Translation-invariant spaces of ultradistributions and some of their applications

Jasson Vindas
jvindas@cage.Ugent.be

Department of Mathematics
Ghent University

ISAAC 10th Congress
Session Generalized Functions
Macau, August 3, 2015

Introduction

Translation-invariant spaces of functions and generalized functions play a central role in many problems from functional and harmonic analysis.

In this talk we present the construction of a large family of translation-invariant spaces of (ultra)distributions. We consider applications in:

1. Topological properties of spaces of ultradistributions.
2. Convolution of ultradistributions.
3. Boundary values of holomorphic functions.

The talk is based on collaborative works with P. Dimovski, S. Pilipović, and B. Prangoski.

Motivation: Some classical distribution spaces

Schwartz introduced spaces of distributions based on L^p spaces as follows. Let $X = L^q$ with $q \in [1, \infty]$ and $X' = L^p$. Set

$$\mathcal{D}_{L^q} = \mathcal{D}_X := \{\varphi \in \mathcal{D} : \ \varphi^{(\alpha)} \in X, \ \forall \alpha \in \mathbb{N}^n\}$$

and

$$\mathcal{D}'_{L^p} = \mathcal{D}'_{X'} := (\mathcal{D}_X)'.$$

This works if $q < \infty$ and he denoted also $\mathcal{B}' = \mathcal{D}'_{L^\infty}$, the space of bounded distributions. For $q = \infty$, one replaces $\mathcal{D}_{L^\infty}$ by its closed subspace

$$\dot{\mathcal{B}} = \{\varphi \in \mathcal{D} : \lim_{|x| \to \infty} \varphi^{(\alpha)}(x) = 0, \ \forall \alpha \in \mathbb{N}^n\}$$

and defines the space of integrable distributions $\mathcal{D}'_{L^1} = (\dot{\mathcal{B}})'$.

These spaces are crucial in classical distribution theory. Some examples:

- For convolution (Schwartz, Ortner-Wagner, ...).
- For boundary values (Tillmann, Carmichael, ...).
- $\mathcal{B}'$ and $\mathcal{D}_{L^1}$ for Tauberian theory (Beurling, Pilipović-Stanković, ...).

Our goal: to single out properties of a Banach space X preserving the richness of $\mathcal{D}_X$ and $\mathcal{D}'_{X'}$.

Motivation: Some classical distribution spaces

Schwartz introduced spaces of distributions based on L^p spaces as follows. Let $X = L^q$ with $q \in [1, \infty]$ and $X' = L^p$. Set

$$\mathcal{D}_{L^q} = \mathcal{D}_X := \{\varphi \in \mathcal{D} : \ \varphi^{(\alpha)} \in X, \ \forall \alpha \in \mathbb{N}^n\}$$

and

$$\mathcal{D}'_{L^p} = \mathcal{D}'_{X'} := (\mathcal{D}_X)'.$$

This works if $q < \infty$ and he denoted also $\mathcal{B}' = \mathcal{D}'_{L^\infty}$, the space of bounded distributions. For $q = \infty$, one replaces $\mathcal{D}_{L^\infty}$ by its closed subspace

$$\dot{\mathcal{B}} = \{\varphi \in \mathcal{D} : \lim_{|x|\to\infty} \varphi^{(\alpha)}(x) = 0, \ \forall \alpha \in \mathbb{N}^n\}$$

and defines the space of integrable distributions $\mathcal{D}'_{L^1} = (\dot{\mathcal{B}})'$.

These spaces are crucial in classical distribution theory. Some examples:

- For convolution (Schwartz, Ortner-Wagner, ...).
- For boundary values (Tillmann, Carmichael, ...).
- $\mathcal{B}'$ and $\mathcal{D}_{L^1}$ for Tauberian theory (Beurling, Pilipović-Stanković, ...).

Our goal: to single out properties of a Banach space X preserving the richness of $\mathcal{D}_X$ and $\mathcal{D}'_{X'}$.

Motivation: Some classical distribution spaces

Schwartz introduced spaces of distributions based on L^p spaces as follows. Let $X = L^q$ with $q \in [1, \infty]$ and $X' = L^p$. Set

$$\mathcal{D}_{L^q} = \mathcal{D}_X := \{\varphi \in \mathcal{D} : \ \varphi^{(\alpha)} \in X, \ \forall \alpha \in \mathbb{N}^n\}$$

and

$$\mathcal{D}'_{L^p} = \mathcal{D}'_{X'} := (\mathcal{D}_X)'.$$

This works if $q < \infty$ and he denoted also $\mathcal{B}' = \mathcal{D}'_{L^\infty}$, the space of bounded distributions. For $q = \infty$, one replaces $\mathcal{D}_{L^\infty}$ by its closed subspace

$$\dot{\mathcal{B}} = \{\varphi \in \mathcal{D} : \lim_{|x| \to \infty} \varphi^{(\alpha)}(x) = 0, \ \forall \alpha \in \mathbb{N}^n\}$$

and defines the space of integrable distributions $\mathcal{D}'_{L^1} = (\dot{\mathcal{B}})'$.
These spaces are crucial in classical distribution theory. Some examples:

- For convolution (Schwartz, Ortner-Wagner, ...).
- For boundary values (Tillmann, Carmichael, ...).
- $\mathcal{B}'$ and $\mathcal{D}_{L^1}$ for Tauberian theory (Beurling, Pilipović-Stanković, ...).

Our goal: to single out properties of a Banach space X preserving the richness of $\mathcal{D}_X$ and $\mathcal{D}'_{X'}$.

Motivation: Some classical distribution spaces

Schwartz introduced spaces of distributions based on L^p spaces as follows. Let $X = L^q$ with $q \in [1, \infty]$ and $X' = L^p$. Set

$$\mathcal{D}_{L^q} = \mathcal{D}_X := \{\varphi \in \mathcal{D} : \ \varphi^{(\alpha)} \in X, \ \forall \alpha \in \mathbb{N}^n\}$$

and

$$\mathcal{D}'_{L^p} = \mathcal{D}'_{X'} := (\mathcal{D}_X)'.$$

This works if $q < \infty$ and he denoted also $\mathcal{B}' = \mathcal{D}'_{L^\infty}$, the space of bounded distributions. For $q = \infty$, one replaces $\mathcal{D}_{L^\infty}$ by its closed subspace

$$\dot{\mathcal{B}} = \{\varphi \in \mathcal{D} : \lim_{|x| \to \infty} \varphi^{(\alpha)}(x) = 0, \ \forall \alpha \in \mathbb{N}^n\}$$

and defines the space of integrable distributions $\mathcal{D}'_{L^1} = (\dot{\mathcal{B}})'$.
These spaces are crucial in classical distribution theory. Some examples:

- For convolution (Schwartz, Ortner-Wagner, ...).
- For boundary values (Tillmann, Carmichael, ...).
- $\mathcal{B}'$ and $\mathcal{D}_{L^1}$ for Tauberian theory (Beurling, Pilipović-Stanković, ...).

