On the sum function of the Möbius function of a Beurling number system

(Jasson Vindas, Budapest, Rényi Institute, 25-7-24)

$\boxed{1}$ Introduction: For the usual natural numbers, it is well-known (due to Landau, 1911) that the PNT in the form

(1.1) $$\pi(x) \sim \frac{x}{\log x}$$

is elementarily equivalent to

(1.2) $$M(x) = o(x).$$

For Beurling primes, this is not the case, here is an example first observed by Zhang (1987). Let us mention that we work with Beurling systems that are not necessarily discrete. For them, Π and N are right continuous non-decreasing functions such that $N(1)=1$, $\Pi(1)=0$, and

$$\zeta(s) = \int_{1^-}^{\infty} x^{-s}\, dN(x) = \exp\left(\int_{1}^{\infty} x^{-s}\, d\Pi(x)\right) \quad \text{(typically for } \operatorname{Re} s > 1\text{)}.$$

$$\frac{1}{\zeta(s)} = \int_{1^-}^{\infty} x^{-s}\, dM(x) \quad \text{(definition of } M\text{)}.$$

Example 1.1 (Beurling, 1937). Consider

$$\Pi(x) = \int_{1}^{x} \frac{1-\cos(\log u)}{\log u}\, du, \quad \text{so that}$$

$$\log \zeta(s) = -\log(s-1) + \frac{1}{2}\log(s-1-i) + \frac{1}{2}\log(s-1+i).$$

The zeta function is then

$$\zeta(s) = \left(1 + \frac{1}{(s-1)^2}\right)^{\frac{1}{2}},$$

having a simple pole at $s=1$ and zeros at $s = 1 \pm i$, of order $\frac{1}{2}$. This Π does not satisfy the PNT

$$\Pi(x) = \frac{x}{\log x}\left(1 - \frac{\sqrt{2}}{2}\cos\left(\log x - \frac{\pi}{4}\right)\right) + O\left(\frac{x}{\log^2 x}\right).$$

Thus, the PNT breaks here.

The N satisfies

$$N(x) = x + O\left(\frac{x}{\log^{3/2} x}\right).$$

We see that

$$\int_0^\infty x^{-s}\, dM(x) = \frac{1}{\zeta(s)}$$ converges to an L^1_{loc} function on $\mathrm{Re}\, s = 1$. So does $\int_0^\infty x^{-s}[dN(x) + dM(x)] - \frac{1}{s-1}$.

By the Wiener-Ikehara theorem,

$$M(x) = o(x).$$

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We see that (1.2) $\not\Rightarrow$ (1.1). We want to discuss:

<u>Q 1</u>: Does (1.1) $\Rightarrow$ (1.2)?

<u>Q 2</u>: Find mild conditions so that (1.2) holds.

For Q1 we know no answer, but we discuss in Section 7 a situation where (1.2) always holds and therefore no counterexample could possibly be constructed. The other sections are dedicated to Q2.

Other definition we need is $d\psi = \log \cdot d\Pi$, in terms of Mellin-Stieltjes transforms

$$\int_1^\infty x^{-s} d\psi(x) = -\frac{\zeta'(s)}{\zeta(s)}.$$

$\boxed{2}$ $N(x) \ll x$ and $\Pi(x) \sim \frac{x}{\log x} \Rightarrow M(x) = o(x)$.

We show in this section the statement from the header. Joint work with Debruyne and Diamond (2018). We call

(2.1) $N(x) \ll x$.

Lemma 2.1 Under (1.1) and (2.1),

$$\frac{M(x)}{x} = -\frac{1}{\log x}\int_1^x \frac{M(t)}{t^2}\,dt + o(1).$$

Proposition:

(2.2) $\log dM = -dM * d\psi$. ///

Proof of Lemma 1: Our assumption of the PNT is of course equivalent to the fact that the primitive of

$$\delta_1 + dx - d\psi$$

is $O(x)$. So we add and subtract $(\delta_1 + dt) * dM$ to (2.2) and then integrate:

$$(2.3) \quad \int_1^x \log t \, dM(t) = \int_1^x dM * (\delta_1 + dt - d\psi) - \int_{1^-}^x dM * (\delta_1 + dt).$$

