Tauberian class estimates for wavelet and non-wavelet transforms of vector-valued distributions

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In this talk we study vector-valued distributions via integral transforms of the form

$$M_{\varphi}\mathbf{f}(x,y) = (\mathbf{f} * \varphi_y)(x), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+, \tag{1}$$

where

$$\varphi_y(t) = y^{-n}\varphi(t/y).$$

We call such transforms regularizing transforms.

Two important cases can be distinguished:

1. The wavelet case: $\int_{\mathbb{R}^n} \varphi(t)\mathrm{d}t = 0$.
2. The non-wavelet case: $\int_{\mathbb{R}^n} \varphi(t)\mathrm{d}t \neq 0$.

Our aim is:

- To present several precise characterizations of the spaces of distributions with values in Banach spaces in terms of norm size estimates for (1).

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General Notation

- E always denotes a fixed Banach space with norm $\|\cdot\|_E$.
- X stands for a (Hausdorff) locally convex topological vector space.
- $\mathcal{S}'(\mathbb{R}^n, X) = L_b(\mathcal{S}(\mathbb{R}^n), X)$, the space of X-valued tempered distributions.
- $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$, the upper half-space.
- $\hat{\varphi}$ denotes the Fourier transform.

Statement of the problem

Suppose that **f** a priori takes values in the "broad" space X, i.e.,

- $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$.

Suppose that the "narrower" space

- E is continuously embedded in X.

If we know that **f** takes values in E, that is, $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$, then (for some k, l, C):

$$\|M_{\varphi}\mathbf{f}(x,y)\|_{E} \leq C\frac{(1+y)^{k}(1+|x|)^{l}}{y^{k}}, \quad (x,y) \in \mathbb{H}^{n+1}. \qquad (2)$$

We call (2) a (Tauberian) class estimate.

Converse problem: Up to what extend does the class estimate (2) allow one to conclude that **f** actually takes values in E? The problem has a Tauberian character.

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Motivation

The stated problem was first raised and studied by Drozhzhinov and Zav'yalov (2002,2003). It gives a general setting to attack problems such as:

1. Classical Hardy-Littlewood-Karamata type Tauberian theorems for various integral transforms (e.g., the Laplace transform).
2. Stabilization in time for certain Cauchy problems (e.g., for the heat equation).
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Local and global class estimates

We shall consider local and global versions of the Tauberian class estimate:

- Global class estimate:

$$\|M_{\varphi}\mathbf{f}(x,y)\|_{E} \leq C\frac{(1+y)^{k}\,(1+|x|)^{l}}{y^{k}}, \quad \text{for almost all } (x,y)\in\mathbb{H}^{n+1} \tag{GCE}$$

- Local class estimate:

$$\|M_{\varphi}\mathbf{f}(x,y)\|_{E} \leq C\frac{(1+|x|)^{l}}{y^{k}}, \quad \text{for almost all } (x,y)\in\mathbb{R}^{n}\times(0,1]. \tag{LCE}$$

for some $k, l \in \mathbb{N}$ and $C > 0$.

Furthermore, we assume from now on that:

- The Banach space E is continuously embedded in the locally convex space X.

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Non-degenerate test functions

Naturally, not all kernels φ will be well-suited to our problem.
The good ones are:

Definition

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. It is said to be degenerate if there is a ray through the origin along which $\hat{\varphi}$ identically vanishes. In contrary case, the test function it is said to be non-degenerate.

Our Tauberian kernels are the non-degenerate test functions.

- In Wiener Tauberian theory the Tauberian kernels are those φ such that $\hat{\varphi}$ do not vanish at any point.
- In our theory the Tauberian kernels will be those φ such that $\hat{\varphi}$ do not identically vanish on any ray through the origin.

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The non-wavelet case

For the non-wavelet case, we always obtain a full characterization of $\mathcal{S}'(\mathbb{R}^n, E)$.

Theorem

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$ *and let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *be such that* $\int_{\mathbb{R}^n} \varphi(t)\mathrm{d}t \neq 0$. *Then,* $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ *if and only if*

1. $M_\varphi \mathbf{f}(x, y)$ *takes values in* E *for almost all* $(x, y) \in \mathbb{R}^n \times (0, 1]$ *and is measurable as an* E*-valued function on* $\mathbb{R}^n \times (0, 1]$*, and,*
2. *A (LCE)*

$$\|M_\varphi \mathbf{f}(x, y)\|_E \leq C\frac{(1 + |x|)^l}{y^k}$$

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The wavelet case

The analysis of the wavelet case is more complicated. We only obtain characterizations of $\mathcal{S}'(\mathbb{R}^n, E)$ up to a correction term that is totally controlled by the wavelet.