Our goal: to single out properties of a Banach space X preserving the richness of $\mathcal{D}_X$ and $\mathcal{D}'_{X'}$.

Weight sequences

Let $(M_p)_p$ be a sequence of positive numbers satisfying:

$(M.1)$ (Logarithmic convexity)
$M_p^2 \leq M_{p-1}M_{p+1}$ for $p \in \mathbb{N}$,

$(M.2)$ (Stability under ultradifferential operators)
$\exists A > 0, H > 0,\ M_p \leq AH^p \min_{0\leq q\leq p} M_q M_{p-q}$ for $p \in \mathbb{N}$,

$(M.3)$ (Strong non-quasi-analyticity)

$$\exists A > 0,\ \sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} \leq Aq \frac{M_q}{M_{q+1}} \text{ for } q > 0.$$

Its associated function M is defined as:

$$M(t) = \sup_{p\in\mathbb{N}} \log_+ \frac{M_0 t^p}{M_p}, \quad t \in [0, \infty).$$

We also write $M_\alpha := M_{|\alpha|}$ and $M(x) := M(|x|)$ for $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$.

Weight sequences

Let $(M_p)_p$ be a sequence of positive numbers satisfying:

$(M.1)$ (Logarithmic convexity)
$M_p^2 \leq M_{p-1}M_{p+1}$ for $p \in \mathbb{N}$,

$(M.2)$ (Stability under ultradifferential operators)
$\exists A > 0, H > 0,\ M_p \leq AH^p \min\limits_{0\leq q\leq p} M_q M_{p-q}$ for $p \in \mathbb{N}$,

$(M.3)$ (Strong non-quasi-analyticity)

$$\exists A > 0,\ \sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} \leq Aq\frac{M_q}{M_{q+1}} \text{ for } q > 0.$$

Its associated function M is defined as:

$$M(t) = \sup_{p\in\mathbb{N}} \log_+ \frac{M_0 t^p}{M_p},\quad t \in [0, \infty).$$

We also write $M_\alpha := M_{|\alpha|}$ and $M(x) := M(|x|)$ for $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$.

The basic spaces of ultradistributions

We consider the standard spaces of test functions (over $\mathbb{R}^n$)

$$\mathcal{D}^*, \quad \mathcal{E}^*, \quad \text{and} \quad \mathcal{S}^*$$

and ultradistributions

$$\mathcal{D}'^*, \quad \mathcal{E}'^*, \quad \text{and} \quad \mathcal{S}'^*,$$

where we use the convention $* = \varnothing$, (M_p), or $\{M_p\}$. More concretely,

- $\varnothing$ stands for the Schwartz case (C^∞ case).
- (M_p) stands for the Beurling case.
- $\{M_p\}$ stands for the Roumieu case.

$P(D) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha D^\alpha$ is an ultradifferential operator of class $*$ if:

- $* = \varnothing$: $P(D)$ is a usual differential operator of finite order.
- $* = (M_p)$ ($\{M_p\}$): the coefficients satisfy the estimate

$$|a_\alpha| \leq C \frac{L^{|\alpha|}}{M_\alpha}, \quad \forall \alpha,$$

for some $L > 0$ and $C > 0$ (for every $L > 0$ and some $C_L > 0$.)

The basic spaces of ultradistributions

We consider the standard spaces of test functions (over $\mathbb{R}^n$)

$$\mathcal{D}^*, \quad \mathcal{E}^*, \quad \text{and} \quad \mathcal{S}^*$$

and ultradistributions

$$\mathcal{D}'^*, \quad \mathcal{E}'^*, \quad \text{and} \quad \mathcal{S}'^*,$$

where we use the convention $* = \varnothing$, (M_p), or $\{M_p\}$. More concretely,

- $\varnothing$ stands for the Schwartz case (C^∞ case).
- (M_p) stands for the Beurling case.
- $\{M_p\}$ stands for the Roumieu case.

$P(D) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha D^\alpha$ is an ultradifferential operator of class $*$ if:

- $* = \varnothing$: $P(D)$ is a usual differential operator of finite order.
- $* = (M_p)$ ($\{M_p\}$): the coefficients satisfy the estimate

$$|a_\alpha| \leq C \frac{L^{|\alpha|}}{M_\alpha}, \quad \forall \alpha,$$

for some $L > 0$ and $C > 0$ (for every $L > 0$ and some $C_L > 0$.)

The basic spaces of ultradistributions

We consider the standard spaces of test functions (over $\mathbb{R}^n$)

$$\mathcal{D}^*, \quad \mathcal{E}^*, \quad \text{and} \quad \mathcal{S}^*$$

and ultradistributions

$$\mathcal{D}'^*, \quad \mathcal{E}'^*, \quad \text{and} \quad \mathcal{S}'^*,$$

where we use the convention $* = \varnothing$, (M_p), or $\{M_p\}$. More concretely,

- $\varnothing$ stands for the Schwartz case (C^∞ case).
- (M_p) stands for the Beurling case.
- $\{M_p\}$ stands for the Roumieu case.

$P(D) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha D^\alpha$ is an ultradifferential operator of class $*$ if:

- $* = \varnothing$: $P(D)$ is a usual differential operator of finite order.
- $* = (M_p)$ ($\{M_p\}$): the coefficients satisfy the estimate

$$| a_\alpha | \leq C \frac{L^{|\alpha|}}{M_\alpha}, \quad \forall \alpha,$$

for some $L > 0$ and $C > 0$ (for every $L > 0$ and some $C_L > 0$.)

TIB spaces of tempered (ultra)distributions

T_h is the translation operator, namely, $T_h f = f(\,\cdot\, + h)$.

Definition

A Banach space E is said to be a translation-invariant Banach space (TIB) of tempered (ultra)distributions of class $*$ if

(I) $\mathcal{D}^* \hookrightarrow E \hookrightarrow \mathcal{D}'^*$.

(II) $T_h(E) \subseteq E$ for each $h \in \mathbb{R}^n$.

(*III*) The function $\omega(h) = \|T_{-h}\|_E$ satisfies the estimates:

- If $* = \emptyset$, there are $C, \tau > 0$

$$\omega(h) \leq C(1 + |h|)^\tau, \quad \forall h \in \mathbb{R}^n.$$

- If $* = (M_p)$ ($\{M_p\}$), there exist $C, \tau > 0$ ($\forall \tau > 0$, $\exists C = C_\tau > 0$) such that

$$\omega(h) \leq C e^{M(\tau|h|)}, \quad \forall h \in \mathbb{R}^n.$$

TIB spaces of tempered (ultra)distributions

T_h is the translation operator, namely, $T_h f = f(\,\cdot\, + h)$.

Definition

A Banach space E is said to be a translation-invariant Banach space (TIB) of tempered (ultra)distributions of class $*$ if

(I) $\mathcal{D}^* \hookrightarrow E \hookrightarrow \mathcal{D}'^*$.

(II) $T_h(E) \subseteq E$ for each $h \in \mathbb{R}^n$.