The LHS of (2.3) is

$$M(x)\lg x - \int_1^x \frac{M(t)}{t}\,dt = M(x)\lg x + O(x).$$

The first term of the RHS is (we use $|dM| \le dN$)

$$\int_1^x \left(\frac{x}{t} - \psi\left(\frac{x}{t}\right)\right) dM(t) \ll \int_1^x O\left(\frac{x}{t}\right) dN(t) = O(x \lg x).$$

the last term is

$$-\int_{1^-}^x \frac{x}{t}\,dM(t) = M(x) + x\int_1^x \frac{M(t)}{t^2}\,dt.$$

<u>Theorem 2.2</u> (1.1) and (2.1) $\Rightarrow$ (1.2).

<u>Proof.</u> Claim 1: We may assume $M(x)$ has infinitely many sign changes, otherwise, by Lemma 2.1, if M has the same sign for $x > z$,

$$\frac{M(x)}{x} + \frac{1}{\lg x}\int_z^x \frac{M(t)}{t^2}\,dt = o(1) \Rightarrow \frac{M(x)}{x} = o(1).$$

and we would be done.

Suppose M changes sign at X. We show $\frac{M(x)}{x} = \mathcal{O}(1)$.

Suppose $M(x) > 0$ (similarly $M(x) < 0$, and $M(x) = 0$ is trivial).

There is then $y \in (x-1, x)$ with $M(y) \le 0$.

$$\frac{M(x)}{x} = \mathcal{O}(1) - \frac{1}{\log x} \int_y^x \frac{M(t)}{t^2}\,dt - \frac{\log y}{\log x} \frac{1}{\log y} \int_1^y \frac{M(t)}{t^2}\,dt$$

$$= \mathcal{O}(1) + \frac{1}{\log x} \int_y^x O\left(\frac{1}{t}\right) dt + \frac{\log y}{\log x} \frac{M(y)}{y}$$

$$\Rightarrow \frac{M(x)}{x} - \frac{M(y)}{y} = \mathcal{O}(1) \Rightarrow \frac{M(x)}{x} = \mathcal{O}(1).$$

Finally if M changes sign at y and has the same sign on $(y, z]$, we have for $x \in (y, z]$ via the same calculation

$$\frac{M(x)}{x} + \frac{1}{\log x} \int_y^x \frac{M(t)}{t^2} = \frac{\log y}{\log x} \frac{M(y)}{y} + \mathcal{O}(1) = \mathcal{O}(1),$$

which implies $\frac{M(x)}{x} = \mathcal{O}(1)$ because the two terms in the LHS have the same sign. ///

$\boxed{3}$ A remainder in the PNT leading to $M(x) = \mathcal{O}(x)$.

Diamond proved in 1977 that the condition

$$(3.1) \quad \int_1^\infty \left| \Pi(x) - \frac{x}{\log x} \right| \frac{dx}{x^2} < \infty$$

always yields

$$(3.2) \quad N(x) \sim a \cdot x \qquad (a > 0).$$

Combining this fact with Theorem 2.2 gives:

Theorem 3.1: Suppose that

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^{\alpha} x}\right)$$

with $\alpha > 1$. Then $M(x) = o(x)$ holds. ///

So if we have slightly more than the PNT (1.1), we get (1.2).

$\boxed{4}$ **Do $N(x) = O(x)$ and $\pi(x) = O\left(\frac{x}{\log x}\right)$ imply $M(x) = o(x)$?**

The answer is no. Here a counterexample found together with Debruyne and Diamond. The bound $\pi(x) \ll \frac{x}{\log x}$ is of course equivalent to $\psi(x) \ll x$. We need the following lemma

Lemma 4.1. If $M(x) = o(x)$, then $\zeta(s)$ has no right-hand zero of order ≥ 1 on $\operatorname{Re} s = 1$.