From now on, we assume that φ is a non-degenerate wavelet, namely,

$$\int_{\mathbb{R}^n} \varphi(t)\mathrm{d}t = 0 \quad \text{and } \varphi \text{ is non-degenerate.}$$

Global class estimate

The wavelet case

We begin with global class estimates:

Theorem

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$ *and let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *be a non-degenerate wavelet. The two conditions:*

1. $M_\varphi \mathbf{f}(x, y)$ *takes values in* E *for almost all* $(x, y) \in \mathbb{H}^{n+1}$ *and is measurable as an* E*-valued function on* $\mathbb{H}^{n+1}$*, and,*
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are necessary and sufficient for the existence of $\mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, X)$ *such that* $\mathbf{f} - \mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, E)$ *and* $\operatorname{supp} \hat{\mathbf{G}} \subseteq \{0\}$*.*

Corollary

If X *is a normed space, the function* $\mathbf{G} = \mathbf{P}$*, a polynomial with values in* X*.*

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If X is a normed space, the function $\mathbf{G} = \mathbf{P}$, a polynomial with values in X.

Local class estimates

For local class estimates, the support of the correction term $\hat{\mathbf{G}}$ is not any longer the origin, but it is still controlled by φ. We first need a definition.

Definition

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be non-degenerate. Given $\omega \in \mathbb{S}^{n-1}$, we consider $\hat{\varphi}_\omega(r) := \hat{\varphi}(r\omega)$ as a function of one variable r. We define its index of non-degenerateness as

$$\tau = \inf \left\{ r \in \mathbb{R}_+ : \ \operatorname{supp} \hat{\varphi}_\omega \cap [0, r] \neq \emptyset, \forall \omega \in \mathbb{S}^{n-1} \right\}.$$

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If we replace the (GCE) by merely a (LCE) in the previous theorem, then: for every $r > \tau$, there is an X-valued entire function $\mathbf{G}$ such that

$$\mathbf{f} - \mathbf{G} \in \mathcal{S}'(\mathbb{R}^n, E)$$

and

$$\operatorname{supp} \hat{\mathbf{G}} \subseteq \{t \in \mathbb{R}^n : \ |t| \leq r\} .$$

The result is optimal, namely, in general, $\hat{\mathbf{G}}$ cannot be taken with support in $\{t \in \mathbb{R}^n : \ |t| \leq \tau\}$.

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Strongly non-degenerate wavelets

It is still possible to strengthen the previous result, but one should use the following kind of wavelets:

Definition

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a wavelet. We call φ strongly non-degenerate if there exist constants $N \in \mathbb{N}$, $r > 0$, and $C > 0$ such that

$$C\left|u\right|^N \leq \left|\hat{\varphi}(u)\right|, \quad \text{for all } \left|u\right| \leq r.$$

The above property is equivalent to the following one. There exists N such that P_N, the Taylor polynomial of φ of order N at the origin, satisfies: for any given $\omega \neq 0$, the polynomial of one variable

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Eliminating the correction term
Generalized Littlewood-Paley pairs

It is possible to eliminate the correction term in the the wavelet case, provided that one counts with additional convolution data.

Definition

Let $\theta, \varphi \in \mathcal{S}(\mathbb{R}^n)$. The pair (θ, φ) is said to be a Littlewood-Paley pair (LP-pair) if:

1. φ is non-degenerate with index of non-degenerateness τ.
2. $\hat{\theta}(u) \neq 0$ on the ball $|u| \leq \tau$.

Example. Let $\theta \in \mathcal{S}(\mathbb{R}^n)$ be a radial function such that $\hat{\theta}$ is nonnegative, $\hat{\theta}(u) = 1$ for $|u| < 1/2$ and $\hat{\theta}(u) = 0$ for $|u| > 1$. Set $\hat{\varphi}(u) = -u \cdot \nabla\hat{\theta}(u)$. Then (θ, φ) is a LP-pair.

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2. *A (LCE) is satisfied.*
3. $(\mathbf{f} * \theta)(x)$ *takes values in* E *for almost all* $x \in \mathbb{R}^n$*, it is measurable, and it is of slow growth, i.e.,*

$$||(\mathbf{f} * \theta)(x)||_E \leq C(1 + |x|)^a.$$

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Comments on the (Tauberian) theorems

The theorems we have discussed improve several earlier results of Drozhzhinov and Zav'ylov.

Main improvements:

- Enlargement of the Tauberian kernels. Actually, our class of non-degenerate kernels is the optimal one.
- Our results are valid for general locally convex spaces X (Drozhzhinov and Zav'ylov only considered normed spaces in the multidimensional case).

References

For further results see our preprint (joint with S. Pilipović):

- Multidimensional Tauberian theorems for wavelets and non-wavelet transforms, preprint (arXiv:1012.5090v2).

See also:

- Y. N. Drozhzhinov, B. I. Zav'yalov, Multidimensional Tauberian theorems for Banach-space valued generalized functions, Sb. Math. 194 (2003), 1599–1646.
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