(*III*) The function $\omega(h) = \|T_{-h}\|_E$ satisfies the estimates:

- If $* = \varnothing$, there are $C, \tau > 0$

$$\omega(h) \leq C(1 + |h|)^{\tau}, \quad \forall h \in \mathbb{R}^n.$$

- If $* = (M_p)$ ($\{M_p\}$), there exist $C, \tau > 0$ ($\forall \tau > 0$, $\exists C = C_\tau > 0$) such that

$$\omega(h) \leq C e^{M(\tau|h|)}, \quad \forall h \in \mathbb{R}^n.$$

Properties of TIB of (ultra)distributions

The function $\omega(h) = \|T_{-h}\|_{L(E)}$ will be called the weight function of E. The associated Beurling algebra of E is the convolution algebra $L^1_\omega = \{u : \|u\|_{L^1_\omega} := \|u\omega\|_{L^1} < \infty\}$.

Theorem

We have the following properties:

(a) $\mathcal{S}^* \hookrightarrow E \hookrightarrow \mathcal{S}'^*$.

(b) $\lim_{h\to 0} \|T_h g - g\|_E = 0, \forall g \in E$.

(c) The convolution mapping $* : \mathcal{S}^* \times \mathcal{S}^* \to \mathcal{S}^*$ extends to $* : L^1_\omega \times E \to E$ and E becomes a Banach module over the Beurling algebra L^1_ω, i.e.

$$\|u * g\|_E \leq \|u\|_{L^1_\omega} \|g\|_E.$$

(d) For $\varphi \in \mathcal{S}^*$ with $\int_{\mathbb{R}^n} \varphi(x)dx = 1$, set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$.

$$\lim_{\varepsilon\to 0^+} \|g - \varphi_\varepsilon * g\|_E = 0, \quad \forall g \in E.$$

Properties of TIB of (ultra)distributions

The function $\omega(h) = \|T_{-h}\|_{L(E)}$ will be called the weight function of E. The associated Beurling algebra of E is the convolution algebra $L^1_\omega = \{u : \|u\|_{L^1_\omega} := \|u\omega\|_{L^1} < \infty\}$.

Theorem

We have the following properties:

(a) $\mathcal{S}^* \hookrightarrow E \hookrightarrow \mathcal{S}'^*$.

(b) $\lim_{h\to 0} \|T_h g - g\|_E = 0$, $\forall g \in E$.

(c) The convolution mapping $* : \mathcal{S}^* \times \mathcal{S}^* \to \mathcal{S}^*$ extends to $* : L^1_\omega \times E \to E$ and E becomes a Banach module over the Beurling algebra L^1_ω, i.e.

$$\|u * g\|_E \leq \|u\|_{L^1_\omega} \|g\|_E.$$

(d) For $\varphi \in \mathcal{S}^*$ with $\int_{\mathbb{R}^n} \varphi(x)dx = 1$, set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$.

$$\lim_{\varepsilon\to 0^+} \|g - \varphi_\varepsilon * g\|_E = 0, \quad \forall g \in E.$$

Properties of TIB of (ultra)distributions

The function $\omega(h) = \|T_{-h}\|_{L(E)}$ will be called the weight function of E. The associated Beurling algebra of E is the convolution algebra $L^1_\omega = \{u : \|u\|_{L^1_\omega} := \|u\omega\|_{L^1} < \infty\}$.

Theorem

We have the following properties:

(a) $\mathcal{S}^* \hookrightarrow E \hookrightarrow \mathcal{S}'^*$.

(b) $\lim_{h\to 0} \|T_h g - g\|_E = 0$, $\forall g \in E$.

(c) The convolution mapping $* : \mathcal{S}^* \times \mathcal{S}^* \to \mathcal{S}^*$ extends to $* : L^1_\omega \times E \to E$ and E becomes a Banach module over the Beurling algebra L^1_ω, i.e.

$$\|u * g\|_E \leq \|u\|_{L^1_\omega} \|g\|_E.$$

(d) For $\varphi \in \mathcal{S}^*$ with $\int_{\mathbb{R}^n} \varphi(x)dx = 1$, set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$.

$$\lim_{\varepsilon\to 0^+} \|g - \varphi_\varepsilon * g\|_E = 0, \quad \forall g \in E.$$

Properties of TIB of (ultra)distributions

The function $\omega(h) = \|T_{-h}\|_{L(E)}$ will be called the weight function of E. The associated Beurling algebra of E is the convolution algebra $L^1_\omega = \{u : \|u\|_{L^1_\omega} := \|u\omega\|_{L^1} < \infty\}$.

Theorem

We have the following properties:

(a) $\mathcal{S}^* \hookrightarrow E \hookrightarrow \mathcal{S}'^*$.

(b) $\lim_{h\to 0} \|T_h g - g\|_E = 0, \forall g \in E$.

(c) The convolution mapping $* : \mathcal{S}^* \times \mathcal{S}^* \to \mathcal{S}^*$ extends to $* : L^1_\omega \times E \to E$ and E becomes a Banach module over the Beurling algebra L^1_ω, i.e.

$$\|u * g\|_E \leq \|u\|_{L^1_\omega} \|g\|_E.$$

(d) For $\varphi \in \mathcal{S}^*$ with $\int_{\mathbb{R}^n} \varphi(x)dx = 1$, set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$.

$$\lim_{\varepsilon\to 0^+} \|g - \varphi_\varepsilon * g\|_E = 0, \quad \forall g \in E.$$

Properties of TIB of (ultra)distributions

The function $\omega(h) = \|T_{-h}\|_{L(E)}$ will be called the weight function of E. The associated Beurling algebra of E is the convolution algebra $L^1_\omega = \{u : \|u\|_{L^1_\omega} := \|u\omega\|_{L^1} < \infty\}$.

Theorem

We have the following properties:

(a) $\mathcal{S}^* \hookrightarrow E \hookrightarrow \mathcal{S}'^*$.

(b) $\lim_{h\to 0} \|T_h g - g\|_E = 0, \forall g \in E$.

(c) The convolution mapping $* : \mathcal{S}^* \times \mathcal{S}^* \to \mathcal{S}^*$ extends to $* : L^1_\omega \times E \to E$ and E becomes a Banach module over the Beurling algebra L^1_ω, i.e.

$$\|u * g\|_E \leq \|u\|_{L^1_\omega} \|g\|_E.$$

(d) For $\varphi \in \mathcal{S}^*$ with $\int_{\mathbb{R}^n} \varphi(x)dx = 1$, set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$.

$$\lim_{\varepsilon\to 0^+} \|g - \varphi_\varepsilon * g\|_E = 0, \quad \forall g \in E.$$

The dual of a TIB of tempered (ultra)distributions

The weight function of E' is $\check{\omega}(x) := \omega(-x)$, it has Beurling algebra $L^1_{\check{\omega}}$.

The dual space $E' \hookrightarrow \mathcal{S}'^*$ carries two convolution structures.

- $* : L^1_{\check{\omega}} \times E' \to E'$, so that E' is a Banach module over $L^1_{\check{\omega}}$.
- $* : E' \times \check{E} \to L^\infty_\omega$ (where $L^\infty_\omega = (L^1_\omega)'$).

