On the Stieltjes moment problem

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The problem of moments, as its generalizations, is an important mathematical problem which has attracted much attention for more than a century.

It was first raised and solved by Stieltjes for positive measures.

Problem (Stieltjes, 1894)

Find conditions over $\{a_n\}_{n=0}^{\infty}$ which ensure the existence of solutions μ to the infinity system of equations

$$a_n = \int_0^{\infty} x^n d\mu(x), \quad n = 0, 1, 2, \dots ,$$

where μ is a positive measure.

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The classical Stieltjes moment problem

Stieltjes found a necessary and sufficient condition for the existence of solutions. Define the sequence of matrices

$$\Delta_n = \begin{pmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \dots & a_{2n} \end{pmatrix} \text{ and } \Delta_n^{(1)} = \begin{pmatrix} a_1 & a_2 & \dots & a_{n+1} \\ a_2 & a_3 & \dots & a_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & a_{n+2} & \dots & a_{2n+1} \end{pmatrix}$$

Theorem (Stieltjes, 1894-1895)

The Stieltjes moment problem

$$a_n = \int_0^\infty x^n d\mu(x), \quad n = 0, 1, 2, \dots ,$$

has solution if and only if

$$\det(\Delta_n) > 0 \textit{ and } \det(\Delta_n^{(1)}) > 0, \quad n = 0, 1, 2, \dots .$$

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Ideas connected with the Stieltjes moment problem

Stieltjes' influential papers led to many important ideas:

- The theory of Stieltjes integrals

$$a_n = \int_0^\infty x^n dF(x), \quad F \nearrow .$$

- The Stieltjes transform, $\Re e\, z \notin (-\infty, 0]$,

$$S(z) = \int_0^\infty \frac{dF(x)}{x+z} \sim \sum_{n=0}^\infty \frac{(-1)^n a_n}{z^{n+1}} .$$

- Continued fraction approximations.

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- Carleman (1923-1926): connections with the theory of quasi-analytic functions.

Other moment problems:

- Hamburger (1920):

$$a_n = \int_{-\infty}^{\infty} x^n dF(x), \quad n = 0, 1, 2, \dots .$$

- Hausdorff (1923):

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Moment problems for arbitrary sequences

Theorem (Boas and Pólya, independently, 1939)

Given an arbitrary sequence $\{a_n\}_{n=0}^{\infty}$, there is always a function of bounded variation F such that

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A. Durán achieved a major improvement to this result:

Theorem (A. Durán, 1989)

Every Stieltjes moment problem

$$a_n = \int_0^{\infty} x^n \phi(x) dx, \quad n = 0, 1, 2, \ldots ,$$

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- A. L. Durán and Estrada found a simple proof (1994):

$$a_n = \int_0^\infty x^n \phi(x)dx, \quad n = 0, 1, 2, \dots, \tag{1}$$

iff $\widehat{\phi}^{(n)}(0) = (-i)^n a_n$. Then, the Borel-Ritt theorem ...
- Chung-Chung-Kim (2003) exploited the method to show that (1) has solutions $\phi \in \mathcal{S}^\beta(0,\infty)$, $\beta > 1$.
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Abstract moment problem

We want to replace

$$a_n = \int_0^\infty x^n \phi(x)dx, \quad n = 0, 1, 2, \dots,$$

by the infinite system of linear equations

$$a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \dots \ , \tag{2}$$

where the sought solution ϕ is an element of a (topological!) vector space E and $f_n \in E'$.

Problem

Conditions over E and $\{f_n\}_{n=0}^\infty$ such that every generalized moment problem (4) has a solution $\phi \in E$.

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Particular cases

- The Borel problem:

$$a_n = \phi^{(n)}(0), \quad n = 0, 1, 2, \ldots.$$

Here $E = C^\infty(\mathbb{R})$ and $f_n = (-1)^n \delta^{(n)}$, elements of $\mathcal{E}'(\mathbb{R})$.

