General Stieltjes moment problems for rapidly decreasing smooth functions

Jasson Vindas
jvindas@cage.Ugent.be

Department of Mathematics
Ghent University

ISAAC 9th Congress
Session Generalized Functions
Krakow, August 6, 2013
The problem of moments, as its generalizations, is an important mathematical problem which has attracted much attention for more than a century.

It was first raised and solved by Stieltjes for positive measures.

Problem (Stieltjes, 1894)

Find conditions over \( \{a_n\}_{n=0}^{\infty} \) which ensure the existence of solutions to the infinity system of equations

\[
a_n = \int_0^\infty x^n \, d\mu(x), \quad n = 0, 1, 2, \ldots ,
\]

where \( \mu \) is a positive measure.

We will discuss several generalizations of this problem.
The problem of moments, as its generalizations, is an important mathematical problem which has attracted much attention for more than a century.

It was first raised and solved by Stieltjes for positive measures.

Problem (Stieltjes, 1894)

Find conditions over \( \{a_n\}_{n=0}^{\infty} \) which ensure the existence of solutions to the infinity system of equations

\[
a_n = \int_{0}^{\infty} x^n d\mu(x), \quad n = 0, 1, 2, \ldots ,
\]

where \( \mu \) is a positive measure.

We will discuss several generalizations of this problem.
The classical Stieltjes moment problem

Stieltjes found a necessary and sufficient condition for the existence of solutions. Define the sequence of matrices

\[
\Delta_n = \begin{pmatrix}
    a_0 & a_1 & \cdots & a_n \\
    a_1 & a_2 & \cdots & a_{n+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_n & a_{n+1} & \cdots & a_{2n}
\end{pmatrix}
\quad \text{and} \quad
\Delta_n^{(1)} = \begin{pmatrix}
    a_1 & a_1 & \cdots & a_{n+1} \\
    a_2 & a_3 & \cdots & a_{n+2} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n+1} & a_{n+2} & \cdots & a_{2n+1}
\end{pmatrix}
\]

Theorem (Stieltjes, 1894-1895)

The Stieltjes moment problem

\[
a_n = \int_0^\infty x^n \, d\mu(x), \quad n = 0, 1, 2, \ldots,
\]

has solution if and only if

\[
\det(\Delta_n) > 0 \quad \text{and} \quad \det(\Delta_n^{(1)}) > 0, \quad n = 0, 1, 2, \ldots.
\]
The classical Stieltjes moment problem

Stieltjes found a necessary and sufficient condition for the existence of solutions. Define the sequence of matrices

$$\Delta_n = \begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ a_1 & a_2 & \ldots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \ldots & a_{2n} \end{pmatrix}$$

and

$$\Delta_n^{(1)} = \begin{pmatrix} a_1 & a_1 & \ldots & a_{n+1} \\ a_2 & a_3 & \ldots & a_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & a_{n+2} & \ldots & a_{2n+1} \end{pmatrix}$$

**Theorem (Stieltjes, 1894-1895)**

The Stieltjes moment problem

$$a_n = \int_0^\infty x^n d\mu(x), \quad n = 0, 1, 2, \ldots,$$

has solution if and only if

$$\det(\Delta_n) > 0 \text{ and } \det(\Delta_n^{(1)}) > 0, \quad n = 0, 1, 2, \ldots.$$
The classical Stieltjes moment problem

Stieltjes found a necessary and sufficient condition for the existence of solutions. Define the sequence of matrices

\[ \Delta_n = \begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ a_1 & a_2 & \ldots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \ldots & a_{2n} \end{pmatrix} \quad \text{and} \quad \Delta_n^{(1)} = \begin{pmatrix} a_1 & a_1 & \ldots & a_{n+1} \\ a_2 & a_3 & \ldots & a_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & a_{n+2} & \ldots & a_{2n+1} \end{pmatrix} \]

Theorem (Stieltjes, 1894-1895)