Proof. Indeed, (1.2) $\Rightarrow$ $\frac{1}{\zeta(\sigma + it)} = o\left(\frac{1}{1-\sigma}\right)$. ///

The counterexample is

$$\pi(x) = \sum_{2^{k+\frac{1}{2}} \leq x} \frac{2^{k+\frac{1}{2}}}{k}.$$

Here

$$\psi(x) = 2^{\lfloor \log x / \log 2 + \frac{1}{2} \rfloor + \frac{1}{2}} \log 2 + O\left(\frac{x}{\log x}\right),$$

so that $x \ll \psi(x) \ll x$.

$$\log \zeta(s) = 2^{-(s-1)/2} \sum_{k=1}^{\infty} \frac{2^{-k(s-1)}}{k} = -2^{(s-1)/2} \log\left(1-2^{-(s-1)}\right)$$

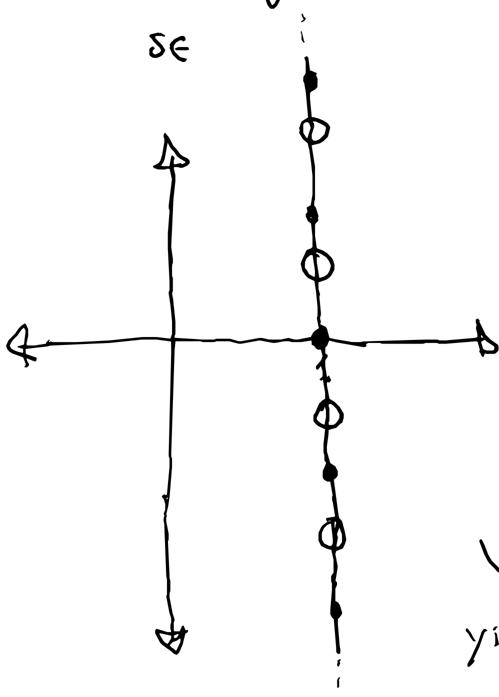
We have that $\zeta(s+4\pi i/\log 2) = \zeta(s)$ also

$$\zeta(s+2\pi i/\log 2) = \frac{1}{\zeta(s)}.$$

A small calculation shows that for $s \to 1$ with $\operatorname{Re} s \geq 1$

$$\zeta(s) = \frac{1}{(s-1)\log s} + O\left(\log\left|\frac{1}{s}\right|\right)$$

So our zeta function has pole-like singularities at $s \in 1 + \frac{4\pi i}{\log 2}\mathbb{Z}$ and right hand zeros at $s \in 1 + \frac{2\pi i}{\log 2} + \frac{4\pi i}{\log 2}\mathbb{Z}$



Therefore, by Lemma 4.1,

$$M(x) = \Omega(x).$$

Since $\zeta(s) - \frac{1}{(s-1)\log 2}$ has L^1_{loc}-boundary behavior near $s=1$, a version of the Wiener-Ikehara theorem yields $N(x) = O(x)$.

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5 $N(x) \sim ax$ and $\Pi(x) \ll \frac{x}{\log x} \Rightarrow M(x) = o(x)$

The PNT can be relaxed to a Chebyshev upper bound if we strengthen $N(x) \ll x$ to positive asymptotic density. By adopting the proof of the classical mean-valued theorem of Halász, we (Debruyne, Diamond, V.) showed in 2018 this under the extra hypothesis

$$(5.1) \quad \int_1^\infty \frac{|N(x) - ax|}{x^{\sigma+1}}\,dx \ll \frac{1}{(\sigma-1)^\beta},$$

for some $\beta \in (0, \frac{1}{2})$. In collaboration with Debruyne and Maes, we showed in 2020 that (5.1) could be removed:

__Theorem 5.1__ If $N(x) \sim ax$ for some $a > 0$ and $\Pi(x) \ll \frac{x}{\log x}$, then $M(x) = o(x)$ holds. ///

We will only sketch the proof of Theorem 5.1.

__Lemma 5.2__ Assume that $N(x) \sim ax$, then $M(x) = o(x)$ iff

$$(5.2) \quad f(x) := \int_1^x \left(\int_1^u \log t\, dM(t) \right) \frac{du}{u} = o(x \log x). \quad ///$$

__Proof.__ This follows via standard elementary Tauberian arguments, we therefore omit details. ///

The following lemma is also classical and follows from the 3-4-1 inequality and Dini's theorem on uniform convergence of monotone pointwise convergence of sequences of continuous functions on compact sets.