- The Borel-Ritt problem. Given a sector $S : \alpha < \arg z < \beta$, $|z| < r$. Find an analytic function ϕ on S such that on any subsector $S_1 : \alpha_1 < \arg z < \beta_1$ one has

$$\phi(z) \sim \sum_{n=0}^{\infty} a_n z^n, \quad z \to 0^+. \tag{3}$$

In our setting, one may consider E the space of analytic functions on S having expansions of the form (3). The f_n are the linear functionals sending φ to its n-th coefficient of the expansion.

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Particular case: General Stieltjes moment problems for rapidly decreasing smooth functions

Direct generalization of Pólya-Boas-Durán problem,

$$a_n = \int_0^{\infty} x^n \phi(x) dx, \quad n = 0, 1, 2, \ldots,$$

where $\phi \in \mathcal{S}(0, \infty)$.

Distribution moment problem:

$$a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \ldots, \tag{4}$$

where $f_n \in \mathcal{S}'[0, \infty)$ (= $f_n \in \mathcal{S}'(\mathbb{R})$ with supp $f_n \subseteq [0, \infty)$).

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Particular cases

1. Continuous generalized moment problem

$$a_n = \int_0^\infty f_n(x)\phi(x)dx, \quad n = 0, 1, 2, \ldots .$$

2. Discrete problem: Let $(B_{k,n})$ be an infinite matrix

$$a_n = \sum_{k=1}^{\infty} B_{k,n}\phi(k), \quad n = 0, 1, 2, \ldots ,$$

or, more generally, $0 < \lambda_n \to \infty$,

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3. Let $\{F_n\}_{n=0}^{\infty}$ be a sequence of functions of local bounded variation (having at most polynomial growth)

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Back to the abstract moment problem

We now consider the abstract moment problem

$$a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \ldots,$$

where E is an FS-space. So, $f_n \in E'$ and $\phi \in E$.

- Fréchet space: locally convex, metrizable, and complete TVS.
- Every Fréchet space E is the projective limit of a decreasing sequence of Banach spaces

$$E = \operatorname{proj}\varprojlim E_j \to \cdots \to E_{n+1} \to E_n \to \ldots \to E_1,$$

with $E_{n+1} \to E_n$ continuous and dense.

- Fréchet-Schwartz (FS): $E_{n+1} \to E_n$ are compact.

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Silva's duality theory for FS-spaces

- A DFS-space (or Silva space) is the inductive limit of an increasing sequence of Banach spaces,

$$X_1 \to X_2 \to \dots X_n \to X_{n+1} \to \cdots \to \operatorname{ind}\lim_{\to} X_j = X,$$

  where each $X_n \to X_{n+1}$ is compact and injective.
- Silva's Lemma: $Y \subset X$ is closed if and only if $Y \cap X_n$ is closed in X_n, $\forall n$.
- Silva's Duality Theorem:
  - The dual of an FS-space is a DFS-space.
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  - The FS- and DFS-spaces are Montel (hence reflexive).
  - If $E = \operatorname{proj}\lim_{\leftarrow} E_n$, with $E_{j+1} \to E_j$ compact and dense, then

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  - The FS- and DFS-spaces are Montel (hence reflexive).
  - If $E = \operatorname{proj} \lim_{\leftarrow} E_n$, with $E_{j+1} \to E_j$ compact and dense, then

$$E' = \operatorname{ind} \lim_{\to} E'_n.$$

Simplest examples of FS- and DFS-spaces

Let $\mathcal{P}_n$ the space of polynomials of degree $\leq n$ (in one variable). So, $\mathcal{P}_n \cong \mathbb{C}^{n+1}$.

Consider the canonical injections

$$\iota_n : \mathcal{P}_n \to \mathcal{P}_{n+1}$$

and the projections

$$\pi_n : \mathcal{P}_{n+1} \to \mathcal{P}_n.$$

- The space of polynomials $\mathcal{P} = \operatorname{ind} \lim\limits_{\to} \mathcal{P}_n$ is DFS.
- The space of formal power series $\mathbb{C}[[\xi]] = \operatorname{proj} \lim\limits_{\leftarrow} \mathcal{P}_n$ is FS.
- Duality $\mathcal{P}' = \mathbb{C}[[\xi]]$ and $(\mathbb{C}[[\xi]])' = \mathcal{P}$.

