# Tauberian theorems for distributions and applications

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Generalized Solutions of Evolution Equations: Theory, Numerical Approximation, and Applications

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We discuss several Tauberian aspects of a class of integral transforms. To a given (vector-valued) distribution $\mathbf{f}$, we assign a smooth function of two variables $(x, y) \in \mathbb{R}^n \times \mathbb{R}_+$,

$$F_\phi \mathbf{f}(x, y) = \langle \mathbf{f}(x + ty), \phi(t) \rangle = \int_{\mathbb{R}^n} \mathbf{f}(t) \frac{1}{y^n} \phi\left(\frac{t - x}{y}\right) \mathrm{d}t \,, \qquad (1)$$

where the kernel $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi(t)\mathrm{d}t = 1$. We call (1) the $\phi-$transform. Our aims are:

1. To present a precise characterization of the spaces of distributions with values in Banach spaces in terms of norm size estimates for (1).
2. To give a general Tauberian theorem for scaling asymptotic properties of distributions.
3. To illustrate our results with some applications:
   - Conditions for stabilization in time of solutions to a class of Cauchy problems.
   - Tauberian theorems for Laplace transforms.

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# Outline

# General Notation

- $E$ always denotes a fixed Banach space with norm $\|\cdot\|_E$.
- $X$ stands for a (Hausdorff) locally convex topological vector space.
- $\mathcal{S}'(\mathbb{R}^n, X) = L_b(\mathcal{S}(\mathbb{R}^n), X)$, the space of $X$-valued tempered distributions.
- $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$, the upper half-space.
- The kernel $\phi \in \mathcal{S}(\mathbb{R}^n)$ is fixed and satisfies $\int_{\mathbb{R}^n} \phi(t)\mathrm{d}t = 1$
- We use the Fourier transform

$$\hat{\psi}(u) = \int_{\mathbb{R}^n} \psi(t) e^{-iu\cdot t} \mathrm{d}t.$$

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# Statement of the problem

Suppose that $\mathbf{f}$ takes a priori values in the “broad” space $X$, i.e.,

- $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$.

Suppose that the “narrower” space

- $E$ is continuously embedded in $X$.

If we know that $\mathbf{f}$ takes values in $E$, $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$, then (for some $k, l, M$):

$$\|F_\phi \mathbf{f}(x,y)\|_E \leq C \frac{(1+y)^k (1+|x|)^l}{y^k}, \quad (x,y) \in \mathbb{H}^{n+1}. \qquad (2)$$

We call (2) a (Tauberian) class estimate.

Converse problem: Does the class estimate (2) allow one to conclude that $\mathbf{f}$ actually takes values in $E$? The problem has a Tauberian character.

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# Motivation

The stated problem was first raised and studied by Drozhzhinov and Zav'yalov. It gives a general setting to attack problems such as:

1. Classical Hardy-Littlewood-Karamata type Tauberian theorems for various integral transforms (e.g., the Laplace transform).
2. Stabilization in time for certain Cauchy problems (e.g., for the heat equation).
3. Norm estimates for solutions to certain PDE (e.g., the Schrödinger equation)
4. Wavelet characterizations of important Banach spaces of functions and distributions (e.g., Besov type spaces).
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# Characterization of distributions with values in Banach spaces: Local class estimates

A local version of the Tauberian class estimate suffices to characterize the spaces of distributions with values in Banach spaces:

**Theorem**

*Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, X)$. Then, $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ if and only if*

1. *$F_\phi \mathbf{f}(x, y)$ takes values in $E$ for almost all $(x, y) \in \mathbb{R}^n \times (0, 1)$ and is measurable as an $E$-valued function on $\mathbb{R}^n \times (0, 1)$, and,*
2. *There are $k, l \in \mathbb{N}$ and $C > 0$ such that*

$$\|F_\phi(x, y)\|_E \leq C \frac{(1 + |x|)^l}{y^k}, \quad (x, y) \in \mathbb{R}^n \times (0, 1).$$

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# Scaling weak-asymptotics

We are interested in asymptotic representations

$$f(at) \sim \rho(a)g(t),$$

as $a \to 0^+$ or $a \to \infty$, in the distributional sense, i.e.,

$$\langle f(at), \psi(t) \rangle \sim \rho(a) \langle g(t), \psi(t) \rangle, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^n). \qquad (3)$$

If (3) holds, then, for some $\alpha \in \mathbb{R}$,

- $g$ is homogeneous of degree $\alpha$, i.e., $g(at) = a^\alpha g(t)$,
- $\rho(a) = a^\alpha L(a)$, where $L$ is a Karamata slowly varying function, i.e.,

$$L(ca) \sim L(a), \quad \forall c > 0.$$

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# Weak-asymptotics

## Notation

- We write $f(at) \sim a^{\alpha} L(a) g(t)$ in $\mathcal{S}'(\mathbb{R}^n)$ if

$$\lim \frac{\langle f(at), \psi(t) \rangle}{a^{\alpha} L(a)} = \langle g(t), \psi(t) \rangle, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^n).$$

- For small $a = \varepsilon \to 0^+$ and large $a = \lambda \to \infty$.

