,L}_{w}(x_0,E) = C^{\alpha,L}_{*,w}(x_0,E)$.

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# Properties of these pointwise spaces

When $\alpha \in \mathbb{N}$, we have $C_w^{\alpha,L}(x_0, E) \subsetneq C_{*,w}^{\alpha,L}(x_0, E)$.
In fact $\mathbf{f} \in C_{*,w}^{\alpha,L}(x_0, E)$ if and only if it has a weak asymptotic expansion

$$\mathbf{f}(x_0 + \varepsilon t) = \mathbf{P}(\varepsilon t) + \varepsilon^{\alpha} \sum_{|m|=\alpha} t^m \mathbf{c}_m(\varepsilon) + O\left(\varepsilon^{\alpha} L(\varepsilon)\right), \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E)$$

where $\mathbf{P}$ is an $E$-valued polynomial and the $\mathbf{c}_m : (0, \infty) \mapsto E$ are continuous $E$-valued functions such that

$$\mathbf{c}_m(a\varepsilon) = \mathbf{c}_m(\varepsilon) + O(L(\varepsilon)), \quad \forall a > 0.$$

# The $\phi-$ and wavelet transforms

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$. We denote $\mathbb{H}^{n+1} = \mathbb{R}^n \times (0, \infty)$.
The moments of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ are denoted by

$$\mu_m(\varphi) = \int_{\mathbb{R}^n} t^m \varphi(t) \mathrm{d}t, \quad m \in \mathbb{N}^n.$$

$\phi-$transform: We always assume $\mu_0(\phi) = \int_{\mathbb{R}^n} \phi(t) \mathrm{d}t = 1$.

$$F_\phi \mathbf{f}(x, y) := \langle \mathbf{f}(x + yt), \phi(t) \rangle = (\mathbf{f} * \check{\phi}_y)(x) \in E, \quad (x, y) \in \mathbb{H}^{n+1}.$$

Wavelet transform: Assume $\psi$ is a wavelet, meaning $\mu_0(\psi) = \int_{\mathbb{R}^n} \psi(t) \mathrm{d}t = 0$.

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# Non-degenerate wavelets

## Definition

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. It is said to be degenerate if there is a ray through the origin along which $\varphi$ identically vanishes. In contrary case, the test function it is said to be non-degenerate.

Our Tauberian kernels are the non-degenerate test functions.

- In Wiener Tauberian theory the Tauberian kernels are those $\varphi$ such that $\hat{\varphi}$ do not vanish at any point.
- In our theory the Tauberian kernels will be those $\varphi$ such that $\hat{\varphi}$ does not identically vanish on any ray through the origin.

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# Comments on the Tauberian theorems

The Tauberians to be presented improve several results of Drozhzhinov and Zavilov, and Y. Meyer (see references at the end).

Main improvements:

- Enlargement of the Tauberian kernels. Actually, our class of non-degenerate wavelets is the optimal one.
- Analysis of critical degrees, i.e., $\alpha \in \mathbb{N}$.

# Characterization of $C^{\alpha,L}_{*,w}(x_0,E)$

Let $\psi$ be non-degenerate with moments $\mu_m(\psi)=0$, $\forall\,|m|\leq[\alpha]$.

**Theorem**

*The following are equivalent:*

- $\mathbf{f}\in C^{\alpha,L}_{*,w}(x_0,E)$
- *There exists* $k\in\mathbb{N}$ *such that*

$$\limsup_{\varepsilon\to0^+}\sup_{|x|^2+y^2=1,\,y>0}\frac{y^k}{\varepsilon^\alpha L(\varepsilon)}\left\|\mathcal{W}_\psi\mathbf{f}\left(x_0+\varepsilon x,\varepsilon y\right)\right\|<\infty.$$

The number $k$ may be arbitrarily large!

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The number $k$ may be arbitrarily large!

# Characterization of $\mathcal{O}^{\alpha,L}(x_0, E)$

Let $\phi$ have $\int_{\mathbb{R}^n} \phi(t)\mathrm{d}t = \mu_0(\phi) = 1$.