The Stieltjes moment problem

\[ a_n = \int_0^\infty x^n d\mu(x), \quad n = 0, 1, 2, \ldots, \]

has solution if and only if

\[ \det(\Delta_n) > 0 \quad \text{and} \quad \det(\Delta_n^{(1)}) > 0, \quad n = 0, 1, 2, \ldots. \]
Stieltjes’ influential papers led to many important ideas:

- The theory Stieltjes integrals

\[ a_n = \int_0^\infty x^n dF(x), \quad F \uparrow. \]

- The Stieltjes transform, \( \Re e z \notin (-\infty, 0] \),

\[ S(z) = \int_0^\infty \frac{dF(x)}{x + z} \sim \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{z^{n+1}}. \]

- Continued fraction approximations
Stieltjes’ influential papers led to many important ideas:

- The theory Stieltjes integrals

\[ a_n = \int_0^\infty x^n dF(x), \quad F \uparrow. \]

- The Stieltjes transform, \( \Re e z \notin (-\infty, 0] \),

\[ S(z) = \int_0^\infty \frac{dF(x)}{x + z} \sim \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{z^{n+1}}. \]

- Continued fraction approximations
Ideas connected with the Stieltjes moment problem

Stieltjes’ influential papers led to many important ideas:

- The theory Stieltjes integrals

  \[ a_n = \int_0^\infty x^n dF(x), \quad F \uparrow. \]

- The Stieltjes transform, \( \mathbb{R}e \, z \notin (-\infty, 0], \)

  \[ S(z) = \int_0^\infty \frac{dF(x)}{x + z} \sim \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{z^{n+1}}. \]

- Continued fraction approximations
Ideas connected with the Stieltjes moment problem

Stieltjes’ influential papers led to many important ideas:

- The theory Stieltjes integrals

\[ a_n = \int_0^\infty x^n dF(x), \quad F \nearrow. \]

- The Stieltjes transform, \( \Re z \notin (-\infty, 0] \),

\[ S(z) = \int_0^\infty \frac{dF(x)}{x + z} \sim \sum_{n=0}^\infty \frac{(-1)^n a_n}{z^{n+1}}. \]

- Continued fraction approximations
Stieltjes’ influential papers led to many important ideas:

- The theory Stieltjes integrals

\[
a_n = \int_0^\infty x^n dF(x), \quad F \nearrow.
\]

- The Stieltjes transform, \( \Re z \notin (-\infty, 0] \),

\[
S(z) = \int_0^\infty \frac{dF(x)}{x + z} \sim \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{z^{n+1}}.
\]

- Continued fraction approximations
Modern approach goes back to Marcel Riesz (1921).

Carleman (1923-1926): connections with the theory of quasi-analytic functions.

Other moment problems:

- Hamburger (1920):
  \[ a_n = \int_{-\infty}^{\infty} x^n dF(x), \quad n = 0, 1, 2, \ldots . \]

- Hausdorff (1923):
  \[ a_n = \int_{b}^{c} x^n dF(x), \quad n = 0, 1, 2, \ldots . \]

For results on classical moment problems see the book by Shohat and Tamarkin (The problem of moments, 1943).
Modern approach goes back to Marcel Riesz (1921).
Carleman (1923-1926): connections with the theory of quasi-analytic functions.

Other moment problems:

- Hamburger (1920):
  \[ a_n = \int_{-\infty}^{\infty} x^n dF(x), \quad n = 0, 1, 2, \ldots. \]

- Hausdorff (1923):
  \[ a_n = \int_b^c x^n dF(x), \quad n = 0, 1, 2, \ldots. \]

For results on classical moment problems see the book by Shohat and Tamarkin (*The problem of moments*, 1943).
Modern approach goes back to Marcel Riesz (1921).

Carleman (1923-1926): connections with the theory of quasi-analytic functions.