When E is reflexive, E' is also a TIB of tempered (ultra)distribution.

Otherwise, E' is not a TIB of tempered (ultra)distribution, in general. The following properties may fail:

- S^* may not be dense in E'.
- The translation operators may not be a C_0-group.
- E' may not have 'nice' approximative units for convolution structure on $L^1_{\check{\omega}}$.

The dual of a TIB of tempered (ultra)distributions

The weight function of E' is $\check{\omega}(x) := \omega(-x)$, it has Beurling algebra $L^1_{\check{\omega}}$.

The dual space $E' \hookrightarrow \mathcal{S}'^*$ carries two convolution structures.

- $* : L^1_{\check{\omega}} \times E' \to E'$, so that E' is a Banach module over $L^1_{\check{\omega}}$.
- $* : E' \times \check{E} \to L^\infty_\omega$ (where $L^\infty_\omega = (L^1_\omega)'$).

When E is reflexive, E' is also a TIB of tempered (ultra)distribution.

Otherwise, E' is not a TIB of tempered (ultra)distribution, in general. The following properties may fail:

- S^* may not be dense in E'.
- The translation operators may not be a C_0-group.
- E' may not have ‘nice’ approximative units for convolution structure on $L^1_{\check{\omega}}$.

The dual of a TIB of tempered (ultra)distributions

The weight function of E' is $\check{\omega}(x) := \omega(-x)$, it has Beurling algebra $L^1_{\check{\omega}}$.

The dual space $E' \hookrightarrow \mathcal{S}'^*$ carries two convolution structures.

- $* : L^1_{\check{\omega}} \times E' \to E'$, so that E' is a Banach module over $L^1_{\check{\omega}}$.
- $* : E' \times \check{E} \to L^\infty_\omega$ (where $L^\infty_\omega = (L^1_\omega)'$).

When E is reflexive, E' is also a TIB of tempered (ultra)distribution.

Otherwise, E' is not a TIB of tempered (ultra)distribution, in general. The following properties may fail:

- S^* may not be dense in E'.
- The translation operators may not be a C_0-group.
- E' may not have 'nice' approximative units for convolution structure on $L^1_{\check{\omega}}$.

The dual of a TIB of tempered (ultra)distributions

The weight function of E' is $\check{\omega}(x) := \omega(-x)$, it has Beurling algebra $L^1_{\check{\omega}}$.

The dual space $E' \hookrightarrow \mathcal{S}'^*$ carries two convolution structures.

- $* : L^1_{\check{\omega}} \times E' \to E'$, so that E' is a Banach module over $L^1_{\check{\omega}}$.
- $* : E' \times \check{E} \to L^\infty_\omega$ (where $L^\infty_\omega = (L^1_\omega)'$).

When E is reflexive, E' is also a TIB of tempered (ultra)distribution.

Otherwise, E' is not a TIB of tempered (ultra)distribution, in general. The following properties may fail:

- S^* may not be dense in E'.
- The translation operators may not be a C_0-group.
- E' may not have 'nice' approximative units for convolution structure on $L^1_{\check{\omega}}$.

The Banach space E'_*

Definition

The Banach space E'_* is defined as $E'_* = L^1_{\check{\omega}} * E'$.

That E'_* is a Banach space follows from the Cohen-Hewitt factorization theorem.

Properties of E'_*

(*i*) E'_* inheres the two convolution structures from E'.

(*ii*) $E'_* = \left\{ f \in E' \middle| \lim_{h \to 0} \|T_h f - f\|_{E'} = 0 \right\}$. In particular, its translation group is a C_0-group.

(*iii*) E'_* has approximative units from S^*,

$$\lim_{\varepsilon \to 0^+} \|f - \varphi_\varepsilon * f\|'_E = 0, \quad \forall f \in E'_*.$$

(*iv*) If E is reflexive, then $E'_* = E'$.

The Banach space E'_*

Definition

The Banach space E'_* is defined as $E'_* = L^1_{\check{\omega}} * E'$.

That E'_* is a Banach space follows from the Cohen-Hewitt factorization theorem.

Properties of E'_*

(*i*) E'_* inheres the two convolution structures from E'.

(*ii*) $E'_* = \left\{ f \in E' \middle| \lim_{h \to 0} \|T_h f - f\|_{E'} = 0 \right\}$. In particular, its translation group is a C_0-group.

(*iii*) E'_* has approximative units from S^*,

$$\lim_{\varepsilon \to 0^+} \|f - \varphi_\varepsilon * f\|'_E = 0, \quad \forall f \in E'_*.$$

(*iv*) If E is reflexive, then $E'_* = E'$.

Examples of E and E'_*

η is an ultrapolynomially bounded weight function of class $*$ if

- $* = \varnothing$: $\dfrac{\eta(x+h)}{\eta(x)} \leq C(1+|h|)^{\tau}$.
- $* = (M_p)$ $(\{M_p\})$: $\dfrac{\eta(x+h)}{\eta(x)} \leq Ce^{M(\tau|h|)}$ for some $C, \tau > 0$ (for every $\tau > 0$ there exists $C > 0$).

Consider $L^p_\eta = \{f : \|f\|_{L^p_\eta} := \|f\eta\|_{L^p} < \infty\}$ for $p \in [1,\infty)$ and $L^\infty_\eta = \{f : \|f\|_{L^\infty_\eta} := \|f/\eta\|_{L^\infty} < \infty\}$ for $p = \infty$.

- $E = L^q_{\eta^{-1}} \Rightarrow E'_* = E' = L^p_\eta$ if $1 < q < \infty$.
- $E = L^1_\eta \Rightarrow E'_* = UC_\eta := \left\{ u \in L^\infty_\eta : \lim_{h\to 0} \|T_h u - u\|_{L^\infty_\eta} = 0 \right\}$.
- $E = C_\eta := \{\varphi \in C : \lim_{|x|\to\infty} \dfrac{\varphi(x)}{\eta(x)} = 0\} \Rightarrow E'_* = L^1_\eta$.

Examples of E and E'_*

η is an ultrapolynomially bounded weight function of class $*$ if

- $* = \varnothing$: $\dfrac{\eta(x+h)}{\eta(x)} \leq C(1+|h|)^{\tau}$.
- $* = (M_p)$ ($\{M_p\}$): $\dfrac{\eta(x+h)}{\eta(x)} \leq Ce^{M(\tau|h|)}$ for some $C, \tau > 0$ (for every $\tau > 0$ there exists $C > 0$).

Consider $L^p_\eta = \{f : \|f\|_{L^p_\eta} := \|f\eta\|_{L^p} < \infty\}$ for $p \in [1,\infty)$ and $L^\infty_\eta = \{f : \|f\|_{L^\infty_\eta} := \|f/\eta\|_{L^\infty} < \infty\}$ for $p = \infty$.