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- $\mathbf{f} \in \mathcal{O}^{\alpha,L}(x_0, E)$.
- *There exists $k \in \mathbb{N}$ such that*

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} \left\| F_\phi \mathbf{f}\,(x_0 + \varepsilon x, \varepsilon y) \right\| < \infty.$$

# Weak-asymptotic behavior

## Tauberian theorem for the $\phi-$transform

### Theorem

*$\mathbf{f}(x_0 + \varepsilon t) \sim \varepsilon^{\alpha} L(\varepsilon)\mathbf{g}(t)$ in $\mathcal{S}'(\mathbb{R}^n, E)$ if and only if*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{\alpha} L(\varepsilon)} F_{\phi}\mathbf{f}(x_0 + \varepsilon x, \varepsilon y) = \mathbf{F}_{x,y} \in E, \quad \forall (x, y) \in \mathbb{S}^n \cap \mathbb{H}^{n+1},$$

*and the Tauberian condition: $\exists k \in \mathbb{N}$ such that*

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2 + y^2 = 1,\, y > 0} \frac{y^k}{\varepsilon^{\alpha} L(\varepsilon)} \left\| F_{\phi}\mathbf{f}\,(x_0 + \varepsilon x, \varepsilon y)\right\| < \infty.$$

*In such a case, $\mathbf{g}$ is completely determined by $F_{\phi}\mathbf{g}(x, y) = \mathbf{F}_{x,y}$.*

$\mathbb{S}^n$ is the unit sphere in $\mathbb{H}^{n+1}$. As usual $\mu_0(\phi) = 1$.

# Weak-asymptotic behavior

## Tauberian theorem for the wavelet transform

What do the following conditions tell us about pointwise behavior?

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathcal{W}_\psi \mathbf{f}(x_0+\varepsilon x, \varepsilon y) = \mathbf{W}_{x,y} \in E, \quad \forall (x,y) \in \mathbb{S}^n \cap \mathbb{H}^{n+1} \tag{3}$$

$$\limsup_{\varepsilon\to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} \left\| \mathcal{W}_\psi \mathbf{f}\,(x_0+\varepsilon x, \varepsilon y)\right\| < \infty \tag{4}$$

Assume $\psi$ is non-degenerate with $\mu_m(\psi)=0$, $|m|\leq[\alpha]$.

**Theorem**

*If $\alpha\notin\mathbb{N}$. Condition (3) and (4) are necessary and sufficient for the existence of $\mathbf{g}$ and an $E$-valued polynomial $\mathbf{P}$ such that*

$$\mathbf{f}(x_0+\varepsilon t) - \mathbf{P}(\varepsilon t) \sim \varepsilon^\alpha L(\varepsilon)\mathbf{g}(t) \quad \mathcal{S}'(\mathbb{R}^n, E).$$

*$\mathbf{g}$ homogeneous and completely determined by $\mathcal{W}_\psi \mathbf{g}(x,y) = \mathbf{W}_{x,y}$.*

# Weak-asymptotic behavior

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# Weak-asymptotic behavior

## Tauberian theorem for the wavelet transform (continuation)

**Theorem**

*If $\alpha \in \mathbb{N}$. Condition (3) and (4) are necessary and sufficient for the existence of* $\mathbf{g}$*, an $E$-valued polynomial* $\mathbf{P}$*, and $E$-valued continuous functions* $\mathbf{c}_m : (0,\infty) \mapsto E$ *such that in $\mathcal{S}'(\mathbb{R}^n, E)$*

$$\mathbf{f}(x_0 + \varepsilon t) = \mathbf{P}(\varepsilon t) + \varepsilon^{\alpha} L(\varepsilon)\mathbf{g}(t) + \varepsilon^{\alpha} \sum_{|m|=\alpha} t^m \mathbf{c}_m(\varepsilon) + o\left(\varepsilon^{\alpha} L(\varepsilon)\right).$$