Other moment problems:

Hamburger (1920):

\[ a_n = \int_{-\infty}^{\infty} x^n dF(x), \quad n = 0, 1, 2, \ldots . \]

Hausdorff (1923):

\[ a_n = \int_{b}^{c} x^n dF(x), \quad n = 0, 1, 2, \ldots . \]

For results on classical moment problems see the book by Shohat and Tamarkin (The problem of moments, 1943).
Moment problems for arbitrary sequences

Theorem (Boas and Pólya, independently, 1939)

Given an arbitrary sequence \( \{a_n\}_{n=0}^{\infty} \), there is always a function of bounded variation \( F \) such that

\[
a_n = \int_0^\infty x^n dF(x), \quad n = 0, 1, 2, \ldots .
\]

A. Durán achieved a major improvement to this result:

Theorem (A. Durán, 1989)

Every Stieltjes moment problem

\[
a_n = \int_0^\infty x^n \phi(x) dx, \quad n = 0, 1, 2, \ldots ,
\]

admits a solution \( \phi \in S(0, \infty) \), namely, \( \phi \in S(\mathbb{R}) \) with \( \text{supp } \phi \subseteq [0, \infty) \).
Theorem (Boas and Pólya, independently, 1939)

Given an arbitrary sequence \( \{a_n\}_{n=0}^{\infty} \), there is always a function of bounded variation \( F \) such that

\[
a_n = \int_{0}^{\infty} x^n dF(x), \quad n = 0, 1, 2, \ldots.
\]

A. Durán achieved a major improvement to this result:

Theorem (A. Durán, 1989)

Every Stieltjes moment problem

\[
a_n = \int_{0}^{\infty} x^n \phi(x) dx, \quad n = 0, 1, 2, \ldots,
\]

admits a solution \( \phi \in S(0, \infty) \), namely, \( \phi \in S(\mathbb{R}) \) with

\[
\text{supp} \phi \subseteq [0, \infty).
\]
Theorem (Boas and Pólya, independently, 1939)

Given an arbitrary sequence \( \{a_n\}_{n=0}^{\infty} \), there is always a function of bounded variation \( F \) such that

\[
a_n = \int_0^\infty x^n dF(x), \quad n = 0, 1, 2, \ldots.
\]

A. Durán achieved a major improvement to this result:

Theorem (A. Durán, 1989)

*Every* Stieltjes moment problem

\[
a_n = \int_0^\infty x^n \phi(x) dx, \quad n = 0, 1, 2, \ldots,
\]

admits a solution \( \phi \in S(0, \infty) \), namely, \( \phi \in S(\mathbb{R}) \) with

\[\text{supp} \phi \subseteq [0, \infty).\]
A. Durán’s proof: Laguerre expansions, Hankel transform.

A. L. Durán and Estrada found a simple proof (1994):

\[ a_n = \int_0^\infty x^n \phi(x) \, dx, \quad n = 0, 1, 2, \ldots, \quad (1) \]

iff \( \hat{\phi}^{(n)}(0) = (-i)^n a_n \). Then, the Borel-Ritt theorem ...

Chung-Chung-Kim (2003) exploited the method to show that (1) has solutions \( \phi \in S^\beta(0, \infty), \beta > 1 \).

Lastra and Sanz (2009) have considered ultradifferentiable classes \( S^*(0, \infty) \), with \( * = (M_p), \{M_p\} \).
A. Durán’s proof: Laguerre expansions, Hankel transform.

A. L. Durán and Estrada found a simple proof (1994):

\[ a_n = \int_0^\infty x^n \phi(x) \, dx, \quad n = 0, 1, 2, \ldots, \quad (1) \]

iff \( \hat{\phi}^{(n)}(0) = (-i)^n a_n \). Then, the Borel-Ritt theorem ...

Chung-Chung-Kim (2003) exploited the method to show that (1) has solutions \( \phi \in S^\beta(0, \infty), \ \beta > 1 \).