- $E = L^q_{\eta^{-1}} \Rightarrow E'_* = E' = L^p_\eta$ if $1 < q < \infty$.
- $E = L^1_\eta \Rightarrow E'_* = UC_\eta := \left\{ u \in L^\infty_\eta : \lim\limits_{h \to 0} \|T_h u - u\|_{L^\infty_\eta} = 0 \right\}$.
- $E = C_\eta := \{\varphi \in C : \lim\limits_{|x| \to \infty} \dfrac{\varphi(x)}{\eta(x)} = 0\} \Rightarrow E'_* = L^1_\eta$.

Examples of E and E'_*

η is an ultrapolynomially bounded weight function of class $*$ if

- $* = \varnothing$: $\dfrac{\eta(x+h)}{\eta(x)} \leq C(1+|h|)^{\tau}$.
- $* = (M_p)$ $(\{M_p\})$: $\dfrac{\eta(x+h)}{\eta(x)} \leq Ce^{M(\tau|h|)}$ for some $C, \tau > 0$ (for every $\tau > 0$ there exists $C > 0$).

Consider $L^p_\eta = \{f : \|f\|_{L^p_\eta} := \|f\eta\|_{L^p} < \infty\}$ for $p \in [1, \infty)$ and $L^\infty_\eta = \{f : \|f\|_{L^\infty_\eta} := \|f/\eta\|_{L^\infty} < \infty\}$ for $p = \infty$.

- $E = L^q_{\eta^{-1}} \Rightarrow E'_* = E' = L^p_\eta$ if $1 < q < \infty$.
- $E = L^1_\eta \Rightarrow E'_* = UC_\eta := \left\{ u \in L^\infty_\eta : \lim_{h\to 0} ||T_h u - u||_{L^\infty_\eta} = 0 \right\}$.
- $E = C_\eta := \{\varphi \in C : \lim_{|x|\to\infty} \dfrac{\varphi(x)}{\eta(x)} = 0\} \Rightarrow E'_* = L^1_\eta$.

The test function space $\mathcal{D}^*_E$

In the distribution case, we define

$$\mathcal{D}_E = \left\{ \varphi \in \mathcal{D}' : \varphi^{(\alpha)} \in E, \ \forall \alpha \right\}.$$

In the ultradistribution case, we set

$$\mathcal{D}_E^{(M_p)} = \varprojlim_{m\to\infty} \mathcal{D}_E^{\{M_p\},m} \quad \text{and} \quad \mathcal{D}_E^{\{M_p\}} = \varinjlim_{m\to 0} \mathcal{D}_E^{\{M_p\},m}, \quad \textit{where},$$

$$\mathcal{D}_E^{\{M_p\},m} = \{ \varphi \in \mathcal{D}'^* : \varphi^{(\alpha)} \in E, \forall \alpha, \text{ and } \sup_\alpha \frac{m^{|\alpha|} \|\varphi^{(\alpha)}\|_E}{M_\alpha} < \infty \}.$$

Notation: We also consider $\mathcal{B}^*_\eta := \mathcal{D}^*_{L^\infty_\eta}$. We use the special notation $\dot{\mathcal{B}}^*_\eta = \mathcal{D}^*_{C_\eta}$.

The test function space $\mathcal{D}^*_E$

In the distribution case, we define

$$\mathcal{D}_E = \left\{ \varphi \in \mathcal{D}' : \varphi^{(\alpha)} \in E, \ \forall \alpha \right\}.$$

In the ultradistribution case, we set

$$\mathcal{D}_E^{(M_p)} = \varprojlim_{m\to\infty} \mathcal{D}_E^{\{M_p\},m} \quad \text{and} \quad \mathcal{D}_E^{\{M_p\}} = \varinjlim_{m\to 0} \mathcal{D}_E^{\{M_p\},m}, \quad \textit{where,}$$

$$\mathcal{D}_E^{\{M_p\},m} = \{\varphi \in \mathcal{D}'^* : \varphi^{(\alpha)} \in E, \forall \alpha, \text{ and } \sup_{\alpha} \frac{m^{|\alpha|} \|\varphi^{(\alpha)}\|_E}{M_\alpha} < \infty\}.$$

Notation: We also consider $\mathcal{B}^*_\eta := \mathcal{D}^*_{L^\infty_\eta}$. We use the special notation $\dot{\mathcal{B}}^*_\eta = \mathcal{D}^*_{C_\eta}$.

The test function space $\mathcal{D}_E^*$

In the distribution case, we define

$$\mathcal{D}_E = \left\{ \varphi \in \mathcal{D}' : \varphi^{(\alpha)} \in E, \ \forall \alpha \right\}.$$

In the ultradistribution case, we set

$$\mathcal{D}_E^{(M_p)} = \varprojlim_{m \to \infty} \mathcal{D}_E^{\{M_p\},m} \quad \text{and} \quad \mathcal{D}_E^{\{M_p\}} = \varinjlim_{m \to 0} \mathcal{D}_E^{\{M_p\},m}, \quad \textit{where},$$

$$\mathcal{D}_E^{\{M_p\},m} = \{ \varphi \in \mathcal{D}'^* : \varphi^{(\alpha)} \in E, \forall \alpha, \text{ and } \sup_{\alpha} \frac{m^{|\alpha|} \|\varphi^{(\alpha)}\|_E}{M_\alpha} < \infty \}.$$

Notation: We also consider $\mathcal{B}_\eta^* := \mathcal{D}_{L_\eta^\infty}^*$. We use the special notation $\dot{\mathcal{B}}_\eta^* = \mathcal{D}_{C_\eta}^*$.

Properties of $\mathcal{D}^*_E$

- $\mathcal{D}^*_E$ is a Fréchet space for $* = \varnothing, (M_p)$. It is a regular and complete inductive limit for $* = \{M_p\}$.
- It is a topological convolution module over L^1_ω.
- $\mathcal{S}^* \hookrightarrow \mathcal{D}^*_E \hookrightarrow E \hookrightarrow \mathcal{S}'^*$.
- Every element of $\mathcal{D}^*_E$ is a smooth function of class $*$, furthermore, $\mathcal{D}^*_E \hookrightarrow \mathcal{O}^*_C \hookrightarrow \mathcal{E}^*$ and (yet sharper)

$$\mathcal{D}^*_{L^1_\omega} \hookrightarrow \mathcal{D}^*_E \hookrightarrow \dot{\mathcal{B}}^*_{\check{\omega}}.$$

Alternative description for $\{M_p\}$. Let $\mathfrak{R}$ be the set of $(r_p)_p$ increasing to ∞. Set

$$\mathcal{D}_E^{\{M_p\},(r_p)} = \{\varphi \in E : \ D^\alpha\varphi \in E, \forall\alpha, \ \text{and} \ \sup_\alpha \frac{\|D^\alpha\varphi\|_E}{M_\alpha \prod_{j=1}^{|\alpha|} r_j} < \infty\}.$$

Theorem

$$\mathcal{D}_E^{\{M_p\}} = \varinjlim_{(r_p)\in\mathfrak{R}} \mathcal{D}_E^{\{M_p\},(r_p)} \quad \text{as t.v.s..}$$