<u>Lemma 5.3.</u> Suppose $N(x) \sim ax$, with $a > 0$, then

$$\frac{1}{\zeta(\sigma+it)} = o\left(\frac{1}{\sigma-1}\right), \text{ uniformly for } t \text{ on compact sets.}$$

<u>Remark 5.4</u> Lemmas 5.2 and 5.3 are the only places where $N(x) \sim ax$ is used. The rest of the steps only use $N(x) = O(x)$ and $\psi(x) = O(x)$.

If dF is such that $F(x) = O(x)$, then by Plancherel theorem $(\sigma > 1)$

$$\int_{\operatorname{Re} s=\sigma} \left|\frac{\hat{F}(s)}{s}\right|^2 |ds| = 2\pi \int_0^\infty e^{-2\sigma x} |F(e^x)|^2 dx$$

$$\ll \frac{1}{\sigma-1}.$$

<u>Corollary 5.5</u> If $\psi(x) \ll x$ and $N(x) \ll x$, then

$$\int_{\operatorname{Re} s=\sigma} \left|\frac{\zeta'(s)}{s\zeta(s)}\right|^2 |ds| \ll \frac{1}{\sigma-1}$$

and

$$\int_{\operatorname{Re} s=\sigma} \left|\frac{\zeta(s)}{s}\right|^2 |ds| \ll \frac{1}{\sigma-1}$$

Lemma 5.6 (Wiener–Wintner)

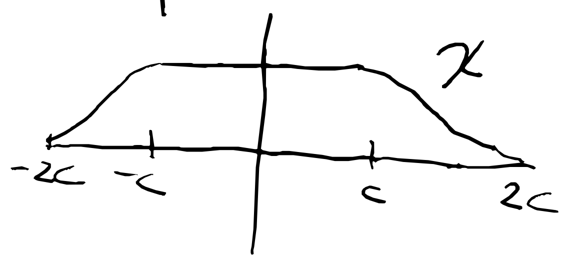
Let F_1 be nondecreasing and F_2 be of bounded $\mathbb{C}$-valued variation, supported on $[1,\infty)$ and with convergent Mellin–Stieltjes transforms $\hat{F}_j$ on $\operatorname{Re} s > \alpha$. If $\alpha < \sigma$ and $|dF_2| \le dF_1$, then

$$\int_b^{b+c} |\hat{F}_2(\sigma+it)|^2\,dt \le \int_{-c}^{c} |\hat{F}_1(\sigma+it)|^2\,dt.$$

Proof. Replacing dF_2 by $x^{-ib}\,dF_2(x)$, we may assume that $b = -\frac{c}{2}$. Replacing then $\frac{c}{2}$ by c, we have to show that

$$\int_{-c}^{c} |F_2(\sigma+it)|^2\,dt \le \int_{-2c}^{2c} |\hat{F}_1(\sigma+it)|^2\,dt.$$

Let χ be a trapezoidal C^∞ function like



$$\phi(u) = \int_{-\infty}^{\infty} e^{iux}\chi(x)\,dx \geq 0$$

Then

$$\int_{-c}^{c} |F_2(\sigma+it)|^2\,dt \le \int_{-2c}^{2c}\int_1^\infty\int_1^\infty (xy)^{-\sigma}(x/y)^{it}\,\chi(t)\,dF_2(x)\,d\bar{F}_2(y)$$

$$\le \int_{1}^{\infty}\int_{1}^{\infty} \phi\left(\log\left(\frac{x}{y}\right)\right)(xy)^{-\sigma} dF_1(x)\,dF_2(y)$$

$$= \int_{-2c}^{2c} \chi(t)\,|\hat{F}_1(\sigma+it)|^2 dt$$

$$\le \int_{-2c}^{2c} |\hat{F}_1(\sigma+it)|^2 dt. \quad ///$$

We are ready to start the proof Theorem 5.1.