Lastra and Sanz (2009) have considered ultradifferentiable classes \( S^*(0, \infty) \), with \( * = (M_p), \{M_p\} \).
A. Durán’s proof: Laguerre expansions, Hankel transform.

A. L. Durán and Estrada found a simple proof (1994):

\[ a_n = \int_0^\infty x^n \phi(x) \, dx, \quad n = 0, 1, 2, \ldots, \quad (1) \]

iff \( \hat{\phi}(n)(0) = (-i)^n a_n \). Then, the Borel-Ritt theorem ...

Chung-Chung-Kim (2003) exploited the method to show that (1) has solutions \( \phi \in S^\beta(0, \infty), \beta > 1 \).

Lastra and Sanz (2009) have considered ultradifferentiable classes \( S^\ast(0, \infty) \), with \( \ast = (M_p), \{M_p\} \).
A. Durán’s proof: Laguerre expansions, Hankel transform.

A. L. Durán and Estrada found a simple proof (1994):

\[ a_n = \int_0^\infty x^n \phi(x) \, dx, \quad n = 0, 1, 2, \ldots, \quad (1) \]

iff \( \hat{\phi}(n)(0) = (-i)^n a_n \). Then, the Borel-Ritt theorem ...

Chung-Chung-Kim (2003) exploited the method to show that (1) has solutions \( \phi \in S^\beta(0, \infty), \beta > 1 \).

Lastra and Sanz (2009) have considered ultradifferentiable classes \( S^* (0, \infty) \), with \( * = (M_p), \{M_p\} \).
We want to replace

\[ a_n = \int_{0}^{\infty} x^n \phi(x) \, dx, \quad n = 0, 1, 2, \ldots, \]

by the infinite system of linear equations

\[ a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \ldots, \quad (2) \]

where \( f_n \in S'[0, \infty) \) (= \( \{ f \in S'(\mathbb{R}) : \text{supp } f \subseteq [0, \infty) \} \)).

**Problem**

*Conditions over \( \{ f_n \}_{n=0}^{\infty} \) such that every generalized moment problem (2) has a solution \( \phi \in S(0, \infty) \).*
We want to replace

$$a_n = \int_0^\infty x^n \phi(x) \, dx, \quad n = 0, 1, 2, \ldots,$$

by the infinite system of linear equations

$$a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \ldots,$$

(2)

where $f_n \in S'[0, \infty)$ (i.e., $f_n \in S'(\mathbb{R}) : \text{supp} \, f \subseteq [0, \infty)$).

**Problem**

*Conditions over $\{f_n\}_{n=0}^\infty$ such that every generalized moment problem (2) has a solution $\phi \in S(0, \infty)$.***
Particular cases

1. **Continuous generalized moment problem**

   \[ a_n = \int_0^\infty f_n(x)\phi(x)\,dx, \quad n = 0, 1, 2, \ldots \]

2. **Discrete problem**: let \((B_{k,n})\) be an infinite matrix

   \[ a_n = \sum_{k=1}^{\infty} B_{k,n}\phi(k), \quad n = 0, 1, 2, \ldots \]

   or, more generally,

   \[ a_n = \sum_{k=1}^{\infty} B_{k,n}\phi(\lambda_k), \quad n = 0, 1, 2, \ldots \]

3. **Let \(\{F_n\}_{n=0}^{\infty}\)** be a sequence of functions of local bounded variation (having at most polynomial growth)

   \[ a_n = \int_0^\infty \phi(x)\,dF_n(x), \quad n = 0, 1, 2, \ldots \]
Particular cases

1. **Continuous generalized moment problem**

\[ a_n = \int_0^\infty f_n(x)\phi(x)\,dx, \quad n = 0, 1, 2, \ldots \]

2. **Discrete problem:** let \((B_{k,n})\) be an infinite matrix

\[ a_n = \sum_{k=1}^\infty B_{k,n}\phi(k), \quad n = 0, 1, 2, \ldots \]

or, more generally,

\[ a_n = \sum_{k=1}^\infty B_{k,n}\phi(\lambda_k), \quad n = 0, 1, 2, \ldots \]