Example:

- Let $x_0 \in \mathbb{R}^n$. We say that $f$ has Łojasiewicz point value $\gamma \in \mathbb{C}$ at $x_0$, and write $f(x_0) = \gamma$, distributionally, if

$$\lim_{\varepsilon \to 0^+} f(x_0 + \varepsilon t) = \gamma \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

- Typical example: $s_{\gamma,\beta}(t) = |t|^{\gamma} \sin(|t|^{-\beta})$ has value $s_{\gamma,\beta}(0) = 0$, for all $\gamma \in \mathbb{R}$ and $\beta > 0$.

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# Tauberian theorems for the $\phi-$transform

## Scaling weak-asymptotic behavior

### Theorem

*The distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ has weak-asymptotic behavior*

$$f(at) \sim a^{\alpha} L(a) g(t) \quad \textit{in } \mathcal{S}'(\mathbb{R}^n)$$

*as $a \to 0^+$ (resp. $a \to \infty$) if and only if*

$$\lim_{a \to 0^+} \frac{1}{a^{\alpha} L(a)} F_{\phi} f(ax, ay) = F_{x,y} \ , \quad \textit{for each } |x|^2 + y^2 = 1, \ y > 0,$$

*and*

$$\limsup_{a \to 0^+} \sup_{|x|^2 + y^2 = 1, \ y > 0} \frac{y^k}{a^{\alpha} L(a)} \left| F_{\phi} f(ax, ay) \right| < \infty, \quad \textit{for some } k \in \mathbb{N},$$

*resp. as $a \to \infty$.*
*In such a case, g is completely determined by $F_{\phi} g(x, y) = F_{x,y}$.*

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# A Generalized Cauchy problem

We will consider the Cauchy problem

$$\frac{\partial}{\partial t}U(x,t)=P\left(\frac{\partial}{\partial x}\right)U(x,t),\quad (x,t)\in\mathbb{H}^{n+1},$$

$$\lim_{t\to 0^{+}}U(x,t)=f(x)\quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

- $\Gamma\subseteq\mathbb{R}^n$ is a closed convex cone with vertex at the origin. Possible situation: $\Gamma=\mathbb{R}^n$.
- $P$ is a homogeneous polynomial of degree $d$. Assume:

$$\Re e\, P(iu)<0,\quad u\in\Gamma,\ u\neq 0.$$

- $f\in\mathcal{S}'(\mathbb{R}^n)$. Assume $\operatorname{supp}\hat{f}\subseteq\Gamma$.

# Asymptotic stabilization in time for solutions

We ask for conditions which ensure the existence of a function $T : (A, \infty) \to \mathbb{R}_+$ such that the following limit exists

$$\lim_{t \to \infty} \frac{U(x,t)}{T(t)} = \ell,$$

uniformly for $x$ in compacts of $\mathbb{R}^n$.

# Generalized Cauchy problem

## Solution

If $U$ is required to have slow growth over $\mathbb{H}^{n+1}$, i.e.,

$$\sup_{(x,t)\in\mathbb{H}^{n+1}} |U(x,t)| \left(t+\frac{1}{t}\right)^{-k_1} (1+|x|)^{-k_2} < \infty, \quad \text{for some } k_1, k_2 \in \mathbb{N},$$

then the Cauchy problem has a unique solution. Moreover,

$$U(x,t) = \frac{1}{(2\pi)^n} \left\langle \hat{f}(u), e^{ix\cdot u} e^{P\left(it^{1/d}u\right)} \right\rangle .$$

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# Relation with the $\phi-$transform

Choose a test function $\eta \in \mathcal{S}(\mathbb{R}^n)$ with the property

$$\eta(u) = e^{P(iu)}, \text{ for } u \in \Gamma;$$

setting $\phi(\xi) = (2\pi)^{-n}\hat{\eta}(\xi)$, we express $U$ as a $\phi-$transform,

$$U(x,t) = \left\langle f(\xi), \frac{1}{t^{n/d}}\phi\left(\frac{\xi - x}{t^{1/d}}\right)\right\rangle = F_\phi f(x,y), \quad \text{with } y = t^{1/d},$$

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# Stabilization along *d*-curves

We say $U$ *stabilizes along $d$-curves* (at infinity), relative to $\lambda^{\alpha}L(\lambda)$, if the following two conditions hold:

1. there exist the limits

$$\lim_{\lambda\to\infty}\frac{U(\lambda x,\lambda^{d}t)}{\lambda^{\alpha}L(\lambda)}=U_0(x,t),\quad (x,t)\in\mathbb{H}^{n+1};$$

2. there are constants $C\in\mathbb{R}_+$ and $k\in\mathbb{N}$ such that

$$\left|\frac{U(\lambda x,\lambda^{d}t)}{\lambda^{\alpha}L(\lambda)}\right|\leq\frac{C}{t^{k}},\quad |x|^2+t^2=1,\ t>0.$$