Properties of $\mathcal{D}^*_E$

- $\mathcal{D}^*_E$ is a Fréchet space for $* = \varnothing, (M_p)$. It is a regular and complete inductive limit for $* = \{M_p\}$.
- It is a topological convolution module over L^1_ω.
- $\mathcal{S}^* \hookrightarrow \mathcal{D}^*_E \hookrightarrow E \hookrightarrow \mathcal{S}'^*$.
- Every element of $\mathcal{D}^*_E$ is a smooth function of class $*$, furthermore, $\mathcal{D}^*_E \hookrightarrow \mathcal{O}^*_C \hookrightarrow \mathcal{E}^*$ and (yet sharper)

$$\mathcal{D}^*_{L^1_\omega} \hookrightarrow \mathcal{D}^*_E \hookrightarrow \dot{\mathcal{B}}^*_{\check{\omega}}.$$

Alternative description for $\{M_p\}$. Let $\mathfrak{R}$ be the set of $(r_p)_p$ increasing to ∞. Set

$$\mathcal{D}_E^{\{M_p\},(r_p)} = \{\varphi \in E : \ D^\alpha\varphi \in E, \forall\alpha, \text{ and } \sup_\alpha \frac{\|D^\alpha\varphi\|_E}{M_\alpha \prod_{j=1}^{|\alpha|} r_j} < \infty\}.$$

Theorem

$$\mathcal{D}_E^{\{M_p\}} = \varprojlim_{(r_p)\in\mathfrak{R}} \mathcal{D}_E^{\{M_p\},(r_p)} \quad \text{as t.v.s..}$$

Properties of $\mathcal{D}^*_E$

- $\mathcal{D}^*_E$ is a Fréchet space for $* = \varnothing, (M_p)$. It is a regular and complete inductive limit for $* = \{M_p\}$.
- It is a topological convolution module over L^1_ω.
- $\mathcal{S}^* \hookrightarrow \mathcal{D}^*_E \hookrightarrow E \hookrightarrow \mathcal{S}'^*$.
- Every element of $\mathcal{D}^*_E$ is a smooth function of class $*$, furthermore, $\mathcal{D}^*_E \hookrightarrow \mathcal{O}^*_C \hookrightarrow \mathcal{E}^*$ and (yet sharper)

$$\mathcal{D}^*_{L^1_\omega} \hookrightarrow \mathcal{D}^*_E \hookrightarrow \dot{\mathcal{B}}^*_{\check{\omega}}.$$

Alternative description for $\{M_p\}$. Let $\mathfrak{R}$ be the set of $(r_p)_p$ increasing to ∞. Set

$$\mathcal{D}_E^{\{M_p\},(r_p)} = \{\varphi \in E : D^\alpha\varphi \in E, \forall\alpha, \text{ and } \sup_\alpha \frac{\|D^\alpha\varphi\|_E}{M_\alpha \prod_{j=1}^{|\alpha|} r_j} < \infty\}.$$

Theorem

$$\mathcal{D}_E^{\{M_p\}} = \varprojlim_{(r_p)\in\mathfrak{R}} \mathcal{D}_E^{\{M_p\},(r_p)} \quad \text{as t.v.s..}$$

Properties of $\mathcal{D}^*_E$

- $\mathcal{D}^*_E$ is a Fréchet space for $* = \varnothing, (M_p)$. It is a regular and complete inductive limit for $* = \{M_p\}$.
- It is a topological convolution module over L^1_ω.
- $\mathcal{S}^* \hookrightarrow \mathcal{D}^*_E \hookrightarrow E \hookrightarrow \mathcal{S}'^*$.
- Every element of $\mathcal{D}^*_E$ is a smooth function of class $*$, furthermore, $\mathcal{D}^*_E \hookrightarrow \mathcal{O}^*_C \hookrightarrow \mathcal{E}^*$ and (yet sharper)

$$\mathcal{D}^*_{L^1_\omega} \hookrightarrow \mathcal{D}^*_E \hookrightarrow \dot{\mathcal{B}}^*_{\check{\omega}}.$$

Alternative description for $\{M_p\}$. Let $\mathfrak{R}$ be the set of $(r_p)_p$ increasing to ∞. Set
$\mathcal{D}^{\{M_p\},(r_p)}_E = \{\varphi \in E : D^\alpha\varphi \in E, \forall\alpha, \text{ and } \sup_\alpha \frac{\|D^\alpha\varphi\|_E}{M_\alpha \prod_{j=1}^{|\alpha|} r_j} < \infty\}.$

Theorem

$$\mathcal{D}^{\{M_p\}}_E = \varinjlim_{(r_p)\in\mathfrak{R}} \mathcal{D}^{\{M_p\},(r_p)}_E \quad \text{as t.v.s..}$$

Properties of $\mathcal{D}^*_E$

- $\mathcal{D}^*_E$ is a Fréchet space for $* = \varnothing, (M_p)$. It is a regular and complete inductive limit for $* = \{M_p\}$.
- It is a topological convolution module over L^1_ω.
- $\mathcal{S}^* \hookrightarrow \mathcal{D}^*_E \hookrightarrow E \hookrightarrow \mathcal{S}'^*$.
- Every element of $\mathcal{D}^*_E$ is a smooth function of class $*$, furthermore, $\mathcal{D}^*_E \hookrightarrow \mathcal{O}^*_C \hookrightarrow \mathcal{E}^*$ and (yet sharper)

$$\mathcal{D}^*_{L^1_\omega} \hookrightarrow \mathcal{D}^*_E \hookrightarrow \dot{\mathcal{B}}^*_{\check{\omega}}.$$

Alternative description for $\{M_p\}$. Let $\mathfrak{R}$ be the set of $(r_p)_p$ increasing to ∞. Set

$$\mathcal{D}_E^{\{M_p\},(r_p)} = \{\varphi \in E : \ D^\alpha\varphi \in E, \forall \alpha, \ \text{and} \ \sup_\alpha \frac{\|D^\alpha \varphi\|_E}{M_\alpha \prod_{j=1}^{|\alpha|} r_j} < \infty\}.$$

Theorem

$$\mathcal{D}_E^{\{M_p\}} = \varprojlim_{(r_p)\in\mathfrak{R}} \mathcal{D}_E^{\{M_p\},(r_p)} \quad \text{as t.v.s..}$$

The (ultra)distribution space $\mathcal{D}'^{*}_{E'_{*}}$

Definition

The space $\mathcal{D}'^{*}_{E'_{*}}$ is the strong dual of $\mathcal{D}^{*}_{E}$.

Theorem

Let $B \subseteq \mathcal{S}'^{*}$. The following statements are equivalent:

(*i*) B is a bounded subset of $\mathcal{D}'^{*}_{E'_{*}}$.

(*ii*) $\{f * \psi | \, f \in B\}$ is a bounded subset of E' for each $\psi \in \mathcal{S}^{*}$.

(*iii*) $\{f * \psi | \, f \in B\}$ is a bounded subset of E'_{*} for each $\psi \in \mathcal{S}^{*}$.

(*iv*) $B = P(D)B_1$ for some bounded subset B_1 of E' and an ultradifferential operator $P(D)$ of class $*$.