3. Let \(\{F_n\}_{n=0}^\infty\) be a sequence of functions of local bounded variation (having at most polynomial growth)

\[ a_n = \int_0^\infty \phi(x)\,dF_n(x), \quad n = 0, 1, 2, \ldots \]
Particular cases

1. **Continuous** generalized moment problem

\[ a_n = \int_0^\infty f_n(x)\phi(x)\,dx, \quad n = 0, 1, 2, \ldots \]

2. **Discrete** problem: let \((B_{k,n})\) be an infinite matrix

\[ a_n = \sum_{k=1}^{\infty} B_{k,n}\phi(k), \quad n = 0, 1, 2, \ldots \]

or, more generally,

\[ a_n = \sum_{k=1}^{\infty} B_{k,n}\phi(\lambda_k), \quad n = 0, 1, 2, \ldots \]

3. Let \(\{F_n\}_{n=0}^\infty\) be a sequence of functions of local bounded variation (having at most polynomial growth)

\[ a_n = \int_0^\infty \phi(x)\,dF_n(x), \quad n = 0, 1, 2, \ldots \]
Particular cases

1. Continuous generalized moment problem

\[ a_n = \int_0^\infty f_n(x)\phi(x)\,dx, \quad n = 0, 1, 2, \ldots \]

2. Discrete problem: let \((B_{k,n})\) be an infinite matrix

\[ a_n = \sum_{k=1}^\infty B_{k,n}\phi(k), \quad n = 0, 1, 2, \ldots \]

or, more generally,

\[ a_n = \sum_{k=1}^\infty B_{k,n}\phi(\lambda_k), \quad n = 0, 1, 2, \ldots \]

3. Let \(\{F_n\}_{n=0}^\infty\) be a sequence of functions of local bounded variation (having at most polynomial growth)

\[ a_n = \int_0^\infty \phi(x)dF_n(x), \quad n = 0, 1, 2, \ldots \]
Let \( \{F_n\}_{n=0}^{\infty} \) be a sequence of functions of local bounded variation on \([0, \infty)\).

**Theorem**

*Every generalized Stieltjes moment problem*

\[
a_n = \int_0^\infty \phi(x) dF_n(x), \quad n = 0, 1, 2, \ldots,
\]

*has a solution* \( \phi \in S(0, \infty) \), *provided that:*

- \( F_1, F_2, F_3 \ldots \) are linearly independent
- \( 1 \notin \text{span}\{F_n\} \) (equivalently, \( \delta \notin \text{span}\{dF_n\} \)).
- For every \( \alpha > 0 \), there is \( N \) such that
  \[
  F_n(x) = o(x^\alpha), \quad n < N, \quad \text{but} \quad F_n(x) = \Omega(x^\alpha), \quad N \leq n.
  \]
Let \( \{F_n\}_{n=0}^{\infty} \) be a sequence of functions of local bounded variation on \([0, \infty)\).

**Theorem**

*Every* generalized Stieltjes moment problem

\[
a_n = \int_0^\infty \phi(x) dF_n(x), \quad n = 0, 1, 2, \ldots,
\]

**has a solution** \( \phi \in S(0, \infty) \), **provided that:**

- \( F_1, F_2, F_3 \ldots \) are linearly independent
- \( 1 \notin \text{span}\{F_n\} \) (equivalently, \( \delta \notin \text{span}\{dF_n\} \)).
- For every \( \alpha > 0 \), there is \( N \) such that

\[
F_n(x) = o(x^\alpha), \quad n < N, \quad \text{but} \quad F_n(x) = \Omega(x^\alpha), \quad N \leq n.
\]
Let \( \{F_n\}_{n=0}^\infty \) be a sequence of functions of local bounded variation on \([0, \infty)\).