- $\mathbf{g}$ determined by $\mathcal{W}_{\psi}\mathbf{g}(x,y) = \mathbf{W}_{x,y}$ up to homogeneous polynomials of degree $\alpha$.
- The $\mathbf{c}_m$ satisfy for some vector $\mathbf{v}_m \in E$

$$\mathbf{c}_m(a\varepsilon) = \mathbf{c}_m(\varepsilon) + L(\varepsilon)\log a\, \mathbf{v}_m + o(L(\varepsilon)), \quad \forall a > 0.$$

# Weak-asymptotic behavior

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# Riemann type distributions

Using our Tauberian theorems, we fully described the pointwise weak properties of the family of Riemann distributions

$$R_\beta(t) = \sum_{n=1}^{\infty} \frac{e^{i\pi n^2 t}}{n^{2\beta}} \in \mathcal{S}'(\mathbb{R}), \quad \beta \in \mathbb{C},$$

at points of $\mathbb{Q}$.

We split $\mathbb{Q}$ into two disjoint subsets $S_0$ and $S_1$ where

$$S_0 = \left\{ \frac{2\nu + 1}{2j} : \nu, j \in \mathbb{Z} \right\} \cup \left\{ \frac{2j}{2\nu + 1} : \nu, j \in \mathbb{Z} \right\}$$

and

$$S_1 = \left\{ \frac{2\nu + 1}{2j + 1} : \nu, j \in \mathbb{Z} \right\}.$$

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# Generalized Riemann zeta function

Interestingly, the pointwise behavior of $R_\beta$ is intimately related to the analytic continuation properties of the zeta-type function

$$\zeta_r(z) := \sum_{n=1}^{\infty} \frac{e^{i\pi r n^2}}{n^z}, \quad \Re e\, z > 1, \tag{5}$$

where $r \in \mathbb{Q}$. If $r = 0$, (5) reduces to $\zeta_0 = \zeta$, the familiar Riemann zeta function.

# Case $r \in S_1 = \left\{ \frac{2\nu+1}{2j+1} : \nu, j \in \mathbb{Z} \right\}$

## Point behavior of Riemann distributions

**Theorem**

*Let $r \in S_1$. The following Dirichlet series is entire in $z$,*

$$\zeta_r(z) = \sum_{n=1}^{\infty} \frac{e^{i\pi r n^2}}{n^z} \quad (\mathrm{C}), \quad z \in \mathbb{C}, \tag{6}$$

*where the sums for $\Re e\, z < 1$ are taken in the Cesàro sense.*

**Theorem**

*Let $r \in S_1$. Then $R_\beta \in C_w^\infty(r)$ for any $\beta \in \mathbb{C}$. Moreover,*

$$R_\beta(r + \varepsilon t) \sim \sum_{m=0}^{\infty} \frac{\zeta_r(2\beta - 2m)}{m!} (i\varepsilon\pi t)^m \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}).$$

# Case $r \in S_0$

## Analytic continuation of generalized Riemann zeta function

**Theorem**

*Let $r \in S_0$. Then, $\zeta_r$ admits an analytic continuation to $\mathbb{C} \setminus \{1\}$, it has a simple pole at $z = 1$ with residue $\mathfrak{p}_r$, and the entire function*

$$A_r(z) = \zeta_r(z) - \frac{\mathfrak{p}_r}{z-1}$$

*can be expressed as the Cesàro limit*

$$A_r(z) = \lim_{x \to \infty} \sum_{1 \le n < x} \frac{e^{i\pi r n^2}}{n^z} - \mathfrak{p}_r \int_1^x \frac{\mathrm{d}\xi}{\xi^z} \quad (\mathrm{C}).$$

The $\mathfrak{p}_r$ are completely determined by the transformation equations:

$$\mathfrak{p}_0 = 1, \quad \mathfrak{p}_{r+2} = \mathfrak{p}_r, \quad \text{and} \quad \mathfrak{p}_{-\frac{1}{r}} = \sqrt{-\frac{i}{r}}\, \mathfrak{p}_r.$$

# Case $r \in S_0$

## Point behavior of Riemann distributions

We define the generalized gamma constant as

$$\gamma_r := A_r(1).$$

Observe that in fact $\gamma_0 = \gamma$, the familiar Euler gamma constant.