By Lemma 5.2, we have to prove that

$$\frac{f(x)}{x} = o(\log x).$$

The Mellin-Stieltjes transform of f is

$$-\frac{1}{s}\left(\frac{1}{\zeta(s)}\right)'$$

We have, using Perron inversion with $\sigma_x = 1 + \frac{1}{\log x}$

$$\left|\frac{f(x)}{x}\right| = \frac{1}{2\pi}\left|\int_{\mathrm{Re}\, s=\sigma} \frac{x^{s-1}}{s^2}\,\frac{\zeta'(s)}{\zeta(s)}\cdot\frac{1}{\zeta(s)}\,ds\right|$$

$$\ll (\log x)^{1/2}\left(\int_{\mathrm{Re}=\sigma_x} \left|\frac{1}{s\,\zeta(s)}\right|^2 |ds|\right)^{1/2} = \log^{\frac{1}{2}} x\,(I(x))^{\frac{1}{2}}$$

where we have used the first inequality in

Corollary 5.5. Our problem then reduces to show that

$$(5.3) \quad I(x) = \int_{\operatorname{Re} s = \sigma_x} \left| \frac{1}{s\zeta(s)} \right|^2 |ds| = O(\log x).$$

Let λ be a large but for the moment fixed number. We split $I(x) = I_1(x) + I_2(x)$, where $I_1(x)$ corresponds to integration over $\{\sigma_x + it : |t| \geq \lambda\}$ and $I_2(x)$ to $\{\sigma_x + it : |t| \leq \lambda\}$.

Now, using Lemma 5.6 and the second bound in Corollary 5.5., we get that (since $|dM| \leq dN$).

$$\int_{\lambda+m \leq |t| \leq \lambda+m+1} \left| \frac{1}{\zeta(\sigma_x+it)} \right|^2 dt \leq 2 \int_{-1}^{1} |\zeta(\sigma_x+it)|^2 dt$$

$$\ll \frac{1}{\sigma_x - 1} = \log x.$$

Therefore,

$$(5.4) \quad I_1(x) \ll \log x \sum_{m=0}^{\infty} \frac{1}{1+m^2+\lambda^2} \ll \frac{\log x}{\lambda}$$

Let $dG = \exp^*(-\frac{3}{4} d\Pi)$, so that $\hat{G}(s) = \left(\frac{1}{\zeta(s)}\right)^{3/4}$.

Using Lemma 5.3,

$$(5.5) \quad I_2(x) = O_\lambda((\log x)^{\frac{1}{2}}) \cdot \int_{\operatorname{Re} s = \sigma_x} \left| \frac{\hat{G}(s)}{s} \right|^2 |ds|.$$

Therefore, (5.3) would follow from (5.4) and (5.5) if one is able to show that

$$(5.6)\quad \int_{\operatorname{Re} s=\sigma_x} \left|\frac{\hat{G}(s)}{s}\right|^2 |ds| \ll \log^{\frac{1}{2}} x.$$

Observe that $|dG(x)| \leq dN(x)$ so that $G(x)=O(x)$. This this bound combined with the simple observation above Corollary 5.5 does not yield (5.6). Nevertheless, we still get

$$(5.7)\quad \int_{\operatorname{Re} s=\sigma_x} \left|\frac{\hat{G}(s)}{s}\right|^2 |ds| \ll \int_0^\infty e^{-2y(\sigma-1)} \left|\frac{G(e^y)}{e^y}\right|^2 dy.$$

So, to make progress, we need to get a subtle bound for $\frac{G(x)}{x}$. We will achieve this by applied an elementary but powerful technique, known as Rankin's method.

<u>Lemma 5.7.</u> $\frac{G(x)}{x} \ll \frac{1}{(\log x)^{\frac{1}{4}}}$. ///

Before proving Lemma 5.7, notice that inserting it into (5.7) yields

$$\int_{\operatorname{Re} s=\sigma_x} \left|\frac{\hat{G}(s)}{s}\right|^2 |ds| \ll \int_0^\infty e^{-2y(\sigma_x-1)} y^{-\frac{1}{2}} dy$$

$$\ll \frac{1}{(\sigma_x - 1)^{1/2}} = (\log x)^{1/2},$$

showing the desired bound (5.6).