**Theorem**

*Every* generalized Stieltjes moment problem

\[
a_n = \int_0^\infty \phi(x) dF_n(x), \quad n = 0, 1, 2, \ldots,
\]

has a solution \( \phi \in S(0, \infty) \), provided that:

- \( F_1, F_2, F_3 \ldots \) are linearly independent
- \( 1 \notin \text{span}\{F_n\} \) (equivalently, \( \delta \notin \text{span}\{dF_n\} \)).
- **For every** \( \alpha > 0 \), there is \( N \) such that

\[
F_n(x) = o(x^\alpha), \quad n < N, \quad \text{but} \quad F_n(x) = \Omega(x^\alpha), \quad N \leq n.
\]
The following generalized moment problems always have a solution $\phi \in S(0, \infty)$:

Let $\{\alpha_n\}_{n=0}^{\infty}$ be such that $\Re\alpha_n \uparrow \infty$.

\[
a_n = \sum_{k=1}^{\infty} k^{\alpha_n} \phi(k), \quad n = 0, 1, 2, \ldots.
\]

\[
a_n = \sum_{p \text{ prime}} p^{\alpha_n} \phi(p), \quad n = 0, 1, 2, \ldots.
\]

\[
a_n = \int_0^\infty x^{\alpha_n} \sin \left( \frac{1}{x^\beta} \right) \phi(x) \, dx, \quad n = 0, 1, 2, \ldots,
\]

where $\beta \geq 0$. 
The following generalized moment problems always have a solution \( \phi \in S(0, \infty) \)

Let \( \{\alpha_n\}_{n=0}^{\infty} \) be such that \( \Re e \alpha_n \uparrow \infty \).

\[
a_n = \sum_{k=1}^{\infty} k^{\alpha_n} \phi(k), \quad n = 0, 1, 2, \ldots
\]

\[
a_n = \sum_{p \text{ prime}} p^{\alpha_n} \phi(p), \quad n = 0, 1, 2, \ldots
\]

\[
a_n = \int_{0}^{\infty} x^{\alpha_n} \sin \left( \frac{1}{x^{\beta}} \right) \phi(x) \, dx, \quad n = 0, 1, 2, \ldots,
\]

where \( \beta \geq 0 \).
The following generalized moment problems always have a solution $\phi \in S(0, \infty)$

Let $\{\alpha_n\}_{n=0}^{\infty}$ be such that $\Re \alpha_n \nearrow \infty$.

$$a_n = \sum_{k=1}^{\infty} k^{\alpha_n} \phi(k), \quad n = 0, 1, 2, \ldots$$

$$a_n = \sum_{p \text{ prime}} p^{\alpha_n} \phi(p), \quad n = 0, 1, 2, \ldots$$

$$a_n = \int_{0}^{\infty} x^{\alpha_n} \sin \left( \frac{1}{x^\beta} \right) \phi(x) dx, \quad n = 0, 1, 2, \ldots,$$

where $\beta \geq 0$. 
Let $f \in S'[0, \infty)$ and $\alpha > -1$. We write

$$f(x) = O(x^\alpha) \ (C, m), \quad x \to \infty$$

if $f^{(-m)}$, the primitive of order $m$ of $f$, is locally integrable for large $x$, and

$$f^{(-m)}(x) = O(x^{\alpha+m}), \quad x \to \infty,$$

in the ordinary sense.
Let $f \in S'[0, \infty)$ and $\alpha > -1$. We write

$$f(x) = O(x^\alpha) \quad (C, m), \quad x \to \infty$$

if $f^{(-m)}$, the primitive of order $m$ of $f$, is locally integrable for large $x$, and

$$f^{(-m)}(x) = O(x^{\alpha+m}), \quad x \to \infty,$$

in the ordinary sense.
Let \( \{f_n\}_{n=0}^{\infty} \) be a sequence of distributions with \( \text{supp} \ f_n \subseteq [0, \infty) \).