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Abstract moment problem in FS-spaces

Theorem

Let $E = \operatorname{proj}\lim_{\leftarrow} E_j$ *be an FS-space, where* $E_{n+1} \to E_n$ *is compact and dense. Consider* $\{f_n\}_{n=0}^{\infty} \subset E'$*. Every arbitrary abstract moment problem*

$$\langle f_n, \phi \rangle = a_n, \quad n \in \mathbb{N},$$

has solution $\phi \in E$ *if and only if*

1. $f_0, f_1, \dots, f_n, \dots,$ *are linearly independent.*
2. $\operatorname{span}\{f_n : n \in \mathbb{N}\} \cap E_j'$ *is finite dimensional,* $\forall j \in \mathbb{N}$.

Proof. Sketch on the blackboard. We use the following lemma:
Lemma. *Let* $X = \operatorname{ind}\lim_{\rightarrow} X_j$ *be a DFS-space, with* $X_j \to X_{j+1}$ *compact and injective. A continuous injective mapping* $L : P \to X$ *has closed range if and only if* $L(\mathcal{P}) \cap X_j$ *is finite dimensional* $\forall j$.

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Applications

- For the Borel problem:

$$a_n = \phi^{(n)}(0) = \langle (-1)^n \delta^{(n)}, \phi \rangle, \quad n = 0, 1, 2, \ldots,$$

one takes $E = C^\infty(\mathbb{R}) = \operatorname{proj} \varprojlim C^j[-j, j]$. Since all elements of the dual of $C^j[-j, j]$ are derivatives of order $\leq j + 1$ of measures, the last theorem implies that every Borel problem has solution.

- A similar argument shows that every Borel-Ritt problem has a solution.
- For the Stieltjes moment problem, one writes

$$\mathcal{S}(0, \infty) = \operatorname{proj} \varprojlim \mathcal{S}_p(0, \infty),$$

where $\mathcal{S}_p(0, \infty)$ is

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- $\{x^n\}_{n=0}^{\infty} \cap \mathcal{S}'_p(0,\infty)$, $\forall p$. The theorem yields that

$$\int_0^{\infty} x^n \phi(x) \mathrm{d}x = a_n, \quad n = 0, 1, 2, \dots ,$$

has always a solution $\phi \in \mathcal{S}(0,\infty)$.

- More generally, it is possible to characterize the $\{f_n\}_{n=0}^{\infty} \subset \mathcal{S}[0,\infty)$ for which every generalized Stieltjes moment problem

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The weighted Stieltjes moment problem

Let $0 \leq F \nearrow$ on $[0, \infty)$ and let $\{\alpha_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$.

Theorem

Every weighted Stieltjes moment problem

$$a_n = \int_0^\infty \phi(x) x^{\alpha_n} dF(x), \quad n \in \mathbb{Z},$$

has a solution $\phi \in \mathcal{S}(0,\infty)$, provided that:

1. *The sets $\{n \in \mathbb{Z} : -M \leq \Re e\, \alpha_n \leq M\}$ are finite $\forall M > 0$.*
2. *If $\lim_{n\to\infty} \Re e\, \alpha_n = \infty$, then*

$$-\infty < \limsup_{x\to\infty} \frac{\log \int_0^x F(t)\mathrm{d}t}{\log x}.$$

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The following generalized moment problems always have a solution $\phi \in \mathcal{S}(0,\infty)$.

Let $\{\alpha_n\}_{n=0}^{\infty}$ be such that $\Re e\, \alpha_n \nearrow \infty$.

$$a_n = \sum_{k=1}^{\infty} k^{\alpha_n}\phi(k), \quad n = 0, 1, 2, \dots .$$

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