General Stieltjes moment problems for rapidly decreasing smooth functions

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The problem of moments, as its generalizations, is an important mathematical problem which has attracted much attention for more than a century.

It was first raised and solved by Stieltjes for positive measures.

Problem (Stieltjes, 1894)

Find conditions over $\{a_n\}_{n=0}^{\infty}$ which ensure the existence of solutions to the infinity system of equations

$$a_n = \int_0^\infty x^n d\mu(x), \quad n = 0, 1, 2, \dots ,$$

where μ is a positive measure.

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The classical Stieltjes moment problem

Stieltjes found a necessary and sufficient condition for the existence of solutions. Define the sequence of matrices

$$\Delta_n = \begin{pmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \dots & a_{2n} \end{pmatrix} \text{ and } \Delta_n^{(1)} = \begin{pmatrix} a_1 & a_1 & \dots & a_{n+1} \\ a_2 & a_3 & \dots & a_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & a_{n+2} & \dots & a_{2n+1} \end{pmatrix}$$

Theorem (Stieltjes, 1894-1895)

The Stieltjes moment problem

$$a_n = \int_0^\infty x^n d\mu(x), \quad n = 0, 1, 2, \dots ,$$

has solution if and only if

$$\det(\Delta_n) > 0 \text{ and } \det(\Delta_n^{(1)}) > 0, \quad n = 0, 1, 2, \dots .$$

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Ideas connected with the Stieltjes moment problem

Stieltjes' influential papers led to many important ideas:

- The theory Stieltjes integrals

$$a_n = \int_0^\infty x^n dF(x), \quad F \nearrow .$$

- The Stieltjes transform, $\Re e\, z \notin (-\infty, 0]$,

$$S(z) = \int_0^\infty \frac{dF(x)}{x+z} \sim \sum_{n=0}^\infty \frac{(-1)^n a_n}{z^{n+1}} .$$

- Continued fraction approximations

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- Carleman (1923-1926): connections with the theory of quasi-analytic functions.

Other moment problems:

- Hamburger (1920):

$$a_n = \int_{-\infty}^{\infty} x^n dF(x), \quad n = 0, 1, 2, \dots .$$

- Hausdorff (1923):

$$a_n = \int_{b}^{c} x^n dF(x), \quad n = 0, 1, 2, \dots .$$

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Theorem (Boas and Pólya, independently, 1939)

Given an arbitrary sequence $\{a_n\}_{n=0}^{\infty}$, there is always a function of bounded variation F such that

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A. Durán achieved a major improvement to this result:

Theorem (A. Durán, 1989)

Every Stieltjes moment problem

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$$a_n = \int_0^\infty x^n \phi(x) dx, \quad n = 0, 1, 2, \ldots, \tag{1}$$

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- Chung-Chung-Kim (2003) exploited the method to show that (1) has solutions $\phi \in \mathcal{S}^\beta(0, \infty)$, $\beta > 1$.
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General Stieltjes moment problems for rapidly decreasing smooth functions

We want to replace

$$a_n = \int_0^\infty x^n \phi(x) dx, \quad n = 0, 1, 2, \dots,$$

by the infinite system of linear equations

$$a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \dots, \tag{2}$$

where $f_n \in \mathcal{S}'[0,\infty)$ $(= \{f \in \mathcal{S}'(\mathbb{R}) : \operatorname{supp} f \subseteq [0,\infty)\})$.

Problem

Conditions over $\{f_n\}_{n=0}^\infty$ such that every generalized moment problem (2) has a solution $\phi \in \mathcal{S}(0,\infty)$.

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Particular cases

1. Continuous generalized moment problem

$$a_n = \int_0^\infty f_n(x)\phi(x)dx, \quad n = 0, 1, 2, \ldots.$$

2. Discrete problem: let $(B_{k,n})$ be an infinite matrix

$$a_n = \sum_{k=1}^\infty B_{k,n}\phi(k), \quad n = 0, 1, 2, \ldots.$$

or, more generally,

$$a_n = \sum_{k=1}^\infty B_{k,n}\phi(\lambda_k), \quad n = 0, 1, 2, \ldots.$$

3. Let $\{F_n\}_{n=0}^\infty$ be a sequence of functions of local bounded variation (having at most polynomial growth)

$$a_n = \int_0^\infty \phi(x)dF_n(x), \quad n = 0, 1, 2, \ldots.$$

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General Stieltjes moment problem for sequences of measures

Let $\{F_n\}_{n=0}^{\infty}$ be a sequence of functions of local bounded variation on $[0, \infty)$.

Theorem

Every generalized Stieltjes moment problem

$$a_n = \int_0^{\infty} \phi(x) dF_n(x), \quad n = 0, 1, 2, \dots,$$

has a solution $\phi \in \mathcal{S}(0, \infty)$, provided that:

- *$F_1, F_2, F_3 \dots$ are linearly independent*
- $1 \notin \operatorname{span}\{F_n\}$ *(equivalently, $\delta \notin \operatorname{span}\{dF_n\}$).*
- *For every $\alpha > 0$, there is N such that*

$$F_n(x) = o(x^{\alpha}), \quad n < N, \quad \text{but} \;\; F_n(x) = \Omega(x^{\alpha}), \quad N \leq n.$$

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Examples

The following generalized moment problems always have a solution $\phi \in \mathcal{S}(0,\infty)$

Let $\{\alpha_n\}_{n=0}^{\infty}$ be such that $\Re e\,\alpha_n \nearrow \infty$.

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$$a_n = \sum_{p \text{ prime}} p^{\alpha_n}\phi(p), \quad n = 0,1,2,\ldots.$$

$$a_n = \int_0^{\infty} x^{\alpha_n}\sin\left(\frac{1}{x^{\beta}}\right)\phi(x)\mathrm{d}x, \quad n = 0,1,2,\ldots,$$

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Cesàro asymptotics

Let $f \in \mathcal{S}'[0,\infty)$ and $\alpha > -1$. We write

$$f(x) = O(x^{\alpha}) \quad (\mathrm{C}, m), \quad x \to \infty$$

if $f^{(-m)}$, the primitive of order m of f, is locally integrable for large x, and

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Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of distributions with supp $f_n \subseteq [0,\infty)$.

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- $\operatorname{span}\{f_n\} \cap \operatorname{span}\{\delta^{(j)}\} = \{0\}$.
- *There is an increasing sequence of integers* $\{m_j\}_{j=0}^{\infty}$ *such that for every* $\beta > 0$ *there exists* $\alpha = \alpha_{j,\beta} \geq \beta$ *for which one can find* $N = N_\alpha \in \mathbb{N}$ *such that*

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