**Theorem**

*Every generalized Stieltjes moment problem* \[
a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \ldots,
\]

*has a solution* \( \phi \in S(0, \infty) \) *if:*

1. \( f_1, f_2, f_3, \ldots, f_n, \ldots, \) are linearly independent.
2. \( \text{span}\{f_n\} \cap \text{span}\{\delta(i)\} = \{0\} \).
3. There is an increasing sequence of integers \( \{m_j\}_{j=0}^{\infty} \) *such that* for every \( \beta > 0 \) there exists \( \alpha = \alpha_{j,\beta} \geq \beta \) *for which one can find* \( N = N_{\alpha} \in \mathbb{N} \) *such that* \[
\sum_{n=0}^{M} b_n f_n(x) = O(x^\alpha) \quad (C, m_j) \text{ implies } b_N = b_{N+1} = \cdots = b_M = 0.
\]
Let \( \{f_n\}_{n=0}^{\infty} \) be a sequence of distributions with \( \text{supp } f_n \subseteq [0, \infty) \).

**Theorem**

*Every* generalized Stieltjes moment problem

\[ a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \ldots, \]

has a solution \( \phi \in S(0, \infty) \) if:

- \( f_1, f_2, f_3 \ldots, f_n, \ldots \), are linearly independent.
- \( \text{span}\{f_n\} \cap \text{span}\{\delta^{(j)}\} = \{0\} \).
- There is an increasing sequence of integers \( \{m_j\}_{j=0}^{\infty} \) such that for every \( \beta > 0 \) there exists \( \alpha = \alpha_{j,\beta} \geq \beta \) for which one can find \( N = N_{\alpha} \in \mathbb{N} \) such that

\[
\sum_{n=0}^{M} b_n f_n(x) = O(x^\alpha) \quad (C, m_j) \text{ implies } b_N = b_{N+1} = \cdots = b_M = 0.
\]
General Stieltjes moment problem for sequences of distributions

Let \( \{ f_n \}_{n=0}^{\infty} \) be a sequence of distributions with \( \text{supp} \ f_n \subseteq [0, \infty) \).

**Theorem**

*Every* generalized Stieltjes moment problem

\[
a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \ldots,
\]

*has a solution* \( \phi \in S(0, \infty) \) *if:*

- \( f_1, f_2, f_3 \ldots, f_n, \ldots \), *are linearly independent.*
- \( \text{span}\{f_n\} \cap \text{span}\{\delta(j)\} = \{0\} \).
- There is an increasing sequence of integers \( \{ m_j \}_{j=0}^{\infty} \) *such that for every* \( \beta > 0 \) *there exists* \( \alpha = \alpha_{j,\beta} \geq \beta \) *for which one can find* \( N = N_{\alpha} \in \mathbb{N} \) *such that*

\[
\sum_{n=0}^{M} b_n f_n(x) = O(x^{\alpha}) \quad (C, m_j) \implies b_N = b_{N+1} = \cdots = b_M = 0.
\]
Let \( \{f_n\}_{n=0}^{\infty} \) be a sequence of distributions with \( \text{supp} \ f_n \subseteq [0, \infty) \).

**Theorem**

Every generalized Stieltjes moment problem

\[
a_n = \langle f_n, \phi \rangle, \quad n = 0, 1, 2, \ldots,
\]

has a solution \( \phi \in S(0, \infty) \) if:

- \( f_1, f_2, f_3 \ldots, f_n, \ldots \), are linearly independent.
- \( \text{span}\{f_n\} \cap \text{span}\{\delta(j)\} = \{0\} \).
- There is an increasing sequence of integers \( \{m_j\}_{j=0}^{\infty} \) such that for every \( \beta > 0 \) there exists \( \alpha = \alpha_{j,\beta} \geq \beta \) for which one can find \( N = N_\alpha \in \mathbb{N} \) such that

\[
\sum_{n=0}^{M} b_nf_n(x) = O(x^\alpha) \quad (C, m_j) \text{ implies } b_N = b_{N+1} = \cdots = b_M = 0.
\]