# Stabilization in time for Cauchy problems

## Theorem

*The solution U to the Cauchy problem stabilizes along d-curves if and only if f has weak-asymptotic behavior at infinity, relative to $\lambda^{\alpha}L(\lambda)$.*

## Corollary

*If U stabilizes along d-curves, relative to $\lambda^{\alpha}L(\lambda)$, then U stabilizes in time with respect to $T(t) = t^{\alpha/d}L(t^{1/d})$. That is, there is a constant $\ell$ such that*

$$\lim_{t\to\infty} \frac{U(x,t)}{T(t)} = \ell,$$

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# Example: The heat equation

We immediately recover a result of Drozhzhinov and Zavialov for the heat equation.

Let $U$ be the solution to the Cauchy problem (here actually $\Gamma = \mathbb{R}^n$)

$$\frac{\partial}{\partial t} U = \Delta_x U,$$

$$\lim_{t \to 0^+} U(x,t) = f(x) \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

If $U$ stabilizes along parabolas (i.e., d=2), then it stabilizes in time.

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# Multidimensional Laplace transforms

Let $\Gamma$ be a closed convex acute cone with vertex at the origin. Acute means that the conjugate cone

$$\Gamma^{*}=\{\xi\in\mathbb{R}^{n}:\ \xi\cdot u\geq 0,\forall u\in\Gamma\}\ \text{has non-empty interior.}$$

Set

$$\mathcal{S}'_{\Gamma}=\left\{h\in\mathcal{S}'(\mathbb{R}^{n}):\ \operatorname{supp} h\subseteq\Gamma\right\}$$

$$C_{\Gamma}=\operatorname{int}\Gamma^{*}\ \text{and}\ T^{C_{\Gamma}}=\mathbb{R}^{n}+iC_{\Gamma}.$$

Given $h\in\mathcal{S}'_{\Gamma}$, its Laplace transform is defined as

$$\mathcal{L}\left\{h;z\right\}=\left\langle h(u),e^{iz\cdot u}\right\rangle,\quad z\in T^{C_{\Gamma}};$$

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# Laplace transforms as $\phi-$transforms

We may express the Laplace transform as a $\phi-$transform if we fix a direction in $C_\Gamma$.

- Fix $\omega \in C_\Gamma$
- Choose $\eta_\omega \in \mathcal{S}(\mathbb{R}^n)$ such that $\eta_\omega(u) = e^{-\omega\cdot u}$, $\forall u \in \Gamma$
- Set

$$\phi_\omega = 1/(2\pi)^n \hat{\eta}_\omega \text{ and } \hat{f} = (2\pi)^n h$$

Then,

$$\mathcal{L}\left\{h; x + i\sigma\omega\right\} = F_{\phi_\omega} f(x,\sigma), \quad x \in \mathbb{R}^n,\ \sigma \in \mathbb{R}_+.$$

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# Tauberian theorem for Laplace transforms

## Corollary

*Let $h \in \mathcal{S}'_\Gamma$. Then, an estimate (for some $\omega \in C_\Gamma$, $k \in \mathbb{N}$)*

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2+\sigma^2=1} \frac{\sigma^k \varepsilon^{n+\alpha}}{L(1/\varepsilon)} \left|\mathcal{L}\left\{h; \varepsilon\left(x + i\sigma\omega\right)\right\}\right| < \infty, \tag{4}$$

*and the existence of an open subcone $C' \subset C_\Gamma$ such that*

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon^{\alpha+n}}{L(1/\varepsilon)} \mathcal{L}\left\{h; i\varepsilon\xi\right\} = G(i\xi), \quad \text{for each } \xi \in C', \tag{5}$$

*are necessary and sufficient for*

$$h(\lambda u) \sim \lambda^\alpha L(\lambda) g(u) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}^n), \quad \text{for some } g \in \mathcal{S}'_\Gamma.$$

*In such a case $G(z) = \mathcal{L}\left\{g; z\right\}$, $z \in T^{C_\Gamma}$.*

# Further results: Wavelets

Versions of the discussed Tauberian theorems are also valid if one replaces the $\phi-$transform by a wavelet transform

$$\mathcal{W}_{\psi}\mathbf{f}(x,y) = \langle \mathbf{f}(x+yt), \overline{\psi}(t) \rangle = \int_{\mathbb{R}^n} \mathbf{f}(t) \frac{1}{y^n} \overline{\psi}\left(\frac{t-x}{y}\right) \mathrm{d}t$$

where $\int_{\mathbb{R}^n} \psi(t)\mathrm{d}t = 0$. The wavelet must be non-degenerate:

**Definition**

$\psi \in \mathcal{S}(\mathbb{R}^n)$ is non-degenerate if $\hat{\psi}$ does not identically vanish along any ray through the origin.

The Tauberian theorems then hold up to polynomial corrections.

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