(*v*) $B = P(D)B_2$ for some $B_2 \subseteq E'_{*} \cap UC_{\omega}$ which is simultaneously bounded in E'_{*} and in UC_{ω} and an ultradifferential operator $P(D)$ of class $*$. Moreover, if E is reflexive, we may choose $B_2 \subseteq E'_{*} \cap C_{\omega}$.

The following theorem provides a Shiraishi type characterization of the convolution of Roumieu ultradistributions

Theorem

Let $f, g \in \mathcal{D}'^{\{M_p\}}$. Then the following statements are equivalent:

(*i*) The convolution of f and g exists.

(*ii*) $(\varphi * \check{f})g \in \mathcal{D}'^{\{M_p\}}_{L^1}$ for all $\varphi \in \mathcal{D}^{\{M_p\}}$.

(*iii*) $(\varphi * \check{g})f \in \mathcal{D}'^{\{M_p\}}_{L^1}$ for all $\varphi \in \mathcal{D}^{\{M_p\}}$.

(*iv*) $(\varphi * \check{f})(\psi * g) \in L^1$ for all $\varphi, \psi \in \mathcal{D}^{\{M_p\}}$.

Boundary values

- C open convex cone and $C(r) = C \cap B(0, r)$.
- $d_C(y) = \text{dist}(y, \partial C)$.
- $T^{C(r)} = \mathbb{R}^n + iC(r)$.

Problem: Characterize those holomorphic functions F on $T^{C(r)}$ such that F has boundary values in $\mathcal{D}'_{E'_*}$, i.e., $\exists f \in \mathcal{D}'_{E'_*}$ such that

$$f = \lim_{\substack{y \to 0 \\ y \in C}} F(\,\cdot\, + iy), \quad \text{strongly in } \mathcal{D}'_{E'_*}. \tag{1}$$

Theorem

F has boundary values in $\mathcal{D}'_{E'_*}$ if and only if $\exists r' < r$ such that

(a) $F(\,\cdot\, + iy) \in E'$, $\forall y \in C(r')$.

(b) There are $\kappa, M > 0$

$$\|F(\,\cdot\, + iy)\|_{E'} \leq \frac{M}{(d_C(y))^{\kappa}}, \quad |y| < r'.$$

Boundary values

- C open convex cone and $C(r) = C \cap B(0, r)$.
- $d_C(y) = \mathrm{dist}(y, \partial C)$.
- $T^{C(r)} = \mathbb{R}^n + iC(r)$.

Problem: Characterize those holomorphic functions F on $T^{C(r)}$ such that F has boundary values in $\mathcal{D}'_{E'_*}$, i.e., $\exists f \in \mathcal{D}'_{E'_*}$ such that

$$f = \lim_{\substack{y \to 0 \\ y \in C}} F(\,\cdot\, + iy), \quad \text{strongly in } \mathcal{D}'_{E'_*}. \tag{1}$$

Theorem

F has boundary values in $\mathcal{D}'_{E'_*}$ if and only if $\exists r' < r$ such that

(a) $F(\,\cdot\, + iy) \in E'$, $\forall y \in C(r')$.

(b) There are $\kappa, M > 0$

$$\|F(\,\cdot\, + iy)\|_{E'} \leq \frac{M}{(d_C(y))^{\kappa}}, \quad |y| < r'.$$

Boundary values

- C open convex cone and $C(r) = C \cap B(0,r)$.
- $d_C(y) = \text{dist}(y, \partial C)$.
- $T^{C(r)} = \mathbb{R}^n + iC(r)$.

Problem: Characterize those holomorphic functions F on $T^{C(r)}$ such that F has boundary values in $\mathcal{D}'_{E'_*}$, i.e., $\exists f \in \mathcal{D}'_{E'_*}$ such that

$$f = \lim_{\substack{y \to 0 \\ y \in C}} F(\,\cdot\, + iy), \quad \text{strongly in } \mathcal{D}'_{E'_*}. \tag{1}$$

Theorem

F has boundary values in $\mathcal{D}'_{E'_*}$ if and only if $\exists r' < r$ such that

(a) $F(\,\cdot\, + iy) \in E'$, $\forall y \in C(r')$.

(b) There are $\kappa, M > 0$

$$\|F(\,\cdot\, + iy)\|_{E'} \leq \frac{M}{(d_C(y))^{\kappa}}, \quad |y| < r'.$$

Let $C_1, \ldots, C_m$ be open convex cones of $\mathbb{R}^n$ with $\mathbb{R}^n = \bigcup_{j=1}^m C_j^*$.

Theorem

Every $f \in \mathcal{D}'_{E'_*}$ admits the boundary value representation

$$f = \sum_{j=1}^{m} \lim_{\substack{y \to 0 \\ y \in C_j}} F_j(\,\cdot\, + iy) \tag{2}$$

strongly in $\mathcal{D}'_{E'_*}$, where each F_j is holomorphic in the tube T^{C_j}.

Other results

- "Edge of the wedge" theorems.
- Hyperfunctional represention of $\mathcal{D}'_{E'_*}$.

Analytic representations of $\mathcal{D}'_{E'_*}$

Let $C_1, \ldots, C_m$ be open convex cones of $\mathbb{R}^n$ with $\mathbb{R}^n = \bigcup_{j=1}^m C_j^*$.

Theorem

Every $f \in \mathcal{D}'_{E'_*}$ admits the boundary value representation

$$f = \sum_{j=1}^{m} \lim_{\substack{y \to 0 \\ y \in C_j}} F_j(\,\cdot\, + iy) \tag{2}$$

strongly in $\mathcal{D}'_{E'_*}$, where each F_j is holomorphic in the tube T^{C_j}.

Other results

- "Edge of the wedge" theorems.
- Hyperfunctional represention of $\mathcal{D}'_{E'_*}$.

Quasianalytic case

We have obtained analogous results for the quasianalytic case.

- $(M_p)_p$ and $(A_p)_p$: weight sequences. A : associated function of $(A_p)_p$.
- $\mathcal{S}^*_\dagger$ mixed Gelfand-Shilov type space, where $\dagger = (A_p)$ or $\{A_p\}$.

Definition

E is said to be a TIB of (ultra)distributions of class $* - \dagger$ if

(I) $\mathcal{S}^*_\dagger \hookrightarrow E \hookrightarrow \mathcal{S}'^*_\dagger$

(II) $T_h(E) \subseteq E, \forall h \in \mathbb{R}^n$.

(*III*) If $\dagger = (A_p)$ ($\{A_p\}$), $\exists C, \tau > 0$ ($\forall \tau > 0, \exists C = C_\tau > 0$) such that

$$\omega(h) \leq Ce^{A(\tau|h|)}, \quad \forall h \in \mathbb{R}^n.$$

Assumptions on the weight sequences:

- $(A_p)_p$ satisfies $(M.1)$, $(M.2)$ and $p! \subset A_p \quad (A(x) = O(e^{|x|}))$.
- $(M_p)_p$ satisfies $(M.1)$, $(M.2)$ and there is $s > 0$ such that

$$\sum_{q=p+1}^{\infty} M^s_{q-1}/M^s_q \leq c_0 p M^s_p/M^s_{p+1}, \ \forall p \in \mathbb{Z}_+.$$

Quasianalytic case

We have obtained analogous results for the quasianalytic case.