**Theorem.** Let $r \in S_0$. We have the expansions as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R})$.
(i) If $\beta \in \mathbb{C} \setminus \{1/2\}$, then

$$R_\beta(r+\varepsilon t) \sim \frac{(-i\pi)^{\beta-\frac{1}{2}}\Gamma\left(\frac{1}{2}-\beta\right)\mathfrak{p}_r}{2}(\varepsilon t+i0)^{\beta-\frac{1}{2}}+\sum_{m=0}^{\infty}\frac{\zeta_r(2\beta-2m)}{m!}(i\varepsilon\pi t)^m.$$

(ii) When $\beta = 1/2$, we have

$$R_{\frac{1}{2}}(r+\varepsilon t) \sim \gamma_r+\frac{\mathfrak{p}_r}{2}\left(-\log\left(\frac{\varepsilon\,|t|}{\pi}\right)+\frac{i\pi}{2}\operatorname{sgn} t-\gamma\right)+\sum_{m=1}^{\infty}\frac{\zeta_r(1-2m)}{m!}(i\varepsilon\pi t)^m.$$

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# Application

## Regularity theorem in the tempered generalized function algebra

We show a regularity Theorem of G. Hörmann for the algebra of regular tempered generalized functions

$$\mathcal{G}_{\tau}^{\infty}(\mathbb{R}^n) \subset \mathcal{G}_{\tau}(\mathbb{R}^n)$$

**Theorem**

$\mathcal{S}'(\mathbb{R}^n) \cap \mathcal{G}_{\tau}^{\infty}(\mathbb{R}^n) = \mathcal{O}_M(\mathbb{R}^n)$.

This equality means that if $f \in \mathcal{S}'(\mathbb{R}^n)$ and the net $f_{\varepsilon} = f * \phi_{\varepsilon}$ determines an element of $\mathcal{G}_{\tau}^{\infty}(\mathbb{R}^n)$, i.e.,

$$(\exists a \in \mathbb{R})(\forall m \in \mathbb{N}^n)(\exists N \in \mathbb{N})(\sup_{x \in \mathbb{R}^n}(1+|x|)^{-N}|f_{\varepsilon}^{(m)}(x)| = O(\varepsilon^{-a})),$$

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# Proof

## Regularity theorem in the tempered generalized function algebra

Fix $m \in \mathbb{N}^n$. It suffices to show $f^{(m)}$ is continuous of polynomial growth. Let $k \in \mathbb{N}$ be such that $\beta = 2k - a > 0$.

Step 1. Set $\psi = \overline{\Delta^k \check{\phi}}$, a non-degenerate wavelet.

$$(1+|x|)^{-N_0}\varepsilon^{2k}\left|((\Delta^k f^{(m)}) * \phi_\varepsilon)(x)\right| = (1+|x|)^{-N_0}\left|\mathcal{W}_\psi f^{(m)}(x,\varepsilon)\right| = O(\varepsilon^\beta).$$

Step 2. Define the vector-valued distribution $\mathbf{h}$ by $\langle \mathbf{h}, \rho \rangle = f^{(m)} * \check{\rho}$. $\exists N > N_0$ such that $\mathbf{h} \in \mathcal{S}'(\mathbb{R}^n, E)$, where $E$ is the Banach space of continuous functions $v$ with norm

$$\|v\| := \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-N} |v(\xi)| < \infty.$$

Step 3. Since $\mathcal{W}_\psi \mathbf{h}(x,y)(\xi) = \mathcal{W}_\psi f^{(m)}(\xi + x, y)$,

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## Regularity theorem in the tempered generalized function algebra

The conclusion from the Tauberian theorem is: $\mathbf{h} \in C_w^{\beta}(0, E)$. So, the Łojasiewicz point value $\mathbf{h}(0) = v \in E$ exits, i.e.,

$$\lim_{\varepsilon \to 0^+} f^{(m)} * \check{\rho}_\varepsilon = \lim_{\varepsilon \to 0^+} \langle \mathbf{h}(\varepsilon t), \rho(t) \rangle = v \int_{\mathbb{R}^n} \rho(t) \mathrm{d}t, \quad \forall \rho \in \mathcal{S}(\mathbb{R}^n)$$

where the limit holds in $E$. But we take in particular $\rho = \check{\phi}$, so

$$\lim_{\varepsilon \to 0^+} (f^{(m)} * \phi_\varepsilon)(\xi) = v(\xi), \quad \text{uniformly for } \xi \in \mathbb{R}^n,$$

thus $f^{(m)} = v$ is a continuous function of at most polynomial growth.