<u>Proof of Lemma 5.7:</u> First notice that

$$N(x) = O(x) \Rightarrow \zeta(\sigma) \ll \frac{1}{\sigma - 1} \Rightarrow$$

$$\int_1^\infty \frac{d\Pi(x)}{x^\sigma} = \log \zeta(\sigma) \le O(1) + \log\left(\frac{1}{\sigma-1}\right).$$

An elementary Tauberian argument implies

$$(5.8) \quad \int_1^x \frac{d\Pi(u)}{u} \le \log\log x + O(1).$$

We now use that multiplication by log acts as a derivation in the (Mellin) convolution algebra of measures. In particular, $\log \cdot \exp^*(dA) = (\log dA) * \exp^*(dA)$. Applying this to $dG = \exp^*(-\frac{3}{4} d\Pi)$,

$$\int_1^x \log u \, |dG(u)| \le \frac{3}{4} \int_1^x \underbrace{(\log \cdot d\Pi)}_{d\psi} * |dG|$$

$$= \frac{3}{4} \int_1^x \psi\left(\frac{x}{u}\right) |dG(u)| \ll x \int_1^x \frac{|dG(u)|}{u}$$

$$= x \int_1^x \exp^*\left(\frac{3}{4} \frac{d\Pi(u)}{u}\right)$$

$$\le x \exp^*\left(\frac{3}{4} \int^x \frac{d\Pi(u)}{u}\right) = x \log^{3/4} x,$$

where we have used (5.8). Integrating by parts

$$G(x) \leq O(1) + \int_2^x \log u \, d|G(u)| \cdot \frac{1}{\log u}$$

$$\ll 1 + x \frac{\log^{3/4} x}{\log x} + \int_2^x \frac{du}{\log^{5/4} u}$$

$$\ll \frac{x}{(\log x)^{1/4}},$$

completing the proof. ///

6 Sufficient condition for $M(x) = \vartheta(x)$ in terms of N

The counterpart of Theorem 3.1 in terms of N is:

Theorem 6.1 If $N(x) = ax + O\left(\frac{x}{\log^{\alpha} x}\right)$, $\alpha > 1$, $a > 0$, then $M(x) = \vartheta(x)$.

Proof: Diamond showed in 1973 that the assumption from this theorem implies Chebyshev's bounds, the rest follows from Theorem 5.1.

7 Another sufficient condition for $M(x) = \vartheta(x)$ preventing the existence of certain counterexamples.

We have no clue whether PNT $\Rightarrow$ $M(x) = \vartheta(x)$.
The only thing we can say today is a warning for those who choose to attempt to disprove it.

If one would like to give a counterexample to PNT $\Rightarrow$ $M(x) = \vartheta(x)$, Theorem 2.2 tells us that $\frac{N(x)}{x}$ must be unbounded. During a visit to J.-P. Kahane in Paris (2017), he proposed

$$d\Pi(u) = \frac{1-\frac{1}{u}}{\log u}\,du + \frac{1-\frac{1}{u}}{(\log u)(\log\log(e^e+u))}\,du,$$

as a possible counterexample. In fact, here $\Pi(x) \sim \frac{x}{\log x}$ and one can show that $\frac{N(x)}{x} \to \infty$.
We eventually showed that for it $M(x) = \vartheta(x)$, failing to deliver a counterexample. In fact, we showed that any monotone perturbation of $\Pi(x) = \int_1^x \frac{1-\frac{1}{u}}{\log u}\,du$ and of Riemann's counting function of the classical primes that satisfies the PNT also fails to be a counterexample.

The precise statement is the following one.
For a proof, see my latest article listed below.

<u>Theorem 7.1</u> Let $d\Pi$ admit the decomposition

$$d\Pi = d\Pi_0 + dE,$$

where

$$\int_{1^-}^{x} |dE(u)| = o\left(\frac{x}{\log x}\right)$$

and

$$M_0(x) = \int_{1^-}^{x} \exp^*(-d\Pi_0) = O\left(\frac{x}{\log^a x}\right).$$

Then $M(x) = \int_{1^-}^{x} \exp^*(-d\Pi) = O\left(\frac{x}{\log^{a'} x}\right)$ for any $a' < a$. ///

8 References

This tutorial to $M(x) = o(x)$ is based on the following works:

- G. Debruyne, H.G. Diamond, J.V., $M(x) = o(x)$ Estimates for Beurling numbers, J. Théor. Nombres Bordeaux 30 (2018), 469-483.
- G. Debruyne, F. Maes, J.V., Halász's theorem for Beurling generalized numbers, Acta Arith.

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- J. V., On the estimate $M(x) = o(x)$ for Beurling generalized numbers, Analysis Math., to appear.