- $(M_p)_p$ and $(A_p)_p$: weight sequences. A : associated function of $(A_p)_p$.
- $\mathcal{S}^*_\dagger$ mixed Gelfand-Shilov type space, where $\dagger = (A_p)$ or $\{A_p\}$.

Definition

E is said to be a TIB of (ultra)distributions of class $* - \dagger$ if

(I) $\mathcal{S}^*_\dagger \hookrightarrow E \hookrightarrow \mathcal{S}'^*_\dagger$

(II) $T_h(E) \subseteq E, \forall h \in \mathbb{R}^n$.

(III) If $\dagger = (A_p)$ ($\{A_p\}$), $\exists C, \tau > 0$ ($\forall \tau > 0, \exists C = C_\tau > 0$) such that

$$\omega(h) \leq Ce^{A(\tau|h|)}, \quad \forall h \in \mathbb{R}^n.$$

Assumptions on the weight sequences:

- $(A_p)_p$ satisfies $(M.1)$, $(M.2)$ and $p! \subset A_p \quad (A(x) = O(e^{|x|}))$.
- $(M_p)_p$ satisfies $(M.1)$, $(M.2)$ and there is $s > 0$ such that

$$\sum_{q=p+1}^{\infty} M^s_{q-1}/M^s_q \leq c_0 p M^s_p/M^s_{p+1}, \ \forall p \in \mathbb{Z}_+.$$

Quasianalytic case

We have obtained analogous results for the quasianalytic case.

- $(M_p)_p$ and $(A_p)_p$: weight sequences. A : associated function of $(A_p)_p$.
- $\mathcal{S}^*_\dagger$ mixed Gelfand-Shilov type space, where $\dagger = (A_p)$ or $\{A_p\}$.

Definition

E is said to be a TIB of (ultra)distributions of class $* - \dagger$ if

(I) $\mathcal{S}^*_\dagger \hookrightarrow E \hookrightarrow \mathcal{S}'^*_\dagger$

(II) $T_h(E) \subseteq E, \forall h \in \mathbb{R}^n$.

(*III*) If $\dagger = (A_p)$ ($\{A_p\}$), $\exists C, \tau > 0$ ($\forall \tau > 0$, $\exists C = C_\tau > 0$) such that

$$\omega(h) \leq Ce^{A(\tau|h|)}, \quad \forall h \in \mathbb{R}^n.$$

Assumptions on the weight sequences:

- $(A_p)_p$ satisfies $(M.1)$, $(M.2)$ and $p! \subset A_p \quad (A(x) = O(e^{|x|}))$.
- $(M_p)_p$ satisfies $(M.1)$, $(M.2)$ and there is $s > 0$ such that

$$\sum_{q=p+1}^{\infty} M^s_{q-1}/M^s_q \leq c_0 p M^s_p/M^s_{p+1}, \ \forall p \in \mathbb{Z}_+.$$

Quasianalytic case

We have obtained analogous results for the quasianalytic case.

- $(M_p)_p$ and $(A_p)_p$: weight sequences. A : associated function of $(A_p)_p$.
- $\mathcal{S}^*_\dagger$ mixed Gelfand-Shilov type space, where $\dagger = (A_p)$ or $\{A_p\}$.

Definition

E is said to be a TIB of (ultra)distributions of class $* - \dagger$ if

(I) $\mathcal{S}^*_\dagger \hookrightarrow E \hookrightarrow \mathcal{S}'^*_\dagger$

(II) $T_h(E) \subseteq E$, $\forall h \in \mathbb{R}^n$.

(*III*) If $\dagger = (A_p)$ ($\{A_p\}$), $\exists C, \tau > 0$ ($\forall \tau > 0$, $\exists C = C_\tau > 0$) such that

$$\omega(h) \leq C e^{A(\tau|h|)}, \quad \forall h \in \mathbb{R}^n.$$

Assumptions on the weight sequences:

- $(A_p)_p$ satisfies $(M.1)$, $(M.2)$ and $p! \subset A_p \quad (A(x) = O(e^{|x|}))$.
- $(M_p)_p$ satisfies $(M.1)$, $(M.2)$ and there is $s > 0$ such that

$$\sum_{q=p+1}^{\infty} M^s_{q-1}/M^s_q \leq c_0 p M^s_p / M^s_{p+1}, \ \forall p \in \mathbb{Z}_+.$$

Quasianalytic case

We have obtained analogous results for the quasianalytic case.

- $(M_p)_p$ and $(A_p)_p$: weight sequences. A : associated function of $(A_p)_p$.
- $\mathcal{S}^*_\dagger$ mixed Gelfand-Shilov type space, where $\dagger = (A_p)$ or $\{A_p\}$.

Definition

E is said to be a TIB of (ultra)distributions of class $* - \dagger$ if

(I) $\mathcal{S}^*_\dagger \hookrightarrow E \hookrightarrow \mathcal{S}'^*_\dagger$

(II) $T_h(E) \subseteq E, \forall h \in \mathbb{R}^n$.

(*III*) If $\dagger = (A_p)$ $(\{A_p\})$, $\exists C, \tau > 0$ ($\forall \tau > 0$, $\exists C = C_\tau > 0$) such that

$$\omega(h) \leq C e^{A(\tau |h|)}, \quad \forall h \in \mathbb{R}^n.$$

Assumptions on the weight sequences:

- $(A_p)_p$ satisfies $(M.1)$, $(M.2)$ and $p! \subset A_p \quad (A(x) = O(e^{|x|}))$.
- $(M_p)_p$ satisfies $(M.1)$, $(M.2)$ and there is $s > 0$ such that

$$\textstyle\sum_{q=p+1}^{\infty} M^s_{q-1}/M^s_q \leq c_0 p M^s_p / M^s_{p+1}, \ \forall p \in \mathbb{Z}_+.$$

References

For more details, see:

1. Dimovski, P., Pilipović, S., Prangoski, B., Vindas, J., *Convolution of ultradistributions and ultradistribution spaces associated to translation-invariant Banach spaces,* Kyoto J. Math, to appear.
2. Dimovski, P., Pilipović, S., Vindas, J., *New distribution spaces associated to translation-invariant Banach spaces,* Monatsh. Math. 177 (2015), 495–515.
3. Dimovski, P., Pilipović, S., Vindas, J., *Boundary values of holomorphic functions in translation-invariant distribution spaces,* Complex Var. Elliptic Equ. 60 (2015), 1169–1189.
4. Dimovski, P., Prangoski, B., Vindas, J., *On a class of translation-invariant spaces of quasianalytic ultradistributions,* Novi Sad J. Math. 45 (2015), 143–175 .
5. Pilipović, S., Prangoski, B., Vindas, J., *On quasi-analytic classes of Gelfand-Shilov type. Parametrix and convolution*, preprint (arXiv:1507.08331).