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# Estrada's distributionally small distributions at infinity

$\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ is said to be distributionally small at infinity if it satisfies the Estrada-Kanwal moment asymptotic expansion

$$\mathbf{f}(\lambda t) \sim \sum_{|m|=0}^{\infty} \frac{(-1)^{|m|}}{m!\lambda^{|m|+n}} \delta^{(m)}(t)\mathbf{w}_m \quad \text{as } \lambda \to \infty \ \text{ in } \mathcal{S}'(\mathbb{R}^n, E), \quad (7)$$

for some multi-sequence $\{\mathbf{w}_m\}_{m\in\mathbb{N}^n}$ in $E$, in the sense that $\forall N$

$$\langle \mathbf{f}(\lambda t), \varphi(t)\rangle = \sum_{|m|\leq N} \frac{\varphi^{(m)}(0)}{m!\lambda^{|m|+n}} \mathbf{w}_m + O\left(\frac{1}{\lambda^{N+n+1}}\right) \quad \text{as } \lambda \to \infty.$$

Distributions in many important distribution spaces, such as $\mathcal{E}'(\mathbb{R}^n, E), \mathcal{O}'_M(\mathbb{R}^n, E)$, and $\mathcal{O}'_C(\mathbb{R}^n, E)$, satisfy (7).

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# Estrada characterization

**f** satisfies (7) $\Leftrightarrow$ **f** $\in \mathcal{K}'(\mathbb{R}^n, E)$.

Here $\mathcal{K}(\mathbb{R}^n)$ is the space of symbols of pseudodifferential operators given by

$$\mathcal{K}(\mathbb{R}^n) = \bigcup_{\beta} \mathcal{K}_\beta(\mathbb{R}^n) = \operatorname{ind} \lim_{\beta \to \infty} \mathcal{K}_\beta(\mathbb{R}^n),$$

where $\mathcal{K}_\beta(\mathbb{R}^n)$ consists of those smooth functions $\varphi$ such that

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# Distributionally small distributions at infinity
## Wavelet characterization

Fourier transforming (7), $\mathbf{f} \in \mathcal{K}'(\mathbb{R}^n, E) \Leftrightarrow \hat{\mathbf{f}} \in C^\infty_w(0, E)$, where

$$C^\infty_w(x_0, E) := \bigcap_{\alpha\in\mathbb{R}} C^\alpha_{*,w}(x_0, E) = \bigcap_{\alpha\in\mathbb{R}} C^\alpha_w(x_0, E), \quad x_0 \in \mathbb{R}^n.$$

By the Tauberian theorem for the wavelet transform,

### Theorem

*An $E$-valued tempered distribution $\mathbf{f}$ belongs to $\mathcal{K}'(\mathbb{R}^n, E)$ if and only if $\exists \{\nu_p\}_{p=0}^\infty$ of non-negative integers such that $\forall p \in \mathbb{N}$*

$$\limsup_{\varepsilon\to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^{\nu_p}}{\varepsilon^p} \left\| \mathcal{W}_\psi \hat{\mathbf{f}}(\varepsilon x, \varepsilon y) \right\| < \infty.$$

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# References

For further results see our preprint:

- Multidimensional Tauberian theorems for wavelets and non-wavelet transforms, preprint, 2011.

See also:

- Y. N. Drozhzhinov, B. I. Zavialov, Multidimensional Tauberian theorems for Banach-space valued generalized functions, Sb. Math. 194 (2003), 1599–1646.
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